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FRAME TOLERANCES ARE DIRECTLY DECOMPOSABLE

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A *frame* is a complete lattice satisfying the Join Infinite Distributive Identity $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$. A *lattice tolerance* on a frame (or, generally, on a lattice) is a reflexive symmetric relation on its support compatible with finite meets and joins. A *frame tolerance* on a frame is a lattice tolerance compatible with arbitrary joins (suprema). A lattice (frame) tolerance on the product of frames $\mathfrak{L} = \prod_{i \in I} \mathfrak{L}_i$ is *directly decomposable* if there exist lattice (frame) tolerances T_i on $\mathfrak{L}_i$ such that $T = \prod_{i \in I} T_i = \{[a, b] \in \mathfrak{L} \times \mathfrak{L} \mid \forall (i \in I)\, [p_i(a), p_i(b)] \in T_i\}$ where p_i are projections of $\mathfrak{L}$ onto $\mathfrak{L}_i$ $(i \in I)$.

We know that lattice tolerances on products of finite number of lattices are directly decomposable while those on products of infinite number of nontrivial lattices are not (cf. [1]). The same statement is obviously valid for lattice tolerances on frames. For frame tolerances, we shall prove a stronger result.

Theorem. *Frame tolerances on arbitrary products of frames are directly decomposable.*

Proof. Let T be a frame tolerance on the product of frames $\mathfrak{L} = \prod_{i \in I} \mathfrak{L}_i$. Denote by 0_i the least and by 1_i the greatest elements of $\mathfrak{L}_i$, put $\beta_i(T) = \{[x, y] \in \mathfrak{L}_i \times \mathfrak{L}_i \mid \mid [e_i(x), e_i(y)] \in T\}$ where $e_i(x)$ is defined by $p_j(e_i(x)) = 0_j$ if $i \neq j$, and $p_i(e_i(x)) = x$ $(i, j \in I)$. They are obviously frame tolerances. We shall prove $T = \prod_{i \in I} \beta_i(T)$. Let $[a, b] \in T$. Then $[e_i(p_i(a)), e_i(p_i(b))] = [e_i(1_i) \wedge a, e_i(1_i) \wedge b] \in T$, and so $[p_i(a), p_i(b)] \in \beta_i(T)$ $(i \in I)$. Hence $[a, b] \in \prod_{i \in I} \beta_i(T)$. Conversely, let $[a, b] \in \prod_{i \in I} \beta_i(T)$, i.e. $[p_i(a), p_i(b)] \in \beta_i(T)$ $(i \in I)$. Then $[e_i(p_i(a)), e_i(p_i(b))] \in T$ $(i \in I)$, and consequently $[a, b] = \bigvee_{i \in I} [e_i(p_i(a)), e_i(p_i(b))] \in T$. Q.E.D.

Frame tolerances on a frame form a complete lattice (cf. [2]).

Corollary. *The lattice of all frame tolerances on the product of frames $\mathfrak{L} = \prod_{i \in I} \mathfrak{L}_i$ is isomorphic to the product of lattices of all frame tolerances on the frames $\mathfrak{L}_i$ $(i \in I)$.*

Proof. In fact, the theorem assigns to any frame tolerance T on $\mathfrak{L}$ an element of the product of lattices of frame tolerances on $\mathfrak{L}_i$ $(i \in I)$. This assignment is obviously an injective isotone mapping. It remains to prove its surjectivity. Let T_i $(i \in I)$

be frame tolerances on $\mathfrak{L}_i$ respectively. Put $T = \prod_{i \in I} T_i$.Then $\beta_i(T) = \{[x, y] \in \in \mathfrak{L}_i \times \mathfrak{L}_i \mid [e_i(x), e_i(y)] \in T\} = \{[x, y] \in \mathfrak{L}_i \times \mathfrak{L}_i \mid [p_i(e_i(x)), p_i(e_i(y)) \in T_i\} = = \{[x, y] \in \mathfrak{L}_i \times \mathfrak{L}_i \mid [x, y] \in T\} = T_i$. Q.E.D.

Added in proof. Analogous statements may be *a fortiori* proved for frame congruences. Both proofs work also for frame-compatible reflexive relations.

References

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