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ON DECOMPOSITIONS OF COMPLETE GRAPHS INTO THREE FACTORS WITH GIVEN DIAMETERS

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1. DEFINITIONS AND AUXILIARY RESULTS

All graphs considered in this paper are undirected, without loops and multiple edges. By a factor of a graph G we mean a subgraph of G containing all vertices of G. A system of factors of G such that every edge of G belongs to exactly one of them is called a decomposition of G.

If G is a graph then $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. We shall denote the number of elements of a set M by $|M|$.

The problem of the existence of the decomposition of the complete graph into two factors with given diameters is solved completely in [1]. In [1] many other questions concerning the decompositions of complete graphs into factors with given diameters have been solved and some open problems have been formulated. Later mainly the problem of the decomposition of complete graphs into factors of diameter two was investigated very intensively [3, 4, 5, 6]. In this paper we shall investigate the decompositions of complete graphs into three factors F_1, F_2, F_3 with given diameters d_1, d_2, d_3, respectively. We admit the factor F_3 to be disconnected, i.e. $d_3 = \infty$.

Every shortest path between two vertices of a connected graph G the distance of which is equal to the diameter of G will be called a *diameter path* of G. Note that the diameter path of G may or may not be determined uniquely. In what follows the symbols C_1, C_2, C_3 will denote arbitrarily choosen but fixed diameter paths of connected factors F_1, F_2, F_3, respectively. Further, A_i will be the set of vertices of C_i $(i = 1, 2, 3)$. In some cases the above given indices will be omitted.

In [1, Theorem 9] it has been proved that if none of the diameters d_1, d_2, d_3 equals 1 then there exists a positive integer n such that the complete graph with n vertices (K_n) has a decomposition into 3 factors with the diameters d_1, d_2, d_3. The least such n will be denoted by $F(d_1, d_2, d_3)$ and in the case $d_1 = d_2 = d_3 = d$ by $f(d)$.

The following results (A)–(D) are proved in [1]:

(A) If the complete graph K_n is decomposable into 3 factors with diameters d_1, d_2, d_3

then for $N > n$ the complete graph K_N is also decomposable into 3 factors with the same diameters (it is a special case of Theorem 1 in [1]).

(B) If $5 \leqq d_1 \leqq d_2 \leqq d_3 < \infty$ then

$$F(d_1, d_2, d_3) \leqq d_1 + d_2 + d_3 - 8$$

(it is a part of Theorem 6 in [1]).

(C) If d is an positive integer, $d \geqq 5$ then

$$2{,}366d - 5{,}965 < f(d) \leqq 3d - 8$$

(see Theorem 10 in [1]).

(D) Let $d_1 \leqq d_2 < \infty$. We have:

I. If $d_1 \geqq 5$ then
$\max\{d_2, \frac{2}{3}(d_1 + d_2 - 2)\} < F(d_1, d_2, \infty) \leqq d_1 + d_2 - 4$;

II. If $2 \leqq d_1 \leqq 5$ then $F(d_1, d_2, \infty) = d_2 + 1$ except the values $F(2, 2, \infty) = 5$ and $F(2, 3, \infty) = 6$

(see Theorem 8 in [1]).

Note. In what follows we shall briefly say "the vertices of the graph G are adjacent to k vertices from B" instead of "there are exactly k vertices in B such that each of them is adjacent to at least one vertex of G".

Lemma 1. a) *Each vertex $v \in V(F)$ is adjacent (in the factor F) to at most 3 vertices from A and moreover the distance between any two of them is not greater than two.*

b) *If a connected subgraph G of a factor F has diameter l then the vertices of G are adjacent to at most $l + 3$ vertices from $A - V(G)$ and moreover the distance between any two of them is not greater than $l + 2$.*

Proof. The statements are obvious if we realize that the set A is the set of vertices of some diameter path C in the factor F.

Lemma 2. a) $|A_1 \cap A_2 \cap A_3| \leqq 6$

b) *Let i, j, k be an arbitrary permutation of the numbers 1, 2, 3. If $d_i \geqq 14$ then* $|A_j \cap A_k| \leqq 6$.

Proof. a) Let $|A_1 \cap A_2 \cap A_3| = k$. Every edge of the complete graph K with the vertex set $A_1 \cap A_2 \cap A_3$ must be an edge of some of the diameter paths C_1, C_2, C_3. Each of these paths can contain at most $k - 1$ edges of K and thus we have

$$3(k - 1) \geqq \frac{k(k - 1)}{2}$$

which implies our statement.

b) We shall proceed in indirect way. We suppose that, for instance, $|A_2 \cap A_3| > 6$ (in other cases we can proceed in an analogous way). Let $A = \{v_1, v_2, \ldots, v_7\}$ be an arbitrary subset of the set $A_2 \cap A_3$ and K be the complete graph with the vertex set A. At most 12 edges of K can belong to $E(F_2) \cup E(F_3)$ and thus at least 9 edges of K belong to the factor F_1. Let F_1' be the induced subgraph of F_1 with the vertex set A. F_1' has at least 9 edges and the degree of every vertex of F_1' is obviously at least 2. It is easy to verify that the graph F_1' is either isomorphic to the graph in Fig. 1 or connected and its diameter is not greater than 4. First part of this lemma implies (if $d_1 > 5$) that the set $A_1 - (A_2 \cap A_3)$ is nonempty. We consider the edges $vv_1, vv_2, \ldots, vv_7$, where v is an arbitrary vertex from $A_1 - (A_2 \cap A_3)$. At least one of these edges belongs to F_1.

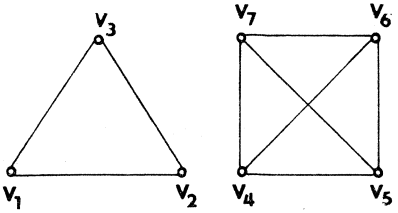

Figure 1

In fact, according to Lemma 1a at most three from these edges belong to F_2 and at most three belong to F_3. If F_1' is isomorphic to the graph in Fig. 1 then the vertices each of its components can have at most 4 adjacent vertices in $A_1 - (A_2 \cap A_3)$ (see Lemma 1b) and therefore $|A_1 - (A_2 \cap A_3)| \leqq 8$. If F_1' is connected then by Lemma 1b we have $|A_1 - (A_2 \cap A_3)| \leqq 7$. Finally, we get $|A_1| \leqq 8 + 6$, i.e. $d_1 \leqq 13$.

Lemma 3. *If $|A_1 \cap A_2| \leqq 6$ then the vertices from $A_1 \cap A_2$ are adjacent in F_3 to at most 12 vertices from $A_3 - (A_1 \cap A_2)$.*

Proof. (i) If $|A_1 \cap A_2| \leqq 4$ then by Lemma 1a the vertices from $A_1 \cap A_2$ are adjacent in F_3 to at most 4.3 vertices from $A_3 - (A_1 \cap A_2)$.

(ii) If $|A_1 \cap A_2| = 5$ then at least 2 edges of the complete graph with the set of vertices $A_1 \cap A_2$ belong to F_3. It suffices to consider the case when the above mentioned edges are exactly 2. If these edges are adjacent then according to Lemma 1 the vertices from $A_1 \cap A_2$ are adjacent to at most $2.3 + (2 + 3) = 11$ vertices from $A_3 - (A_1 \cap A_2)$. If these edges are not adjacent the result is the same since $1.3 + 2 . (1 + 3) = 11$.

(iii) If $|A_1 \cap A_2| = 6$ then the subgraph F_3' of the factor F_3 induced by the set $A_1 \cap A_2$ has at least 5 edges and the degree of every vertex of F_3' is at least 1. If the graph F_3' is connected then its diameter is at most 5 and according to Lemma 1b the vertices from $A_1 \cap A_2$ are adjacent to at most 8 vertices from $A_3 - (A_1 \cap A_2)$. If the graph F_3' is not connected then it has two components and one of them has diameter 1 and the other has diameter 1 or 2. Thus according to Lemma 1b the vertices from $A_1 \cap A_2$ are adjacent to at most $(1 + 3) + (2 + 3) = 9$ vertices from $A_3 - (A_1 \cap A_2)$.

Lemma 4. *Let $i, j \in \{1, 2, 3\}$, $i \neq j$. Let $B_i \subseteq A_i$, $B_j \subseteq A_j$, $|B_i| = k$, $|B_j| = m$ and*

$B_i \cap B_j = \emptyset$. *Let E be the set of all edges uv where $u \in B_i$ and $v \in B_j$. Then at most $3(k + m)$ edges from E belong to $E(F_i) \cup E(F_j)$.*

Proof. According to Lemma 1a each vertex from B_i can be adjacent in F_j to at most 3 vertices from A_j and thus at most $3k$ edges from E can belong to the factor F_j. In an analogous way it can be shown that at most $3m$ edges from E can belong to the factor F_i.

MAIN RESULTS AND PROBLEMS

Corollary of Lemma 2.

$$\lim_{d \to \infty} \frac{f(d)}{d} = 3$$

Proof. For $d \geqq 14$ and for any decomposition of a complete graph K into 3 factors of diameter d according to Lemma 2b we have

$$|V(K)| \geqq |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| -$$
$$- |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \geqq$$
$$\geqq 3(d + 1) - 3.6 = 3d - 15 .$$

Then the upper bound for $f(d)$ given in (C) yields

$$3d - 15 \leqq f(d) \leqq 3d - 8 .$$

Thus Problem 3 from [1] is solved.

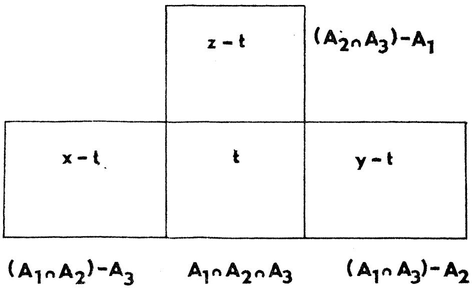

Figure 2

In the sequel we shall use the graphical scheme from Fig. 2 with the following denotations: $|A_1 \cap A_2| = x$, $|A_1 \cap A_3| = y$, $|A_2 \cap A_3| = z$, $|A_1 \cap A_2 \cap A_3| = t$.

Lemma 5. *Let us suppose there are vertices $v_1 \in A_1 - (A_2 \cup A_3)$, $v_2 \in A_2 - (A_1 \cup A_3)$ and $v_3 \in A_3 - (A_1 \cup A_2)$ in a complete graph K such that the following conditions are fulfilled:*

a) $v_1v_2 \in E(F_3)$, $v_1v_3 \in E(F_2)$, $v_2v_3 \in E(F_1)$,

b) *if* $v \in (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3)$ *then* $v_i v \notin E(F_i)$ *for* $i = 1, 2, 3$.
Then $|V(K)| \geqq d_1 + d_2 + d_3 - 8$.

Proof. Since $v_1 v_2 \in E(F_3)$ then according to Lemma 1b the vertices v_1, v_2 are adjacent in F_3 to at most 4 vertices from A_3 and similarly the vertices v_1, v_3 are adjacent in F_2 to at most 4 vertices from A_2 and the vertices v_2, v_3 are adjacent in F_1 to at most 4 vertices from A_1. At first we shall investigate from the point of view of possible distribution into individual factors only the edges of the following three types:

(i) $v_1 w$, where $w \in A_2 \cap A_3$,
(ii) $v_2 w$, where $w \in A_1 \cap A_3$,
(iii) $v_3 w$, where $w \in A_1 \cap A_2$.

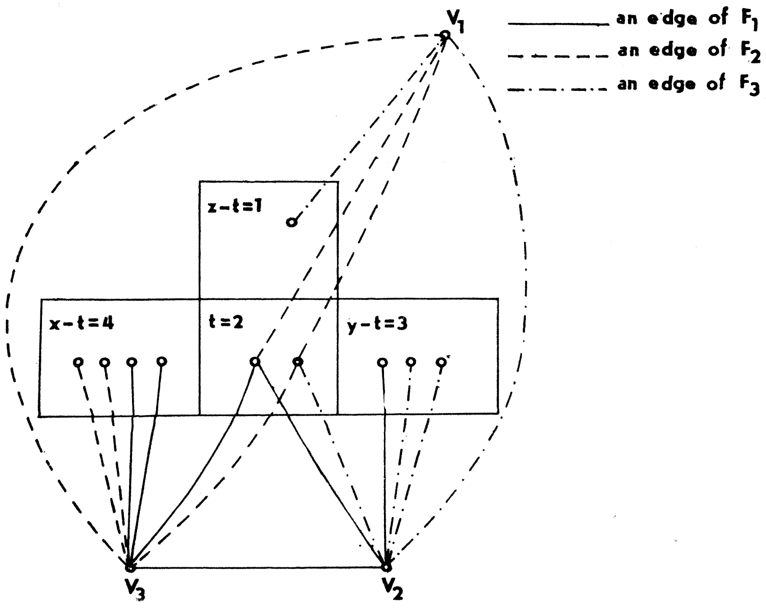

Figure 3

Fig. 3 represents one of the ways of distribution of these edges into individual factors which corresponds to the conditions of the lemma and Lemma 1b. It is easy to verify, if such a way exists the following inequality holds:

(1) $$(x - t) + (y - t) + (z - t) + 2t \leqq 3.4 .$$

It can be the equality in (1) only in the case when the vertices v_2, v_3 are adjacent in F_1 to exactly 4 vertices from $A_1 \cap (A_2 \cup A_3)$, the vertices v_1, v_3 are adjacent in F_2 to exactly 4 vertices from $A_2 \cap (A_1 \cup A_3)$, the vertices v_1, v_2 are adjacent in F_3 to exactly 4 vertices from $A_3 \cap (A_1 \cup A_2)$ and in addition for every vertex

$v \in A_1 \cap A_2 \cap A_3$ the edges vv_1, vv_2, vv_3 belong to exactly two factors. From (1) we have

(2) $$x + y + z - t \leqq 12.$$

Using the inclusion – exclusion principle we get from the inequality (2) the following estimation for $|V(K)|$:

$$|V(K)| \geqq |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = (d_1 + 1) + (d_2 + 1) + (d_3 + 1) - x - y - z + t \geqq d_1 + d_2 + d_3 - 9 .$$

Now we are going to show that the case $|V(K)| = d_1 + d_2 + d_3 - 9$ is not possible. From the preceding considerations it follows that the considered case would happen only under the following assumptions:

(i) the vertices v_2, v_3 are adjacent in F_1 to exactly 4 vertices from $A_1 \cap (A_2 \cup A_3)$; denote them x_1, x_2, x_3, x_4,

(ii) the vertices v_1, v_3 are adjacent in F_2 to exactly 4 vertices from $A_2 \cap (A_1 \cup A_3)$; denote them y_1, y_2, y_3, y_4,

(iii) the vertices v_1, v_2 are adjacent in F_3 to exactly 4 vertices from $A_3 \cap (A_1 \cup A_2)$; denote them z_1, z_2, z_3, z_4.

Without loss of generality the preceding notations can be chosen by Lemma 1 so that the edges $v_3x_1, x_1x_2, x_2x_3, x_3x_4, x_4v_2 \in E(F_1)$ (besides, of course, $x_1 \in A_1 \cap A_2$, $x_4 \in A_1 \cap A_3$), $y_1y_2, y_2y_3, y_3y_4 \in E(F_2)$ and $z_1z_2, z_2z_3, z_3z_4 \in E(F_3)$. The edge $v_3x_4 \notin E(F_3)$ by the assumption. Further, $v_3x_4 \notin E(F_1)$ since in the opposite case the vertices x_1, x_4 would have the distance in F_1 less than 3. Consequently $v_3x_4 \in E(F_2)$. In an analogous way it can be shown that $v_2x_1 \in E(F_3)$. Further $x_1 \neq y_i$ for $i = 1, 2, 3, 4$. In fact, if $x_1 \in (A_1 \cap A_2) - A_3$ the foregoing statement is obvious and if $x_1 \in A_1 \cap A_2 \cap A_3$ then $x_1v_1 \notin E(F_2)$ by the preceding considerations ($v_3x_1 \in E(F_1)$, $v_2x_1 \in E(F_3)$ and the edges v_1x_1, v_2x_1, v_3x_1 can not belong to 3 factors). Similarly,

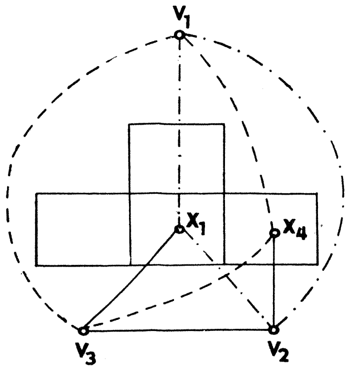

Figure 4

it can be shown that $x_4 \neq z_i$ for $i = 1, 2, 3, 4$. If $x_1 \in A_1 - A_3$ then $v_1x_1 \in E(F_3)$. In fact, by the assumption we have $v_1x_1 \notin E(F_1)$ and in the case $v_1x_1 \in E(F_2)$ the vertices v_1, v_3 would be adjacent in F_2 to 5 vertices y_1, y_2, y_3, y_4, x_1 from A_2 which is impossible (see Lemma 1b). If $x_1 \in A_1 \cap A_2 \cap A_3$ then again $v_1x_1 \in E(F_3)$ because two of the edges v_1x_1, v_2x_1, v_3x_1 must belong to one factor and we know already that $v_2x_1 \in E(F_3)$, $v_3x_1 \in E(F_1)$ and $v_1x_1 \notin E(F_1)$. Thus we have proved that $v_1x_1 \in$ $\in E(F_3)$. Similarly it can be shown that $v_1x_4 \in E(F_2)$ (this is also obvious from symmetry). Now we are going to show that the edge x_1x_4 can not belong to any factor (see Fig. 4).

It is obvious that $x_1x_4 \notin E(F_1)$ since $x_1, x_2, x_3, x_4 \in A_1$ and $x_1x_2, x_2x_3, x_3x_4 \in E(F_1)$. If the edge x_1x_4 belonged to F_2 the distance in F_2 of the vertex x_1 from each of the vertices y_1, y_2, y_3, y_4 would be less than 4, which is impossible, since $x_1, y_1, y_2, y_3, y_4 \in A_2$, $y_1y_2, y_2y_3, y_3y_4 \in E(F_2)$ and $x_1 \neq y_i$ for $i = 1, 2, 3, 4$. If the edge x_1x_4 belonged to F_3 the distance in F_3 of the vertex x_4 from each of the vertices z_1, z_2, z_3, z_4 would be less than 4, which is impossible, too. The proof is finished.

Theorem 1. *If* $65 < d_1 \leqq d_2 \leqq d_3 < \infty$ *then*

$$F(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8\,.$$

Proof. According to (B) it is sufficient to prove that the assumptions of the theorem imply those of Lemma 5. By Lemma 2b we have $|A_1 \cap A_2| \leqq 6$ and by Lemma 3 the vertices from $A_1 \cap A_2$ are adjacent in F_3 to at most 12 vertices from $A_3 - (A_1 \cup A_2)$. Further, any vertex from A_3 can be adjacent in F_3 to at most 2 vertices from A_3. Since according to Lemma 2*b* $|A_1 \cap A_3| \leqq 6$ and $|A_2 \cap A_3| \leqq 6$ the vertices from $(A_1 \cap A_3) \cup (A_2 \cap A_3)$ are adjacent in F_3 to at most 24 vertices from $A_3 - (A_1 \cup A_2)$. Therefore, if $d_3 + 1 > 6 + 6 + 12 + 24 = 48$, i.e. $d_3 > 47$ there is a vertex v_3 belonging to $A_3 - (A_1 \cup A_2)$ for which the condition from the part b) of Lemma 5 is fulfilled. For $d_3 > 65$ there exist at least 19 such vertices. Analogous considerations hold also for the vertices v_1 and v_2. Now we are going to show that there is a triangle $v_1v_2v_3$ for which also the conditions from part a) of Lemma 5 are fulfilled. Let $B_1 \subseteq A_1 - (A_2 \cup A_3)$, $B_2 \subseteq A_2 - (A_1 \cup A_3)$, $B_3 \subseteq$ $\subseteq A_3 - (A_1 \cup A_2)$ and $|B_1| = |B_2| = |B_3| = k$. The total number of triangles with the vertices in the sets B_1, B_2, B_3 (always one vertex in each of the considered sets) is k^3. According to Lemma 4 at most $3.6k.k = 18k^2$ of these triangles do not satisfy the conditions a) of Lemma 5. Thus if $k^3 > 18.k^2$, i.e. $k > 18$ then there exists a triangle with the desired property.

Theorem 2. *If* $5 \leqq d_1 \leqq d_2 < \infty$ *then* $F(d_1, d_2, \infty) = d_1 + d_2 - 4$ *except the value* $F(6, 6, \infty) = 7$.

Proof. According to (D) it is sufficient to show that $F(d_1, d_2, \infty) \geqq d_1 + d_2 - 4$ except the value $F(6, 6, \infty)$. Let K be an arbitrary complete graph and let the factors F_1, F_2, F_3 form its decomposition, where $d_1 \leqq d_2 < \infty$ and the factor F_3 is disconnected, i.e. $d(F_3) = \infty$. We shall consider three cases:

(i) $|A_1 \cap A_2| \leqq 6$.

In this case we have

$$|V(K)| \geqq |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| =$$
$$= (d_1 + 1) + (d_2 + 1) - |A_1 \cap A_2| \geqq d_1 + d_2 - 4 .$$

(ii) $|A_1 \cap A_2| > 6$ and $d_2 > 6$.

Let $B = \{v_1, v_2, \ldots, v_7\} \subseteq A_1 \cap A_2$ and v be an arbitrary vertex from $V(K) - B$. According to Lemma 1a at least one from 7 edges $vv_1, vv_2, \ldots, vv_7$ belongs to the factor F_3. Thus we see that if the subgraph H of the factor F_3 induced by the set B were connected the factor F_3 would be connected too, a contradiction. Thus the graph H is disconnected, has at least $\binom{7}{2} - 12 = 9$ edges and the degree of each of its vertices is at least two. It is obvious that the graph H must be isomorphic to the graph in Fig. 1. Then the subgraph of the factor F_1 induced by the set B is a path of length 6 and similarly the subgraph of the factor F_2 induced by the set B is a path of length 6. Without loss of generality we can assume that the considered paths are $v_4v_1v_5v_2v_6v_3v_7$ in the factor F_1 and $v_5v_3v_4v_2v_7v_1v_6$ in the factor F_2. Take any vertex v from $A_2 - B$; at least one such vertex exists since $d_2 > 6$. It is easy to verify that none of the edges vv_1, vv_2, vv_3 can belong to the factor F_2 and at most two of them can belong to the factor F_1. Then at least one of the edges vv_1, vv_2, vv_3 belongs to the factor F_3. Further, at least of the edges vv_4, vv_5, vv_6, vv_7 belongs to the factor F_3. In fact, it is easy to verify that at most one of these edges namely vv_5 or vv_6 can belong to the factor F_2 and at most two of them can belong to the factor F_1. The above mentioned facts imply that the factor F_3 is connected, a contradiction. Thus we have shown that the case (ii) is impossible.

(iii) $|A_1 \cap A_2| > 6$ and $d_2 \leqq 6$.

These conditions are fulfilled only in the case $d_1 = d_2 = 6$. It is easy to see that the graph in Fig. 1 together with the paths $v_4v_1v_5v_2v_6v_3v_7$ and $v_5v_3v_4v_2v_7v_1v_6$ form the decomposition of the graph K_7 into one disconnected factor and two factors with diameter 6. Hence $F(6, 6, \infty) = 7$ since the inequality $F(6, 6, \infty) \geqq 7$ holds trivially.

We should like to conclude our paper by presenting two open problems.

Problem 1. Is there positive integer d such that $f(d) = 3d - 10$?

Remark. It is possible to show that the inequality $f(d) \geqq 3d - 10$ holds for every positive integer d.

Problem 2. Determine the greatest positive integer d such that $f(d) = 3d - 8$ and $f(d - 1) < 3(d - 1) - 8$.

Remark. $f(5) = 6, f(6) = 9$ (see [1], [2]).

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