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ON LATTICE ORDERED GROUPS HAVING A UNIQUE ADDITION

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Lattice ordered groups with unique addition (for definitions, cf. below) were investigated by P. Conrad and M. Darnel [1]. The case of linear ordered groups having this property was dealt with by T. Ohkuma [2].

In [1] the following open question was proposed:

(*) If G is a lattice ordered group such that the positive cone has a unique addition, then does G have a unique addition as well?

It was remarked in [1] that the answer is yes if G is a linearly ordered group.

In the present paper it will be shown that the answer is positive also in the general case.

1. PRELIMINARIES

Let us recall the following definition (cf. [1]).

A lattice ordered group $G_1 = (G; \leqq, +_1)$ is said to have a *unique addition* if, whenever $G_2 = (G; \leqq, +_2)$ is a lattice ordered group such that the neutral element of the group $(G; +_1)$ is the same as the neutral element of the group $(G; +_2)$, then the operation $+_1$ coincides with the operation $+_2$.

As usual, the positive cone of a lattice ordered group G_1 will be denoted by G_1^+; it is a lattice ordered subsemigroup of G_1 with the underlying set $\{x \in G: x \geqq 0_1\}$, the partial order being inherited from the partial order in G; the symbol 0_1 denotes the neutral element of G_1.

Analogously to the above definition, the positive cone G_1^+ of a lattice ordered group G_1 will be said to have a unique addition if, whenever $G_2 = (G; \leqq, +_2)$ is a lattice ordered group with $0_1 = 0_2$ (where 0_2 is the neutral element of G_2) and $0 \leqq x \in G$, $0 \leqq y \in G$, then $x +_1 y = x +_2 y$.

In what follows, $G_1 = (G; \leqq, +_1)$ is a lattice ordered group. The lattice $(G; \leqq)$ will be denoted by $L(G_1)$.

1.1. Lemma. *Let $H_0 = (H; \leqq, +_0)$ be a lattice ordered group such that the lattice $L(H_0)$ is isomorphic to $L(G_1)$. Assume that the positive cone G_1^+ of G_1 has a unique addition. Then the positive cone H_0^+ of H_0 has a unique addition as well.*

Proof. By way of contradiction, assume that the positive cone H_0^+ of H_0 fails to have a unique addition. Then there is a lattice ordered group $H^* = (H; \leqq, +^*)$ such that

(i) the neutral element of $(H; +_0)$ is the same as the neutral element of $(H; +^*)$ (this neutral element will be denoted by 0);

(ii) there are $x, y \in H$ such that $0 \leqq x$, $0 \leqq y$ and $x +_0 y \neq x +^* y$.

From the fact that $L(G_1)$ is isomorphic to $L(H_0) = L(H^*)$ it follows that there exists an isomorphism φ if $L(G_1)$ onto $L(H_0)$ such that

(1) $$\varphi(0_1) = 0 .$$

We define two binary operations $+_2$ and $+_3$ on G by putting, for each $g_1, g_2 \in G$,

$$g_1 +_2 g_2 = \varphi^{-1}(\varphi(g_1) +_0 \varphi(g_2)) ,$$
$$g_1 +_3 g_2 = \varphi^{-1}(\varphi(g_1) +^* \varphi(g_2)) .$$

Then $G_2 = (G; \leqq, +_2)$ and $G_3 = (G; \leqq, +_3)$ are lattice ordered groups. According to (1) we have $0_1 = 0_2 = 0_3$, where 0_2 and 0_3 have the obvious meaning. Next, the condition (ii) yields that

(2) $$0_1 \leqq \varphi^{-1}(x) , \quad 0_1 \leqq \varphi^{-1}(y) , \quad \varphi^{-1}(x) +_2 \varphi^{-1}(y) \neq$$
$$\neq \varphi^{-1}(x) +_3 \varphi^{-1}(y) .$$

From (2) we infer that we have either

$$\varphi^{-1}(x) +_1 \varphi^{-1}(y) \neq \varphi^{-1}(x) +_2 \varphi^{-1}(y) ,$$

or

$$\varphi^{-1}(x) +_1 \varphi^{-1}(y) \neq \varphi^{-1}(x) +_3 \varphi^{-1}(y) .$$

Hence the positive cone G_1^+ of G_1 fails to have a unique addition, which is a contradiction.

1.2. Lemma. *Assume that the positive cone G_1^+ of G_1 has a unique addition. Then G_1 is abelian.*

Proof. By way of contradiction, suppose that G_1 fails to be abelian. Since for each $x \in G$ there are $y, z \in G$ with $0 \leqq y$, $0 \leqq z$ such that $x = y - z$, it follows that G_1^+ fails to be abelian as well. For each $u, v \in G$ we put $u + +_2 v = v +_1 u$. Then $G_2 = (G; \leqq, +_2)$ is a lattice ordered group with $0_2 = 0_1$. The operation $+_2$ on the positive cone of G_1 does not coincide with the operation $+_1$, which is a contradiction.

2. UNIQUE ADDITION IN G_1^+

In this section we assume that the positive cone G_1^+ of G_1 has a unique addition. Thus in view of 1.2, G_1 is abelian. Let $G_2 = (G; \leqq, +_2)$ be a lattice ordered group such that $0_1 = 0_2$.

For each $g \in G$ and each $x, y \in G$ we put

$$x +_i^g y = x -_i g +_i y \quad (i = 1, 2) .$$

We have obviously

2.1. Lemma. *Let $i \in \{1, 2\}$. Then $G_i^g = (G; \leqq, +_i^g)$ is a lattice ordered group with the neutral element g.*

2.2. Lemma. *Let $i \in \{1, 2\}$. Then the positive cone $(G_i^g)^+$ of G_i^g has a unique addition.*

Proof. This is a consequence of 1.1.

Let us denote by G^+ the underlying set of G_1^+; it is, at the same time, the underlying set of G_2^+. Next let G^- have a dual meaning.

2.3. Lemma. *Let $x, y \in G^-$. Then $x +_1 y = x +_2 y$.*

Proof. Let $\leqq'$ be the partial order on G which is dual to $\leqq$. Then $G_1' = (G; \leqq', +_1)$ and $G_2' = (G; \leqq', +_2)$ are lattice ordered groups having the same neutral element. Next, the lattice $L(G_1')$ is isomorphic to the lattice $L(G_1)$. Hence in view of 1.1, the positive cone of G_1' has a unique addition. Since $0 \leqq' x$ and $0 \leqq' y$, we obtain that $x +_1 y = x +_2 y$.

2.4. Lemma. *Let $x \in G$ such that either $x \geqq 0$ or $x \leqq 0$. Next let n be a positive integer. Then the symbol nx has the same meaning for both G_1 nad G_2.*

Proof. This follows by induction from the fact that G_1^+ has a unique addition, or from 2.3, respectively.

2.5. Lemma. *Let $x \in G$. Then $2x$ is the (uniquely determined) relative complement of the element 0_1 in the interval $[2(x \wedge 0_1), 2(x \vee 0_1)]$ of the lattice $(G; \leqq)$.*

The proof can be established by a routine calculation; it will be omitted.

The lemmas 2.4 and 2.5 yield:

2.6. Lemma. *Let $x \in G$. Then the symbol $2x$ has the same meaning in both G_1 and G_2.*

2.7. Lemma. *Let $z \in G^-$. Then $-_1 z = -_2 z$.*

Proof. Denote $x = -_1 z$, $y = -_2 z$. Then $x \geqq 0_1$ and $y \geqq 0_1$. Hence $2z \leqq x$. Clearly $2z \leqq z$. Thus in view of 2.1 and 2.2 we have

(1) $$x +_1^{2z} z = x +_2^{2z} z .$$

According to 2.6 we obtain

$$x +_1^{2z} z = x -_1 2z +_1 z = x -_1 z = x +_1 x = 2x ,$$
$$x +_2^{2z} z = x -_2 2z +_2 z = x -_2 z = x +_2 y .$$

Hence (1) yields that $2x = x +_2 y$ and therefore $x = y$.

2.8. Lemma. *Let $x \in G^+$. Then $-_1 x = -_2 x$.*

Proof. Denote $-_1 x = z$. Then $z \leqq 0$ and $-_1 z = x$. According to 2.7 we have $-_2 z = x$, whence $-_2 x = z$.

2.9. Lemma. *Let $a_1, b_1 \in G^+$, $a_1 \wedge b_1 = 0_1$. Then $a_1 -_1 b_1$ is the (uniquely determined) relative complement of the element 0_1 in the interval $[-_1 b_1, a_1]$ of the lattice $(G; \leqq)$.*

The proof consists in applying standard calculations; we omit it.

2.10. Lemma. *Let a_1 and b_1 be as in 2.9. Then $a_1 -_1 b_1 = a_1 -_2 b_1$.*

Proof. This is a consequence of 2.8 and 2.9.

2.11. Lemma. *Let $a, b \in G^+$. Then $a -_1 b = a -_2 b$.*

Proof. Put $a \wedge b = u$. Then $u \geqq 0_1$. Denote $a_1 = a -_1 u$, $b_1 = b -_1 u$. We have $a_1 \in G^+$, $b_1 \in G^+$, whence $a = u +_1 a_1 = u +_2 a_1$, $b = u +_1 b_1 = u +_2 b_1$. Thus $a -_1 b = a_1 -_1 b_1$ and $a -_2 b = a_1 -_2 b_1$. Clearly $a_1 \wedge b_1 = 0_1$. Therefore in view of 2.10 we obtain $a -_1 b = a -_2 b$.

2.12. Proposition. *Let $a, b \in G$. Then $a +_1 b = a +_2 b$.*

Proof. Denote $a_1 = a \vee 0_1$ and $a_2 = a \wedge 0_1$. Let b_1 and b_2 have analogous meanings with respect to b. Then

$$a_1 +_1 a_2 = a = a_1 +_2 a_2\,, \quad b_1 +_1 b_2 = b = b_1 +_2 b_2\,.$$

Hence

$$a +_1 b = (a_1 +_1 a_2) +_1 (b_1 +_1 b_2) = (a_1 +_1 b_1) +_1 (a_2 +_1 b_2)\,.$$

Because $a_1 \geqq 0_1$ and $b_1 \geqq 0_1$, the relation $a_1 +_1 b_1 = a_1 +_2 b_1$ is valid. Next, since $a_2 \leqq 0_1$ and $b_2 \leqq 0_1$, in view of 2.3 we have $a_2 +_1 b_2 = a_2 +_2 b_2$. Also, $a_2 +_2 b_2 \leqq 0_1$, whence according to 2.7

$$-_1 (a_2 +_2 b_2) = -_2(a_2 +_2 b_2)\,.$$

Therefore

$$a_1 +_1 b = (a_1 +_2 b_1) -_1 (-_1(a_2 +_2 b_2)) =$$
$$= (a_1 +_2 b_1) -_1 (-_2(a_2 +_2 b_2))\,.$$

Now by applying 2.11 we obtain

$$a_1 +_1 b = (a_1 +_2 b_1) -_2 (-_2(a_2 +_2 b_2)) =$$
$$= (a_1 +_2 a_2) +_2 (b_1 +_2 b_2) = a +_2 b\,.$$

Proposition 2.12 shows that the answer to the question (*) above is 'YES'.

References

[1] *P. Conrad, M. Darnel:* l-groups with unique addition. Algebra and Order. Proc. First Int. Symp. Ordered Algebraic Structures, Luminy-Marseille 1984, Helderman Verlag, Berlin 1986, 15—27.

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