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Ján Andres

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PERIODIC DERIVATIVE OF SOLUTIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

JAN ANDRES, Olomouc

1. The problem from the title was worth mentioning for the first time in H. Poincaré's celestial mechanics investigation [1, p. 80].

Over the years, an increasing attention has been paid to the study of this type of solutions, called here D-periodic (derivo-periodic) solutions (cf. [2]), to many (mostly pendulum-type) equations modelling various processes in astronomy [3], engineering [4] or laser physics [5]. For the precise derivation of the related pendulum-type equation and the meaning of the technical parameters as well as its D-periodic solution see e.g. the last cited number. In all of these contributions only autonomous equations have been treated. However, much more actual situation appears, when some perturbation forces are included. That is why M. Farkas has developed a technique [6], [7] allowing him to consider nonautonomous equations involving a small parameter as well. More concretely, assuming the existence of D-periodic solutions to the autonomous equations, he established the sufficient conditions for the existence of those to the perturbed (nonautonomous) equations. A natural question arises (cf. [8]), whether the existence of such solutions can be ensured for the (D-)periodically forced equations, not necessarily involving a small parameter.

The purpose of this paper consists of an attemption to reply this question for the general equation of the n-th order ($n > 1$) via a D-periodic boundary value problem and to discuss the troublesome difficulties connected. Since the topological degree theory represents one of the most effective tools for attacking the similar problems for pure periodic oscillations, we are using the degree arguments close to [9] applied to the Levinson operator, of course under an appropriate modification.

2. Hence, consider the following boundary value problem ($n > 1$)

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \ (\equiv f(t + \theta,\ x + \omega,\ x', \ldots, x^{(n-1)}))\,, \tag{1}$$

$$x(\theta) - x(0) - \omega = x^{(i)}(\theta) - x^{(i)}(0) = 0 \quad (i = 1, 2, \ldots, n - 1)\,, \tag{2}$$

where $f \in C(R^{n+1})$ satisfies locally the Lipschitz condition with respect to $(x, x', \ldots, x^{(n-1)})$

and θ, ω are the positive reals. It is well-known that the continuous dependence on initial values and the uniqueness of solutions $x(t)$ of the Cauchy problem (1)–(3), where

$$(3) \qquad x^{(j)}(0) = x_0^{(j)} \quad (j = 0, 1, \ldots, n - 1),$$

is guaranteed in this way, as far as they exist.

As we have already mentioned, we apply the following modified Levinson operator

$$T_\mu(X_0) := \begin{cases} (x(\mu\theta; X_0) - x_0 - \mu^2\omega, x'(\mu\theta; X_0) - \\ - x_0', \ldots, x^{(n-1)}(\mu\theta; X_0) - x_0^{(n-1)})/\mu\theta & \text{for} \quad \mu \in (0, 1\rangle, \\ (x_0', x_0'', \ldots, f(0, x_0, x_0', \ldots, x_0^{(n-1)}) & \text{for} \quad \mu = 0, \end{cases}$$

in order to prove the solvability of (1)–(2_μ), where

$$(2_\mu) \qquad x(\mu\theta) - x(0) - \mu^2\omega = x^{(i)}(\mu\theta) - x^{(i)}(0) = 0 \quad (i = 1, 2, \ldots, n - 1)$$

(obviously, (2_μ) reduces to (2) for $\mu = 1$, otherwise $\mu \in \langle 0, 1\rangle$), where $x(t; X_0) := x(t; 0, x_0, x_0', \ldots, x_0^{(n-1)})$ is a solution of (1)–(3). Since the only further presumption is the existence of all solutions of (1)–(3) on the whole interval $\langle 0, \theta\rangle$, it is well-known that the mapping $T_\mu(X_0)$ is a homeomorphism and thus we are interested in the existence of some invariant set $I \subset R^n$, symmetrical with respect to the origin O such that $\ker T_1(\mathrm{cl}\, I)$ exists. Let us note that the required existence of solutions on $\langle 0, \theta\rangle$ will be satisfied automatically by the final restrictions due to the a priori estimates technique employed (a linear boundedness of the right-hand side of (1)) and therefore it can be practically omitted.

3. Proposition. *If there exists an open set $I \subset R^n$, symmetrical with respect to the origin O, such that*

$$(4) \qquad T_\mu(X_0) \neq O$$

and

$$(5) \qquad f(0, x, 0, \ldots, 0) \neq (1 - v) f(0, -x, 0, \ldots, 0)$$

hold for all $X_0 \in \partial I$ (a frontier of I), independently to $\mu, v \in (0, 1\rangle$, then problem (1)–(2) admits a solution.

Proof. It is clear that (1)–(2) is solvable iff $T_1(\hat{X}) = O$ for some $\hat{X} \in \mathrm{cl}\, I$. To show this, an essential degree argument (see e.g [10, p. 20]) reads $T_1(X_0) \neq O$ jointly with $d[T_1, I, O] \neq 0$, where „d" denotes the Brouwer degree (in the opposite case there is no point to prove).

Assuming (4), $T_\mu(X_0)$ represents a homotopical bridge between $T_1(X_0)$ and $T_0(X_0)$ and thus $d[T_0\ I, O] \neq 0$ is enough to assume instead of $d[T_1, I, O] \neq 0$ for the same goal with respect to the invariance [10] under homotopy.

Applying the Borsuk antipodal theorem [10, p. 24], saying that $d[T_0(X_0) - T_0(-X_0), I, O] \neq 0$, we can still replace the last requirement by the condition

$$(6) \qquad T_{0,v}(X_0) := T_0(X_0) - (1 - v)\, T_0(-X_0) \neq O \quad \text{for} \quad v \in (0, 1\rangle$$

by the same reason (i.e. $T_{0,v}(X_0)$ is a homotopical bridge between the operators

$T_{0,0}(X_0) := T_0(X_0) - T_0(-X_0)$ and $T_{0,1}(X_0) := T_0(X_0))$. However, (6) is implied by (5) for $X_0 \in \partial I$ immediately. This completes the proof.

Remark 1. The assertion of the above statement remains obviously valid if (4) is reduced to an assumption of the uniform a priori boundedness of solutions $x(t) \in \in (1)-(2_\mu)$, independently to $\mu \in (0, 1\rangle$, jointly with their derivatives up to the $(n-1)$-th order including and (5) is replaced by

$$\frac{f(0, x, 0, \ldots, 0)}{|f(0, x, 0, \ldots, 0)|} \neq \frac{f(0, -x, 0, \ldots, 0)}{|f(0, -x, 0, \ldots, 0)|}$$

for $f(0, x, 0, \ldots, 0) \neq 0$ where $|x| > R \ldots$ a sufficiently large constant or

(5′) $$\liminf_{|x|\to\infty} f(0, x, 0, \ldots, 0) \operatorname{sgn} x > 0 \vee \limsup_{|x|\to\infty} f(0, x, 0, \ldots, 0) \operatorname{sgn} x < 0 .$$

Assuming furthermore the existence of a positive constant Ω such that

(7) $$f(t, x + \omega, x', \ldots, x^{(n-1)}) \equiv f(t, x, x', \ldots, x^{(n-1)}) + \Omega(t, x, x', \ldots, x^{(n-1)}) ,$$

where

$$|\Omega(t + \theta, x, x', \ldots, x^{(n-1)})| \equiv |\Omega(t, x, x', \ldots, x^{(n-1)})| \equiv$$
$$\equiv |\Omega(t, x + \omega, x', \ldots, x^{(n-1)})| \geqq \Omega$$
$$(\Rightarrow f(t + \theta, x, x', \ldots, x^{(n-1)}) \equiv$$
$$\equiv f(t, x, x', \ldots, x^{(n-1)}) - \Omega(t, x, x', \ldots, x^{(n-1)})) ,$$

we can give

Lemma. *If there exists a constant D_{n-1} estimating $(n - 1)$-th order derivatives of all solutions to $(1)-(2_\mu)$, independently to $\mu \in (0, 1\rangle$, then (4) holds.*

Proof. In view of Remark 1, it is enough to prove the uniform a priori boundedness of solutions $x(t) \in (1)-(2_\mu)$, $\mu \in (0, 1\rangle$, together with their derivatives up to the $(n - 2)$-th including. Let $x(t)$ be a solution of $(1)-(2_\mu)$. Letting

(8) $$x(t) := x_0(t) + x_1 + \mu \frac{\omega}{\theta} t ,$$

(9) $$x_1 := \frac{1}{\mu\theta} \int_0^{\mu\theta} x(t)\, dt - \tfrac{1}{2}\mu^2\omega ,$$

(10) $$\|x_0(t)\| := \max_{t\in\langle 0,\mu\theta\rangle} |x_0(t)|$$

and using the Wirtinger-type inequalities (cf. [11], [12, p. 184])

$$\|x_0(t)\| \leqq \sqrt{\left(\frac{\mu\theta}{12}\right)} \left[\int_0^{\mu\theta} \left[x(t) - \mu \frac{\omega}{\theta} t\right]'^2 dt\right]^{1/2} = \sqrt{\left(\frac{\mu\theta}{12}\right)} \left[\int_0^{\mu\theta} x'^2(t)\, dt - \mu^3 \frac{\omega^2}{\theta}\right]^{1/2}$$

and

$$\int_0^{\mu\theta} \left[x(t) - \mu \frac{\omega}{\theta} t\right]^{(i)2} dt \leqq \left(\frac{\mu\theta}{2\pi}\right)^2 \int_0^{\mu\theta} \left[x(t) - \mu \frac{\omega}{\theta} t\right]^{(i+1)2} dt ,$$

i.e.

$$\int_0^{\mu\theta} x^{(i)2}(t)\,\mathrm{d}t \leqq \left(\frac{\mu\theta}{2\pi}\right)^2 \int_0^{\mu\theta} x^{(i+1)2}(t)\,\mathrm{d}t + (1 - \operatorname{sgn}(i-1))\,\mu^3\,\frac{\omega^2}{\theta}$$

$$(i = 1, 2, \ldots, n-1)\,,$$

we arrive for $n > 1$ at the relations

$$\|x_0(t)\| \leqq \frac{(\mu\theta)^{(m-1)/2}}{2^m(\sqrt{3})\,\pi^{m-1}} \left[\int_0^{\mu\theta} x^{(m)2}(t)\,\mathrm{d}t - (1 - \operatorname{sgn}(m-1))\,\mu^3\,\frac{\omega^2}{\theta}\right]^{1/2} \tag{11}$$

$$(0 < m \leqq n)$$

and

$$\|x^{(i)}(t)\| \leqq \frac{(\mu\theta)^{(k-1)/2}}{2^k(\sqrt{3})\,\pi^{k-1}} \left[\int_0^{\mu\theta} x^{(i+k)2}(t)\,\mathrm{d}t\right]^{1/2} + \tag{12}$$

$$+ (1 - \operatorname{sgn}(i-1))\,\mu\,\frac{\omega}{\theta} \quad (0 < k \leqq n - i,\ i = 1, 2, \ldots, n-1)\,,$$

respectively.

So we obtain

$$\|x_0(t)\| \leqq \left(\mu\,\frac{\theta}{2}\right)^{n-1} \frac{\|x^{(n-1)}(t)\|}{(\sqrt{3})\,\pi^{n-2}} \leqq \left(\frac{\theta}{2}\right)^{n-1} \frac{D_{n-1}}{(\sqrt{3})\,\pi^{n-2}} := D_0\,, \tag{13}$$

$$\|x^{(i)}(t)\| \leqq \left(\mu\,\frac{\theta}{2}\right)^{n-i-1} \frac{\|x^{(n-1)}(t)\|}{(\sqrt{3})\,\pi^{n-i-2}} + (1 - \operatorname{sgn}(i-1))\,\mu\,\frac{\omega}{\theta} \leqq \tag{14}$$

$$\leqq \left(\frac{\theta}{2}\right)^{n-i-1} \frac{D_{n-1}}{(\sqrt{3})\,\pi^{n-i-2}} + (1 - \operatorname{sgn}(i-1))\,\frac{\omega}{\theta} := D_i$$

$$(i = 1, \ldots, n-2)\,.$$

Taking $x(0)\,(= x_0(0) + x_1) = r\omega + s$ with a great enough integer r and a real s with $|s| \leqq \omega$ (cf. (8)), one can get not only (cf. (7)–(10))

$$f(t, x_0(t) - x_0(0) + \mu\,\frac{\omega}{\theta}\,t + s\,,$$

$$x'(t), \ldots, x^{(n-1)}(t)) + r\Omega(t, x(t)\,,\ x'(t), \ldots, x^{(n-1)}(t))\,,$$

but also (cf. (13), (14))

$$\Big\|\int_0^{\mu\theta} f(t, x_0(t) - x_0(0) + \mu\,\frac{\omega}{\theta}\,t + s\,,\ x'(t), \ldots, x^{(n-1)}(t))\,\mathrm{d}t + \tag{15}$$

$$+ r\int_0^{\mu\theta} \Omega(t, x(t), x'(t), \ldots, x^{(n-1)}(t))\,\mathrm{d}t\Big| \geqq$$

$$\geqq \Big\|r\int_0^{\mu\theta} \Omega(t, x(t), x'(t), \ldots, x^{(n-1)}(t))\,\mathrm{d}t\Big| -$$

$$- \left\| \int_0^{\mu\theta} f(t, x_0(t) - x_0(0) + \mu \frac{\omega}{\theta} t + s, x'(t), \ldots, x^{(n-1)}(t)) \, \mathrm{d}t \right\| \geqq$$

$$\geqq \mu\theta \left[|r\Omega| - \left\| f\left(t, x_0(t) - x_0(0) + \mu \frac{\omega}{\theta} t + s, x'(t), \ldots, x^{(n-1)}(t)\right) \right\| \right] \geqq$$

$$\geqq \mu\theta[|r\Omega| - \max_{\substack{t \in \langle 0, \theta \rangle, |x| \leqq 2(D_0 + \omega), \\ |x^{(i)}| \leqq D_i, i = 1, \ldots, n-1}} |f(t, x, x', \ldots, x^{(n-1)})|] > 0 \,,$$

a contradiction to

$$\int_0^{\mu\theta} f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \, \mathrm{d}t = 0$$

because of the substitution $x(t)$ into (1) and the integration of the obtained identity from 0 to $\mu\theta$.

Therefore such a constant

$$R := \frac{\omega}{\Omega} \max_{\substack{t \in \langle 0, \theta \rangle, |x| \leqq 2(D_0 + \omega), \\ |x^{(i)}| \leqq D_i, i = 1, \ldots, n-1}} |f(t, x, x', \ldots, x^{(n-1)})|$$

certainly exists (cf. the last inequality in (15)) that $|x(0)| \leqq R + \omega$, and consequently (cf. (8))

$$\|x(t)\| \leqq 2\|x_0(t)\| + |x(0)| + \omega \leqq 2D_0 + R + 2\omega := D \,. \tag{16}$$

This completes the proof.

Remark 2. It could be seen from the proof of Lemma that the assumption $|\Omega(t, x, x', \ldots, x^{(n-1)})| \geqq \Omega > 0$ from (7) might be replaced by the weaker one, namely

$$\int_0^{\mu\theta} \Omega(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \, \mathrm{d}t \lessgtr 0 \,, \quad \mu \in (0, 1\rangle \,.$$

Moreover, the assumption concerning the boundedness of the $(n - 1)$-th order derivatives could be obviously replaced by the one related to those of the n-th order (although the right-hand side of (1) is not bounded according to (7)).

Remark 3. Assuming (instead of (7))

$$\begin{aligned} f(t, x + \omega, x', \ldots, x^{(n-1)}) &\equiv f(t, x, x', \ldots, x^{(n-1)}) \equiv \\ &\equiv f(t + \theta, x, x', \ldots, x^{(n-1)}) \,, \end{aligned} \tag{7'}$$

one cannot employ a priori estimates technique because of the existence of a family of solutions differing from $x(t) \in (1)$ by $k\omega$ with all integers k. In spite of a possibility to apply Proposition without using this manner, namely to verify condition (4) by local methods as e.g. in [13] for pure periodic oscillations, we are not able to do it here. Perhaps, this difficulty is related to the loss of the uniqueness property of solutions $x(t) \in (1)-(2)$ in general, as it has been pointed out in [14].

4. Now we can give the principal result of our paper.

Theorem. *Let* $((5') \Leftarrow)$ (7) *be fulfilled. If there exist nonnegative constants* F, c_j $(j = 0, 1, \ldots, n-1)$ *such that*

$$|f(t, x, x', \ldots, x^{(n-1)})| \leqq F + \sum_{j=0}^{n-1} c_j |x^{(j)}| \tag{17}$$

holds with

$$\frac{\theta^n}{2^{n+1} 3\pi^{n-2}} \left\{ c_0 \left[1 + 2\frac{\omega}{\Omega} \left(c_0 + c_{n-1} \left(\frac{\theta}{2\pi} \right)^{n-1} \right) \right] + \right. \tag{18}$$
$$\left. + 2\left(1 + c_0 \frac{\omega}{\Omega}\right) \sum_{j=1}^{n-2} c_j \left(\frac{\theta}{2\pi} \right)^j \right\} + \frac{\theta}{2\sqrt{3}} c_{n-1} < 1,$$

then equation (1) *admits a D-periodic solution.*

Proof. Let $x(t)$ be a solution of $(1)-(2_\mu)$, $\mu \in (0, 1\rangle$. Using the relations $(11)-(14)$ and (16), (17), we arrive at the following inequalities

$$\|x^{(n-1)}(t)\| \leqq \frac{\theta}{2\sqrt{3}} \left(F + \sum_{j=0}^{n-1} c_j \|x^{(j)}(t)\|\right) + (1 - \operatorname{sgn}(n-2)) \frac{\omega}{\theta}, \tag{19}$$

$$\|x_0(t)\| \leqq \frac{(\theta/2)^{n-1}}{(\sqrt{3})\, \pi^{n-2}} \|x^{(n-1)}(t)\| := D_0^*, \tag{20}$$

$$\left\{ \begin{array}{l} \|x^{(i)}(t)\| \leqq \left(\dfrac{\theta}{2}\right)^{n-i-1} \dfrac{\|x^{(n-1)}(t)\|}{(\sqrt{3})\, \pi^{n-i-2}} + (1 - \operatorname{sgn}(i-1)) \dfrac{\omega}{\theta} := D_i^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, n-2), \\ \|x^{(n-1)}(t)\| := D_{n-1}^* \end{array} \right. \tag{21}$$

$$\|x(t)\| \leqq 2\|x_0(t)\| + \frac{\omega}{\Omega} \max_{\substack{t \in \langle 0, \theta\rangle, \|x(t)\| \leqq 2(D_0^* + \omega), \\ \|x^{(i)}(t)\| D_i^*, i = 1, \ldots, n-1}} |f(t, x(t), x'(t), \ldots, x^{(n-1)}(t))| + \tag{22}$$
$$+ 2\omega \leqq 2D_0^* + \frac{\omega}{\Omega} (F + \max_{\substack{\|x(t)\| \leqq 2(D_0^* + \omega), \\ \|x^{(i)}(t)\| \leqq D_i^*, i = 1, \ldots, n-1}} \sum_{j=0}^{n-1} c_j \|x^{(j)}(t)\|) + 2\omega \leqq$$
$$\leqq \frac{\theta^{n-1} \|x^{(n-1)}(t)\|}{2^n (\sqrt{3})\, \pi^{n-2}} + \frac{\omega}{\Omega} [F + 2c_0(D_0^* + \omega) + \sum_{i=1}^{n-1} c_i D_i^*] + 2\omega \leqq$$
$$\leqq \|x^{(n-1)}(t)\| \frac{\theta^{n-1}}{2^n (\sqrt{3})\, \pi^{n-2}} \left[1 + 2\frac{\omega}{\Omega} \left(c_0 + \sum_{i=1}^{n-1} c_i \left(\frac{\theta}{2\pi} \right)^i \right) \right] +$$
$$+ \frac{\omega}{\Omega} \left[F + \omega \left(2c_0 + \frac{1}{\theta} \sum_{i=1}^{n-1} c_i (1 - \operatorname{sgn}(i-1)) \right) \right] + 2\omega.$$

Denoting

$$A_0 := \frac{\theta^{n-1}}{2^n (\sqrt{3})\, \pi^{n-2}} \left[1 + 2\frac{\omega}{\Omega} \left(c_0 + \sum_{i=1}^{n-1} c_i \left(\frac{\theta}{2\pi} \right)^i \right) \right],$$

$$A_i := \frac{(\theta/2)^{n-i-1}}{(\sqrt{3})\,\pi^{n-i-2}}, \quad A_{n-1} := 1, \quad (i = 1, \ldots, n-2),$$

$$B_0 := \frac{\omega}{\Omega}\left\{F + \omega\left[2c_0 + \frac{1}{\theta}\sum_{i=1}^{n-1} c_i(1 - \operatorname{sgn}(i-1))\right]\right\} + 2\omega,$$

$$B_i := (1 - \operatorname{sgn}(i-1))\frac{\omega}{\theta} \quad (i = 1, \ldots, n-2)$$

and substituting from (21), (22) into (19), we get

$$\|x^{(n-1)}(t)\| \leqq \frac{\theta}{2\sqrt{3}}\left[F + \sum_{j=0}^{n-1} c_j(A_j\|x^{(n-1)}(t)\| + B_j)\right] + (1 - \operatorname{sgn}(n-2))\frac{\omega}{\theta}.$$

Hence, taking the coefficients c_i $(i = 0, 1, \ldots, n-1)$ as follows

$$\frac{\theta}{2\sqrt{3}}\sum_{j=0}^{n-1} c_j A_j < 1,$$

i.e. (18), we obtain the desired estimate

$$\|x^{(n-1)}(t)\| \leqq$$
$$\leqq \left[\frac{\theta}{2\sqrt{3}}\left(F + \sum_{j=0}^{n-1} c_j B_j\right) + (1 - \operatorname{sgn}(n-2))\frac{\omega}{2}\right] \Big/ \left(1 - \frac{\theta}{2\sqrt{3}}\sum_{j=0}^{n-1} c_j A_j\right) := D_{n-1}$$

yielding (4) according to Lemma.

Moreover, this implies jointly with (21), (22) also

$$\|x^{(i)}(t)\| \leqq A_i D_{n-1} + B_i := D_i \quad (i = 1, \ldots, n-2),$$
$$\|x(t)\| \leqq A_0 D_{n-1} + B_0 := D$$

and consequently the uniform a priori boundedness of all solutions $x(t) \in (1) - (2_\mu)$, $\mu \in (0, 1\rangle$, as well as their derivatives up to the $(n-1)$-th order including is verified explicitely. In view of Remark 1 and the assertion of Proposition it is enough to assume (5′), only. This completes the proof.

Remark 4. Obviously, we could consider a periodic boundary problem

$$x^{(n)} = f(t, x + \omega t/\theta, x' + \omega/\theta, x'', \ldots, x^{(n-1)}),$$
$$x^{(j)}(0) = x^{(j)}(\theta) \quad (j = 0, 1, \ldots, n-1)$$

instead of (1) – (2) for the same aim. In this respect our result seems to be comparable with those obtained by the different topological degree techniques in [15], [16] for pure periodic oscillations in (1). Sometimes our growth conditions (17), (18) can be even better than the analogous growth restrictions from [15], [16]. However, considering a suitable structure of the right-hand side of (1), such conditions can be still much more liberal (cf. [13]).

Remark 5. Although our problem could be solvable for a sufficiently small period θ even in the noncoercive case (7′), as it has been pointed out in [14] in general, it remains open yet for noninfinitesimally small periods θ just under (7′).

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Author's address: 771 46 Olomouc, Vídeňská 15, Czechoslovakia (Přírodovědecká fakulta UP).