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HYPERSURFACES IN 4-DIMENSIONAL EUCLIDEAN SPACE

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1. INTRODUCTION

Let $\boldsymbol{H} = \text{span}_{\boldsymbol{R}}\{1, i, j, k\}$ be the quaternions. We shall fix the basis $\{1, i, j, k\}$ throughout this paper. Then, we may regard $\boldsymbol{H}$ as a 4-dimensional Euclidean space $\boldsymbol{R}^4$ in the natural way. An oriented hypersurface M^3 in $\boldsymbol{H}$ admits a global orthonormal frame field as follows. Let (M^3, f) be an oriented hypersurface of $\boldsymbol{H}$ and ξ a unit normal vector field on M^3. Then $\{\xi i, \xi j, \xi k\}$ is a global orthonormal frame field of $f(M^3)$. We shall call this orthonormal frame field an associated one on $f(M^8)$. So, it is natural to study oriented hypersurfaces in $\boldsymbol{H}$ by using the associated one. The purpose of this paper is to prove the following Theorems A and B.

Theorem A. *Let (M^3, f) be an oriented hypersurface in the quaternions and ξ the unit normal vector field of M^3 in $\boldsymbol{H}$. If one of the vector fields of the associated frame field of $f(M^3)$ is an infinitesimal affine transformation, then*

(1) *M^3 is locally isometric to a 3-dimensional round sphere in $\boldsymbol{H}$ and the immersion f is totally umbilic,*

or

(2) *M^3 is locally isometric to $M^1 \times \boldsymbol{R}^2$ (M^1 is a 1-dimensional Riemannian manifold) and the immersion f is a locally product one.*

Theorem B. *Let (M^3, f) be an oriented hypersurface in the quaternions $\boldsymbol{H}$ and ξ the unit normal vector field of M^3 in $\boldsymbol{H}$. If the associated frame field of $f(M^3)$ is a Ricci adapted frame (i.e., $\varrho(\xi i, \xi j) = \varrho(\xi j, \xi k) = \varrho(\xi k, \xi i) = 0$ on M^3 where ϱ is the Ricci tensor of M^3), then*

(1) *M^3 is locally isometric to a 3-dimensional round sphere in $\boldsymbol{H}$ and the immersion f is totally umbilic,*

or

(2) *M^3 is locally isometric to $M^1 \times \boldsymbol{R}^2$ (M^1 is a 1-dimensional Riemannian manifold) and the immersion f is a locally product one.*

In particular, (M^3, f) is an Einstein hypersurface in $\boldsymbol{H}$.

Remark. In the case (2) of Theorem A, the vector field ξi is an infinitesimal affine transformation which is not a killing vector field.

In this paper, all the manifolds are assumed to be connected and class C^∞ unless otherwise stated. The author would like to express his heartly thanks to Professor K. Sekigawa and Professor K. Tsukada for their constant encouragement and many valuable suggestions.

2. PRELIMINARIES

First, we shall recall some elementary properties of the quaternions $\boldsymbol{H} = \mathrm{span}_{\boldsymbol{R}}\{1, i, j, k\}$ with $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Let $\langle\,,\rangle$ be the canonical inner product of $\boldsymbol{H}$. For any $x \in \boldsymbol{H}$, we denote by $\bar{x}$ the conjugate of x. We write down some elementary properties of $\boldsymbol{H}$.

$$
\begin{aligned}
(2.1)\qquad & \langle xw, y\rangle = \langle x, y\bar{w}\rangle, \quad \langle wx, y\rangle = \langle x, \bar{w}y\rangle, \\
& \overline{xy} = \bar{y}\bar{x}, \\
& \langle x, y\rangle = (x\bar{y} + y\bar{x})/2, \quad \langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle
\end{aligned}
$$

for any $x, y, w \in \boldsymbol{H}$ (see [3]).

We recall also some elementary formulae of hypersurfaces in the Euclidean space. We denote by $\boldsymbol{R}^{n+1}$ an $(n+1)$-dimensional Euclidean space. Let M^n be an n-dimensional hypersurface in $\boldsymbol{R}^{n+1}$. We denote by ∇, D and $\nabla^\perp$ the Riemannian connection of M^n, $\boldsymbol{R}^{n+1}$ and the normal connection of M^n in $\boldsymbol{R}^{n+1}$ respectively, and σ the second fundamental form of M^n in $\boldsymbol{R}^{n+1}$. Then, the Gauss formula and the Weingarten formula are given respectively by

$$(2.2)\qquad \sigma(X, Y) = D_X Y - \nabla_X Y,$$

$$(2.3)\qquad D_x\xi = -A_\xi(X)$$

for any $X, Y \in \mathfrak{X}(M^n)$ ($\mathfrak{X}(M^n)$ denotes the Lie algebra of all differentiable vector fields on M^n), where ξ is the unit normal vector field of M^n in $\boldsymbol{R}^{n+1}$ and $-A_\xi(X)$ denotes the tangential part of $D_x\xi$.

The tangential part $A_\xi(X)$ is related to the second fundamental form σ as follows:

$$(2.4)\qquad \langle\sigma(X, Y), \xi\rangle = \langle A_\xi(X), Y\rangle \quad \text{for any} \quad X, Y \in \mathfrak{X}(M^n).$$

Then, the Gauss, Codazzi equations are given respectively by

$$(2.5)\qquad \langle R(X, Y)Z, W\rangle = \langle\sigma(X, W), \sigma(Y, Z)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle,$$

$$(2.6)\qquad (\nabla\sigma)(X, Y, Z) = (\nabla\sigma)(Y, X, Z)$$

for any $X, Y, Z, W \in \mathfrak{X}(M^n)$, where $\boldsymbol{R}$ is the Riemannian curvature tensor of M^n defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and $(\nabla\sigma)(X, Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$.

We shall give some elementary formulae of an oriented hypersurface in $\boldsymbol{H}$ for the sake of later uses. Let (M^3, f) be an oriented hypersurface in the quaternions $\boldsymbol{H}$.

We denote by ξ the unit normal vector field of M^3 in $\boldsymbol{H}$. Then, we see that $\{\xi i, \xi j, \xi k\}$ is a global orthonormal frame field on M^3.

By (2.1) and (2.3), we get

$$
\begin{aligned}
(2.7)\qquad \nabla_{\xi i}(\xi i) &= \sigma(\xi i, \xi j)\, k - \sigma(\xi i, \xi k)\, j\,,\\
\nabla_{\xi j}(\xi j) &= \sigma(\xi j, \xi k)\, i - \sigma(\xi j, \xi i)\, k\,,\\
\nabla_{\xi k}(\xi k) &= \sigma(\xi k, \xi i)\, j - \sigma(\xi k, \xi j)\, i\,,\\
\nabla_{\xi i}(\xi j) &= \sigma(\xi i, \xi k)\, i - \sigma(\xi i, \xi i)\, k\,,\\
\nabla_{\xi j}(\xi k) &= \sigma(\xi j, \xi i)\, j - \sigma(\xi j, \xi j)\, i\,,\\
\nabla_{\xi k}(\xi i) &= \sigma(\xi k, \xi j)\, k - \sigma(\xi k, \xi k)\, j\,,\\
\nabla_{\xi j}(\xi i) &= \sigma(\xi j, \xi j)\, k - \sigma(\xi j, \xi k)\, j\,,\\
\nabla_{\xi k}(\xi j) &= \sigma(\xi k, \xi k)\, i - \sigma(\xi k, \xi i)\, k\,,\\
\nabla_{\xi i}(\xi k) &= \sigma(\xi i, \xi i)\, j - \sigma(\xi i, \xi j)\, i\,.
\end{aligned}
$$

From (2.7), it follows that $\operatorname{div}(\xi i) = \operatorname{div}(\xi j) = \operatorname{div}(\xi k) = 0$, that is, (M^3, f) has the divergence property ([1]).

3. PROOF OF THEOREM A

First, we shall prepare some lemmas. Without loss of essentiality, we may assume that the vector field ξi is an infinitesimal affine transformation of M^3 (that is, ξi satisfies $\nabla_X(\nabla_Y(\xi i)) - \nabla_{\nabla_X Y}(\xi i) = R(X, \xi i)\, Y$ for any $X, Y \in \mathfrak{X}(M^3)$ (see [8])).

Lemma 3.1. *The vector field ξi is an infinitesimal affine transformation if and only if*

(a) $\langle \sigma(X, Y), \sigma(\xi i, \xi i)\rangle = \langle \sigma(X, \xi i), \sigma(Y, \xi i)\rangle + \langle \sigma(X, \xi j), \sigma(Y, \xi j)\rangle + \langle \sigma(X, \xi k), \sigma(Y, \xi k)\rangle$,

(b) $\langle (\nabla\sigma)(X, Y, \xi j), \xi\rangle = -\langle \sigma(X, Y), \sigma(\xi i, \xi k)\rangle$,

and

(c) $\langle (\nabla\sigma)(X, Y, \xi k), \xi\rangle = \langle \sigma(X, Y), \sigma(\xi i, \xi j)\rangle$ *for any* $X, Y \in \mathfrak{X}(M^8)$.

Proof. By (2.7), we get

$$
\begin{aligned}
(3.1)\qquad & \nabla_X(\nabla_Y(\xi i)) - \nabla_{\nabla_X Y}(\xi i)\\
&= \nabla_X\{\sigma(Y, \xi j)\, k - \sigma(Y, \xi k)\, j\} - \{\sigma(\nabla_X Y, \xi j)\, k - \sigma(\nabla_X Y, \xi k)\, j\}\\
&= (X\langle \sigma(Y, \xi j), \xi\rangle)\, \xi k + \langle \sigma(Y, \xi j), \xi\rangle\, \nabla_X(\xi k)\\
&\quad - \{(X\langle \sigma(Y, \xi k), \xi\rangle)\, \xi j + \langle \sigma(Y, \xi k), \xi\rangle\, \nabla_X(\xi j)\}\\
&\quad - \{\sigma(\nabla_X Y, \xi j)\, k - \sigma(\nabla_X Y, \xi k)\, j\}\\
&= \langle (\nabla\sigma)(X, Y, \xi j) + \sigma(\nabla_X Y, \xi j) + \sigma(Y, \nabla_X(\xi j)), \xi\rangle\, \xi k\\
&\quad + \langle \sigma(Y, \xi j), \xi\rangle\, \{\sigma(X, \xi i)\, j - \sigma(X, \xi j)\, i\}\\
&\quad - \langle (\nabla\sigma)(X, Y, \xi k) + \sigma(\nabla_X Y, \xi k) + \sigma(Y, \nabla_X(\xi k)), \xi\rangle\, \xi j
\end{aligned}
$$

$$
\begin{aligned}
&- \langle \sigma(Y, \xi k), \xi \rangle \{\sigma(X, \xi k)\, i - \sigma(X, \xi i)\, k\} \\
&- \{\sigma(\nabla_X Y, \xi j)\, k - \sigma(\nabla_X Y, \xi k)\, j\} \\
= &- \{\langle \sigma(X, \xi j), \sigma(Y, \xi j)\rangle + \langle \sigma(X, \xi k), \sigma(Y, \xi k)\rangle\} \xi i \\
&- \{\langle (\nabla\sigma)(X, Y, \xi k), \xi \rangle - \langle \sigma(X, \xi j), \sigma(Y, \xi i)\rangle\} \xi j \\
&+ \{\langle (\nabla\sigma)(X, Y, \xi j), \xi \rangle + \langle \sigma(X, \xi k), \sigma(Y, \xi i)\rangle\} \xi k .
\end{aligned}
$$

On the other hand, by (2.5), we get

$$
\begin{aligned}
(3.2) \qquad & R(X, \xi i)\, Y \\
&= \{\langle \sigma(X, \xi i), \sigma(Y, \xi i)\rangle - \langle \sigma(X, Y), \sigma(\xi i, \xi i)\rangle\} \xi i \\
&\quad + \{\langle \sigma(X, \xi j), \sigma(Y, \xi i)\rangle - \langle \sigma(X, Y), \sigma(\xi i, \xi j)\rangle\} \xi j \\
&\quad + \{\langle \sigma(X, \xi k), \sigma(Y, \xi i)\rangle - \langle \sigma(X, Y), \sigma(\xi i, \xi k)\rangle\} \xi k .
\end{aligned}
$$

From (3.1) and (3.2), we have the desired equalities. □

Lemma 3.2.

$$\sigma(\xi i, \xi j) = \sigma(\xi i, \xi k) = (\nabla\sigma)(X, Y, \xi j) = (\nabla\sigma)(X, Y, \xi k) = 0$$

for any $X, Y \in \mathfrak{X}(M^8)$.

Proof. By (b) and (c) of Lemma 3.1, we get

$$
\begin{aligned}
(3.3) \qquad & \langle (\nabla\sigma)(X, \xi k, \xi j), \xi \rangle = -\langle \sigma(X, \xi k), \sigma(\xi i, \xi k)\rangle , \\
& \langle (\nabla\sigma)(X, \xi j, \xi k), \xi \rangle = \langle \sigma(X, \xi j), \sigma(\xi i, \xi j)\rangle
\end{aligned}
$$

for any $X \in \mathfrak{X}(M^3)$. Therefore, by (2.6) and (3.3), we get

$$(3.4) \qquad \langle \sigma(X, \xi j), \sigma(\xi i, \xi j)\rangle + \langle \sigma(X, \xi k), \sigma(\xi i, \xi k)\rangle = 0$$

for any $X \in \mathfrak{X}(M^3)$. Putting $X = \xi i$ in (3.4), we get

$$(3.5) \qquad \|\sigma(\xi i, \xi j)\|^2 + \|\sigma(\xi i, \xi k)\|^2 = 0 .$$

Hence, we have

$$(3.6) \qquad \sigma(\xi i, \xi j) = \sigma(\xi i, \xi k) = 0 .$$

By (3.6) and (b), (c) of Lemma 3.1, we have the desired equalities. □

From Lemma 3.2, it follows that the shape operator A_ξ takes the form

$$(3.7) \qquad A_\xi = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & \nu \\ 0 & \nu & \gamma \end{bmatrix}$$

with respect to the orthonormal frame field $\{\xi i, \xi j, \xi k\}$, where $\alpha = \langle \sigma(\xi i, \xi i), \xi \rangle$, $\beta = \langle \sigma(\xi j, \xi j), \xi \rangle$, $\gamma = \langle \sigma(\xi k, \xi k), \xi \rangle$ and $\nu = \langle \sigma(\xi j, \xi k), \xi \rangle$. Then, by (2.7) and (3.7), we get

$$
\begin{aligned}
(3.8) \qquad & \nabla_{\xi i}(\xi i) = 0 , && \nabla_{\xi j}(\xi j) = \nu \xi i , \\
& \nabla_{\xi k}(\xi k) = -\nu \xi i , && \nabla_{\xi i}(\xi j) = -\alpha \xi k ,
\end{aligned}
$$

$$\nabla_{\xi j}(\xi k) = -\beta\xi i\,, \qquad \nabla_{\xi k}(\xi i) = \nu\xi k - \gamma\xi j\,,$$
$$\nabla_{\xi j}(\xi i) = \beta\xi k - \nu\xi j\,, \quad \nabla_{\xi k}(\xi j) = \gamma\xi i\,,$$
$$\nabla_{\xi i}(\xi k) = \alpha\xi j\,.$$

Lemma 3.3. *The functions α, β, γ and ν satisfy the following conditions:*
(1) *β and γ are constant functions,*
(2) $\alpha\nu = 0$,
(3) $\nu(\beta + \gamma) = 0$,
(4) $\nu^2 + \beta(\alpha - \gamma) = 0$, $-\nu^2 + \gamma(\beta - \alpha) = 0$,
(5) $\xi i(\nu) + \alpha(\gamma - \beta) = 0$,
(6) $\xi j(\alpha) = \xi\, k(\alpha) = \xi j(\nu) = \xi k(\nu) = 0$.

Proof. Taking account of the definition of $\nabla\sigma$, Lemma 3.2 and (3.8), we get

$$\begin{aligned}(3.9)\qquad 0 &= \langle(\nabla\sigma)(\xi j, \xi i, \xi j), \xi\rangle \\ &= \xi j\langle\sigma\langle\xi i, \xi j), \xi\rangle - \langle\sigma(\nabla_{\xi j}(\xi i), \xi j), \xi\rangle - \langle\sigma(\xi i, \nabla_{\xi j}(\xi j)), \xi\rangle \\ &= -\langle\sigma(\beta\xi k - \nu\xi j, \xi j), \xi\rangle - \langle\sigma(\xi i, \nu\xi i), \xi\rangle \\ &= -\beta\nu + \nu\beta - \alpha\nu = -\alpha\nu\,.\end{aligned}$$

Hence we have (2). From (a) of Lemma 3.1 ($X = \xi j$, $Y = \xi k$), we get

$$\alpha\nu = \nu(\beta + \gamma)\,.$$

By (2), we have (3).

Similarly, from Lemma 3.2, (2.6), (3.8), (2) and the definition of $\nabla\sigma$, we get

$$\begin{aligned}(3.10)\qquad 0 &= \langle(\nabla\sigma)(\xi i, \xi j, \xi j), \xi\rangle = \xi\, i(\beta) + 2\alpha\nu = \xi\, i(\beta)\,, \\ 0 &= \langle(\nabla\sigma)(\xi i, \xi k, \xi k), \xi\rangle = \xi\, i(\gamma) - 2\alpha\nu = \xi\, i(\gamma)\,, \\ 0 &= \langle(\nabla\sigma)(\xi k, \xi j, \xi j), \xi\rangle = \xi\, k(\beta)\,, \\ 0 &= \langle(\nabla\sigma)(\xi j, \xi k, \xi k), \xi\rangle = \xi\, j(\gamma)\,, \\ 0 &= \langle(\nabla\sigma)(\xi j, \xi j, \xi j), \xi\rangle = \xi\, j(\beta)\,, \\ 0 &= \langle(\nabla\sigma)(\xi k, \xi k, \xi k), \xi\rangle = \xi\, k(\gamma)\,.\end{aligned}$$

From (3.10), we have (1).

$$\begin{aligned}(3.11)\qquad 0 &= \langle(\nabla\sigma)(\xi i, \xi j, \xi k), \xi\rangle = \xi\, i(\nu) + \alpha(\gamma - \beta)\,, \\ 0 &= \langle(\nabla\sigma)(\xi j, \xi i, \xi k), \xi\rangle = \nu^2 + \beta(\alpha - \gamma)\,, \\ 0 &= \langle(\nabla\sigma)(\xi k, \xi i, \xi j), \xi\rangle = -\nu^2 + \gamma(\beta - \alpha)\,.\end{aligned}$$

From (3.11), we have (4) and (5).

$$\begin{aligned}(3.12)\qquad 0 &= \langle(\nabla\sigma)(\xi j, \xi i, \xi i), \xi\rangle = \xi\, j(\alpha)\,, \\ 0 &= \langle(\nabla\sigma)(\xi k, \xi i, \xi i), \xi\rangle = \xi\, k(\alpha)\,, \\ 0 &= \langle(\nabla\sigma)(\xi j, \xi j, \xi k), \xi\rangle = \xi\, j(\nu)\,, \\ 0 &= \langle(\nabla\sigma)(\xi k, \xi k, \xi j), \xi\rangle = \xi\, k(\nu)\,.\end{aligned}$$

From (3.12), we have (6). □

Now, we are in a crucial position to prove Theorem A. The proof is divided into the following three cases from Lemma 3.3:

Case (1) $\beta = \gamma = 0$,

Case (2) $\beta = \gamma \neq 0$,

Case (3) $\beta \neq \gamma$.

Case (1). Then, by (4) of Lemma 3.3, we get the function v vanishes identically. In the sequel, we identify M^3 with $f(M^3)$ locally. We denote by D_α and D_0 1-dimensional and 2-dimensional distributions defined by $D_\alpha(p) := \mathrm{span}_{\boldsymbol{R}}\{\xi\, i(p)\}$, $D_0(p) := \mathrm{span}_{\boldsymbol{R}}\{\xi\, j(p), \xi\, k(p)\}$ for each $p \in M^3$, respectively. By $(3.8)_1$, each integral curve of D_α is a geodesic in M^3. By $(3.8)_2$, $(3.8)_3$, $(3.8)_5$, $(3.8)_8$, and taking account of $\beta = \gamma = v = 0$, we get

$$
(3.23) \qquad \begin{aligned} \nabla_{\xi i} D_0 &\subset D_0\,, \\ \nabla_{\xi j} D_0 &\subset D_0\,; \\ \nabla_{\xi k} D_0 &\subset D_0\,. \end{aligned}
$$

By (3.23), each leaf of D_0 is parallel in M^3 and furthermore, by $(3.8)_4$, $(3.8)_9$ and (2.2), each integral manifold of D_0 is locally flat, and hence M^3 is a locally product of a 1-dimensional Riemannian manifold and a 2-dimensional Euclidean space.

Next, we shall determine the immersion f. By (2.2), (3.6), (3.7), we get

$$
(3.24) \qquad D_{\xi j}(\xi j) = D_{\xi j}(\xi k) = D_{\xi k}(\xi j) = D_{\xi k}(\xi k) = 0\,,
$$

$$
(3.25) \qquad D_{\xi i}(\xi j \wedge \xi k) = -\lambda \xi k \wedge \xi k + \xi j \wedge (\eta \xi j) = 0\,.
$$

Let $M_\lambda(p)$ be the integral curve of D_λ through a point $p \in M^3$, then by (3.25), we see that images of the leaves of D_0 (by f) through the points on $f(M_\lambda(p))$ are parallel to each other in $\boldsymbol{H} = \boldsymbol{R}^4$ (and hence $f(M_\lambda(p))$ is a planar curve). Thus, the immersion f is the locally product (cf. [5]).

Case (2). Then, taking account of (3) of Lemma 3.3, we get $v = 0$ on M^3. By (4) of Lemma 3.3, we have $\alpha = \beta = \gamma \neq 0$. Hence, in this case (M^3, f) is a round sphere.

Case (3). We assume that $U := \{p \in M^3 \mid v(p) \neq 0\}$ is non-empty in M^3. By (2) and (3) of Lemma 3.3, $\beta + \gamma = 0$ and $\alpha = 0$ on U. Therefore, by (3) of Lemma 3.3, we get $v^2 + \beta^2 = 0$ on U. This is a contradiction. Hence we have $v = 0$ identically. Since $\beta \neq \gamma$, by (4), (5) of Lemma 3.3, we get $\beta\gamma = 0$ and $\alpha = 0$ on M^3. Hence, the following two cases are possible, (3-1) $\alpha = \beta = 0$ and $\gamma \neq 0$, (3-2) $\alpha = \gamma = 0$ and $\beta \neq 0$. Then, in both cases, applying the same arguments as in the case (1), we see also that (M^3, f) is locally isometric to a generalized cylinder $S^1(r) \times \boldsymbol{R}^2 \to \boldsymbol{R}^2 \times \boldsymbol{R}^2$ for some r where $S^1(r)$ is a 1-dimensional sphere of radius r, and $r = 1/\gamma$ or $1/\beta$. The case (3) is the special case of (2) in Theorem A.

This completes the proof of Theorem A.

4. PROOF OF THEOREM B

First, we assume that the Gauss-Kronecker curvature $\det A_\xi$ does not vanish identically. Let U be a connected component of the set $\{p \in M^3 \mid \det A_\xi(p) \neq 0\}$. We put $\alpha = \langle\sigma(\xi i, \xi i), \xi\rangle$, $\beta = \langle\sigma(\xi j, \xi j), \xi\rangle$, $\gamma = \langle\sigma(\xi k, \xi k), \xi\rangle$, $\lambda = \langle\sigma(\xi i, \xi j), \xi\rangle$, $\mu = \langle\sigma(\xi i, \xi k), \xi\rangle$ and $\nu = \langle\sigma(\xi j, \xi k), \xi\rangle$. Then the shape operator A_ξ is written by

$$A_\xi = \begin{bmatrix} \alpha & \lambda & \mu \\ \lambda & \beta & \nu \\ \mu & \nu & \gamma \end{bmatrix}. \tag{4.1}$$

By (2.5) and (4.1), the Ricci curvature ϱ is given by

$$\begin{bmatrix} \varrho(\xi i, \xi i) & \varrho(\xi i, \xi j) & \varrho(\xi i, \xi k) \\ \varrho(\xi j, \xi i) & \varrho(\xi j, \xi j) & \varrho(\xi j, \xi k) \\ \varrho(\xi k, \xi i) & \varrho(\xi k, \xi j) & \varrho(\xi k, \xi k) \end{bmatrix} \tag{4.2}$$
$$= \begin{bmatrix} \alpha(\beta+\gamma) - \lambda^2 - \mu^2 & \gamma\lambda - \mu\nu & \beta\mu - \nu\lambda \\ \gamma\lambda - \mu\nu & \beta(\gamma+\alpha) - \nu^2 - \lambda^2 & \alpha\nu - \lambda\mu \\ \beta\mu - \nu\lambda & \alpha\nu - \lambda\mu & \gamma(\alpha+\beta) - \mu^2 - \nu^2 \end{bmatrix}.$$

By (4.2), the frame $\{\xi i, \xi j, \xi k\}$ is a Ricci adapted frame if and only if

$$\gamma\lambda - \mu\nu = \beta\mu - \nu\lambda = \alpha\nu - \lambda\mu = 0 \quad \text{on} \quad M^3. \tag{4.3}$$

Lemma 4.1. *$\lambda\mu\nu = 0$ on U.*

Proof. We assume there exists a point $q \in U$ with $(\lambda\mu\nu)(q) \neq 0$. By (4.3), we get

$$\alpha = \lambda\mu/\nu, \quad \beta = \nu\lambda/\mu, \quad \gamma = \mu\nu/\lambda \quad \text{at } q. \tag{4.4}$$

By (4.1) and (4.4), we get

$$\det A_\xi(q) = \alpha\beta\gamma + 2\lambda\mu\nu - \{\alpha\nu^2 + \beta\mu^2 + \gamma\nu^2\} = 0.$$

This is a contradiction. □

By (4.3) and Lemma 4.1, we get

$$\alpha\nu^2 = \beta\mu^2 = \gamma\lambda^2 = \lambda\mu\nu = 0. \tag{4.5}$$

On the other hand, we get

$$\det A_\xi = \alpha\beta\gamma \neq 0. \tag{4.6}$$

By (4.5) and (4.6), we have $\lambda = \mu = \nu = 0$ on U. Hence, the shape operator A_ξ is given by

$$A_\xi = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \quad \text{on} \quad U. \tag{4.7}$$

By (2.7) and (4.7), the connection ∇ of M^3 is given by

$$\nabla_{\xi i}(\xi i) = \nabla_{\xi j}(\xi j) = \nabla_{\xi k}(\xi k) = 0, \tag{4.8}$$

$$\nabla_{\xi i}(\xi j) = -\alpha\xi k, \quad \nabla_{\xi j}(\xi i) = \beta\xi k,$$
$$\nabla_{\xi j}(\xi k) = -\beta\xi i, \quad \nabla_{\xi k}(\xi j) = \gamma\xi i,$$
$$\nabla_{\xi k}(\xi i) = -\gamma\xi j, \quad \nabla_{\xi i}(\xi k) = \alpha\xi j.$$

Lemma 4.2. $\alpha = \beta = \gamma\ (\neq 0)$ *on* U.

Proof. By (2.6) and (4.8), we get

$$\begin{aligned} 0 &= (\nabla_{\xi i}A_\xi)(\xi j) - (\nabla_{\xi j}A_\xi)(\xi i) \\ &= \nabla_{\xi i}(A_\xi(\xi j)) - A_\xi(\nabla_{\xi i}(\xi j)) - \nabla_{\xi j}(A_\xi(\xi i)) + A_\xi(\nabla_{\xi j}(\xi i)) \\ &= \nabla_{\xi i}(\beta\xi j) - A_\xi(-\alpha\xi k) - \nabla_{\xi j}(\alpha\xi i) + A_\xi(\beta\xi k) \\ &= \xi i(\beta)\,\xi j + \beta\nabla_{\xi i}(\xi j) + \alpha\gamma\xi k - \xi j(\alpha)\,\xi i - \alpha\nabla_{\xi j}(\xi i) + \beta\gamma\xi k \\ &= \xi i(\beta)\,\xi j - \alpha\beta\xi k + \alpha\gamma\xi k - \xi j(\alpha)\,\xi i - \alpha\beta\xi k + \beta\gamma\xi k \\ &= \xi i(\beta)\,\xi j - \xi j(\alpha)\,\xi i - \{2\alpha\beta - \gamma(\alpha + \beta)\}\,\xi k. \end{aligned}$$

Hence, we get

$$\xi i(\beta) = \xi j(\alpha) = 0 \quad \text{and} \quad 2\alpha\beta = \gamma(\alpha + \beta). \tag{4.9}$$

Similarly, by (2.6) and (4.8), we get

$$\xi j(\gamma) = \xi k(\beta) = 0 \quad \text{and} \quad 2\beta\gamma = \alpha(\beta + \gamma), \tag{4.10}$$

$$\xi k(\alpha) = \xi i(\gamma) = 0 \quad \text{and} \quad 2\gamma\alpha = \beta(\gamma + \alpha). \tag{4.11}$$

By (4.9), (4.10) and (4.11), we get the desired equality. □

From Lemma 4.2, we may see that each point of U is an umbilical point. Hence U is a non-empty, open and closed subset in M^9. Consequently, M^9 is an open piece of a round sphere.

Next, we assume that the Gauss-Kronecker curvature $\det A_\xi$ and the scalar curvature τ vanishes identically on M^9. In this case, we see that M^9 is an Ricci flat hypersurface and hence locally flat one. Hence, we see that M^9 is locally isometric to $M^1 \times \boldsymbol{R}^2$ (see [5]).

Lastly, we assume that the Gauss-Kronecker curvature $\det A_\xi$ vanishes identically on M^9, the scalar curvature τ is not identically 0 on M^9. Let U be a connected component of the set $(p \in M^3 \mid \tau(p) \neq 0\}$. By the assumption, the characteristic polynomial of A_ξ is given by

$$\det(xI - A_\xi) = x^3 - (\operatorname{tr} A_\xi)\,x^2 + (\tau/2)\,x.$$

Hence, the eigenvalues μ_1, μ_2, μ_3 of A_ξ are given by

$$\mu_1 = 0, \quad \mu_2 = \{\operatorname{tr} A_\xi + \surd((\operatorname{tr} A_\xi)^2 - 2\tau)\}/2, \tag{4.12}$$
$$\mu_3 = \{\operatorname{tr} A_\xi - \surd((\operatorname{tr} A_\xi)^2 - 2\tau)\}/2.$$

Then we have

$$\mu_2\mu_3 = \tau/2 \neq 0. \tag{4.13}$$

Therefore, the following two cases possible

Case (1) $\mu_2 \neq \mu_3$ at some point $q \in U$,

Case (2) $\mu_2 = \mu_3$ identically on U.

Case (1) Let U_0 be the connected component of the set $\{q \in U \mid \mu_2(q) \neq \mu_3(q)\}$. Then μ_1, μ_2, μ_3 are differentiable functions on U_0, and there exists the local orthonormal frame field $\{e_1, e_2, e_3\}$ on some neighborhood U_1 of U_0 such that

$$(4.14) \qquad A_\xi(e_1) = 0\,, \quad A_\xi(e_2) = \mu_2 e_2 \quad \text{and} \quad A_\xi(e_3) = \mu_3 e_3\,.$$

On one hand, we easily see that $\mathrm{span}_{\boldsymbol{R}}\{\xi i, \xi j, \xi k\} = \mathrm{span}_{\boldsymbol{R}}\{e_1, e_2, e_3\}$ at each point of U_1. Hence we can put

$$(4.15) \qquad \xi i = \sum_{i=1}^{3} \alpha_i e_i\,, \quad \xi j = \sum_{i=1}^{3} \beta_i e_i\,, \quad \xi k = \sum_{i=1}^{3} \gamma_i e_i\,,$$

where

$$(4.16) \qquad \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \in O(3)\,.$$

Then, for any $i = 1, 2, 3$, α_i, β_i and γ_i are differentiable functions on U_1.

By the assumption, $\{\xi i, \xi j, \xi k\}$ is a Ricci adapted frame, (4.12) and (4.13), we get

$$\begin{aligned}(4.17) \qquad 0 = \varrho(\xi i, \xi j) &= \mathrm{tr}A_\xi \langle A_\xi(\xi i), \xi j\rangle - \langle A_\xi(\xi i), A_\xi(\xi j)\rangle \\ &= \mathrm{tr}A_\xi\{\mu_2\alpha_2\beta_2 + \mu_3\alpha_3\beta_3\} - \{(\mu_2)^2\,\alpha_2\beta_2 + (\mu_3)^2\,\alpha_3\beta_3\} \\ &= \mu_2\mu_3(\alpha_2\beta_2 + \alpha_3\beta_3) = \tau(\alpha_2\beta_2 + \alpha_3\beta_3)/2\,.\end{aligned}$$

Similarly, $(\varrho(\xi j, \xi k) = \varrho(\xi k, \xi i) = 0)$, by (4.12) and (4.13), we get

$$(4.18) \qquad 0 = \tau(\beta_2\gamma_2 + \beta_3\gamma_3)/2 = \tau(\gamma_2\alpha_2 + \gamma_3\alpha_3)/2\,.$$

By the assumption $(\tau \neq 0)$, (4.16), (4.17) and (4.18), we get

$$(4.19) \qquad \alpha_1\beta_1 = \beta_1\gamma_1 = \gamma_1\alpha_1 = 0\,.$$

If α_1 is not identically 0 on U_1, by (4.16) and (4.19), there exists a neighborhood U_2 of U_1 such that

$$(4.20) \qquad \alpha_1 = \pm 1\,, \quad \beta_1 = \gamma_1 = 0\,, \quad \text{on} \quad U_2\,.$$

Hence, without loss of generality, we may put

$$(4.21) \qquad \xi i = e_1\,, \quad \xi j = ae_2 + be_3\,, \quad \xi k = -be_2 + ae_3\,, \quad \text{on} \quad U_2\,,$$

where $a^2 + b^2 = 1$. By (4.1), (4.14) and (4.19), we get

$$\begin{aligned}(4.22) \qquad &\alpha = \langle\sigma(e_1, e_1), \xi\rangle = 0\,, \quad \beta = a^2\mu_2 + b^2\mu_3\,, \\ &\gamma = b^2\mu_2 + a^2\mu_3\,, \quad \lambda = \langle\sigma(e_1, ae_2 + be_3), \xi\rangle = 0\,, \\ &\mu = \langle\sigma(e_1, -be_2 + ae_3), \xi\rangle = 0\,, \\ &\nu = \langle\sigma(ae_2 + be_3, -be_2 + ae_3), \xi\rangle = ab(\mu_3 - \mu_2)\,, \quad \text{on} \quad U_2\,.\end{aligned}$$

On one hand, by (2.2), we get

$$(4.23) \qquad \langle (\nabla\sigma)(\xi i, \xi j, \xi k), \xi \rangle = \xi i(\nu) + \alpha(\gamma - \beta) + \lambda^2 - \mu^2$$
$$= \xi j(\mu) + \beta(\alpha - \gamma) + \nu^2 - \lambda^2 = \xi k(\lambda) + \gamma(\beta - \alpha) + \mu^2 - \nu^2 \quad \text{on} \quad U_2 .$$

By (4.22) and (4.23), we get

$$\xi i(\nu) = -\beta\gamma + \nu^2 = \beta\gamma - \nu^2 , \quad \text{on} \quad U_2 .$$

Hence, we have

$$(4.24) \qquad \beta\gamma - \nu^2 = 0 , \quad \text{on} \quad U_2 .$$

On the other hand, by (4.22) the scalar curvature τ is given by

$$(4.25) \qquad \tau = 2(\alpha\beta + \beta\gamma + \gamma\alpha - \lambda^2 - \mu^2 - \nu^2) = 2(\beta\gamma - \nu^2) .$$

By (4.24) and (4.25), we get $\tau = 0$. This contradicts the assumption. Hence, $\alpha_1 = \beta_1 = \gamma_1 = 0$ on U_1, this contradicts (4.16). Consequently, the case (1) does not occur.

By the same argument, the case (2) does not occur.

This completes the proof of Theorem B.

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