Josef Niederle

Notes on lattices of frame tolerances

Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 3, 468–471

Persistent URL: `http://dml.cz/dmlcz/102399`

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

NOTES ON LATTICES OF FRAME TOLERANCES

JOSEF NIEDERLE, Brno

(Received September 12, 1988)

DEFINITIONS

By a *frame* we mean a complete lattice that satisfies the Join Infinite Distributive Identity $a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i)$. By a *lattice tolerance* on a lattice (or on a frame) we mean a reflexive and symmetric relation on its support compatible with finite meets and joins. By a *frame tolerance* on a frame we mean a lattice tolerance that is compatible with arbitrary joins. *Congruences* are transitive tolerances. A mapping α of a frame L into itself is *extensive* if $(\forall x \in L)\, x \leqq \alpha(x)$, and *idempotent* if $(\forall x \in L)$ $\alpha(\alpha(x)) = \alpha(x)$. The greatest element in $\{z \in L \mid x \wedge z = 0\}$, if it exists, is called the *pseudocomplement* of the element x and will be denoted by x^*. Elements being pseudocomplements of some elements are called *skeletal*. By a *relative pseudocomplement* of an element x with respect to the element y we mean the pseudocomplement of x in $\langle y)$. It will be denoted $x^*(y)$. We shall assume only the case $y \leqq x$.

PSEUDOCOMPLEMENTS IN LATTICES OF LATTICE TOLERANCES

Skeletal elements and relative pseudocomplements of tolerances with respect to congruences in lattices of lattice tolerances were studied in [3] and [1]. Since the transitive hull $\mathbf{C}(T)$ of a lattice tolerance T is a lattice congruence (cf. [5]), and the operator $\mathbf{C}$ is a supremum-complete lattice homomorphism of $\mathbf{Tol}(L)$ onto $\mathbf{Con}(L)$ (cf. [2]) we obtain that the studied elements in $\mathbf{Tol}(L)$ are identical with those in $\mathbf{Con}(L)$.

1. Theorem. *Let L be a lattice. Relative pseudocomplement of a lattice tolerance T with respect to a lattice congruence U in the lattice* $\mathbf{Tol}(L)$ *always exists and is identical with the relative pseudocomplement of* $\mathbf{C}(T)$ *in the lattice* $\mathbf{Con}(L)$.

Proof. We know that $\mathbf{Con}(L)$ is distributive and compactly generated. It follows that it is relatively pseudocomplemented. $\mathbf{C}(T)^{*\mathrm{Con}}(U) = \bigvee^{\mathrm{Con}} \{Y \in \mathbf{Con}(L) \mid Y \wedge \wedge\, \mathbf{C}(T) = U\}$. In both $\mathbf{Tol}(L)$ and $\mathbf{Con}(L)$, meets coincide with set-theoretical intersection. Let T be a lattice tolerance and U a lattice congruence on L such that

$U \subseteqq T$. Then

$$U \subseteqq T \wedge \mathbf{C}(T)^{*\mathrm{Con}}(U) \subseteqq \mathbf{C}(T) \wedge \mathbf{C}(T)^{*\mathrm{Con}}(U) = U\,.$$

Hence $T \wedge \mathbf{C}(T)^{*\mathrm{Con}}(U) = U$. Let S be a lattice tolerance such that $T \wedge S = U$. Then $U = \mathbf{C}(U) = \mathbf{C}(T \wedge S) = \mathbf{C}(T) \wedge \mathbf{C}(S)$, and we may conclude $S \subseteqq \mathbf{C}(S) \subseteqq \subseteqq \mathbf{C}(T)^{*\mathrm{Con}}(U)$. Q.E.D.

As corollaries, we obtain results already known (cf. [3]):

2. Corollary. *For any lattice L, the pseudocomplement of a lattice tolerance T in* $\mathbf{Tol}(L)$ *is identical with the pseudocomplement of the congruence* $\mathbf{C}(T)$ *in* $\mathbf{Con}(L)$.

3. Corollary. *A lattice tolerance on a lattice L is a skeletal element in* $\mathbf{Tol}(L)$ *if and only if it is a skeletal element in* $\mathbf{Con}(L)$.

PSEUDOCOMPLEMENTS IN LATTICES OF FRAME TOLERANCES

It appears natural to ask whether analogous results may be obtained for frame tolerances and frame congruences. Unfortunately, we don't yet know whether the operator $\mathbf{FC}$ assigning to any frame tolerance T the least frame congruence $\mathbf{FC}(T)$ containing T is a lattice homomorphism of the lattice of all frame tolerances $\mathbf{FTol}(L)$ onto the lattice of all frame congruences $\mathbf{FCon}(L)$. We know that $\mathbf{FTol}(L)$ is a frame (cf. [4]). This frame is order isomorphic to the frame $\mathbf{Ext}(L)$ of all extensive $\wedge$-endomorphisms of the frame L, whereby congruences correspond with idempotent extensive $\wedge$-endomorphisms (cf. [4]).

4. Proposition. *The relative pseudocomplement of an extensive $\wedge$-endomorphism with respect to an idempotent $\wedge$-endomorphism in* $\mathbf{Ext}(L)$ *is idempotent.*

Proof. Let α be an extensive $\wedge$-endomorphism and ι an idempotent $\wedge$-endomorphism of the frame L such that $\iota \leqq \alpha$. Since $\mathbf{Ext}(L)$ is a frame, it is relatively pseudocomplemented. Let β be the relative pseudocomplement of α with respect to ι. We have $\alpha \wedge \beta = \iota$, i.e. $(\forall x \in L)\, \alpha(x) \wedge \beta(x) = \iota(x)$. However, $(\alpha \wedge \beta\beta)(x) = = \alpha(x) \wedge \beta\beta(x) = \alpha(x) \wedge \alpha\beta(x) \wedge \beta\beta(x) = \alpha(x) \wedge (\alpha \wedge \beta)\, \beta(x) = \alpha(x) \wedge \iota\beta(x) = = \alpha(x) \wedge \iota\alpha(x) \wedge \iota\beta(x) = \alpha(x) \wedge \iota(\alpha \wedge \beta)(x) = \alpha(x) \wedge \iota\iota(x) = \alpha(x) \wedge \iota(x) = \iota(x)$. Thus $\beta = \beta\beta$, it is an idempotent extensive $\wedge$-endomorphism. Q.E.D.

5. Corollary. *The relative pseudocomplement of a frame tolerance with respect to a frame congruence in* $\mathbf{FTol}(L)$ *is a frame congruence.*

6. Corollary. *The pseudocomplement of an extensive $\wedge$-endomorphism in* $\mathbf{Ext}(L)$ *is idempotent.*

7. Corollary. *The pseudocomplement of a frame tolerance in* $\mathbf{FTol}(L)$ *is a frame congruence.*

8. Theorem. *Let L be a frame. Relative pseudocomplement of a frame tolerance T*

with respect to a frame congruence U in the lattice **FTol**(L) *is identical with the relative pseudocomplement of* **FC**(T) *with respect to U in* **FCon**(L).

Proof. By Corollary 5, $T^{*\mathrm{FTol}}(U)$, $T^{**\mathrm{FTol}}(U) \in \mathbf{FCon}(L)$. It is now obvious that $U \subseteqq T \subseteqq \mathbf{FC}(T) \subseteqq T^{**\mathrm{FTol}}(U)$. Consequently, $U = T \wedge T^{*\mathrm{FTol}}(U) \subseteqq \mathbf{FC}(T) \wedge \wedge T^{*\mathrm{FTol}}(U) \subseteqq T^{**\mathrm{FTol}}(U) \wedge T^{*\mathrm{FTol}}(U) = U$. Further, $\mathbf{FC}(T) \wedge S = U$ implies $T \wedge S = U$, which yields $S \subseteqq T^{*\mathrm{FTol}}(U)$ for any $S \in \mathbf{FCon}(L)$. We have just shown that $T^{*\mathrm{FTol}}(U)$ is the relative pseudocomplement of $\mathbf{FC}(T)$ in $\mathbf{FCon}(L)$. Q.E.D.

9. Corollary. *For any frame L, the pseudocomplement of a frame tolerance T in* **FTol**(L) *is identical with the pseudocomplement of* **FC**(T) *in* **FCon**(L).

10. Corollary. *A frame tolerance on a frame L is a skeletal element in* **FTol**(L) *if and only if it is a skeletal element in* **FCon**(L).

11. Lemma. *In any frame L, and for any $S, T \in$* **FTol**(L), $S \wedge T = U \in$ **FCon**(L) *implies* $\mathbf{FC}(S) \wedge \mathbf{FC}(T) = U$.

Proof. In view of Theorem 8, it is obvious that $U \subseteqq T \subseteqq \mathbf{FC}(T) \subseteqq \mathbf{FC}(S)^* (U)$, which immediately yields $\mathbf{FC}(S) \wedge \mathbf{FC}(T) = U$. Q.E.D.

12. Proposition. *In any frame L, the operator* **FC** *is a frame homomorphism of* **FTol**(L) *onto* **FCon**(L).

Proof. We have already known that **FC** is a complete supremum homomorphism (cf. [4]). It remains to establish $\mathbf{FC}(S \wedge T) = \mathbf{FC}(S) \wedge \mathbf{FC}(T)$. We obtain that $\mathbf{FC}(S \wedge T) = (S \wedge T) \vee \mathbf{FC}(S \wedge T) = (S \vee \mathbf{FC}(S \wedge T)) \wedge (T \vee \mathbf{FC}(S \wedge T))$ holds in **Tol**(L). By the preceding lemma, $\mathbf{FC}(S \wedge T) = \mathbf{FC}(S \vee \mathbf{FC}(S \wedge T)) \wedge \wedge \mathbf{FC}(T \vee \mathbf{FC}(S \wedge T)) = \mathbf{FC}(S) \wedge \mathbf{FC}(T)$. Q.E.D.

ATOMS IN LATTCES OF FRAME TOLERANCES

13. Theorem. *Lattices of all frame tolerances and of all lattice tolerances on a frame have the same atoms.*

Proof. Principal frame tolerances coincide with principal lattice tolerances (cf. [4]). A lattice tolerance being an atom in **Tol**(L) is principal (cf. [2]), and therefore an atom in **FTol**(L). Conversely, let T be an atom in **FTol**(L). Suppose S be a lattice tolerance on L such that $\Delta \subset S \subseteqq T$. Take $[a, b] \in S \setminus \Delta$. Since the principal lattice tolerance $\mathbf{T}(a, b)$ is a frame tolerance. we obtain $\mathbf{T}(a, b) = S = T$. Q.E.D.

14. Corollary. *A frame tolerance T on a frame L is an atom in* **FTol**(L) *if and only if $T =$* $\mathbf{T}(a, b)$ *where $a \succ b$.*

See [2] for the proof.

References

[1] *H.-J. Bandelt:* Tolerance relations on lattices. Bull. Austral. Math. Soc. *23* (1981), 367—381.
[2] *J. Niederle:* On atoms in tolerance lattices of distributive lattices. Časop. pěst. matem. *106* (1981), 311—315.
[3] *J. Niederle:* On skeletal and irreducible elements in tolerance lattices of finite distributive lattices. Časop. pěst. matem. *107* (1982), 23—29.
[4] *J. Niederle:* Tolerances on frames. Arch. Math. (Brno) [Submitted.]
[5] *B. Zelinka:* Tolerances in algebraic structures II. Czechoslovak Math. J. *25* (1975), 175—178.

Author's address: 615 00 Brno 15, Viniční 60, Czechoslovakia.