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ON A CODIMENSION 3 BIFURCATION OF PLANE VECTOR FIELDS WITH Z_2 SYMMETRY

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1. INTRODUCTION

We consider the following 3-parameter family of plane vector fields

(1) $$\dot{x} = f(x, \lambda) := A(\lambda)\, x + h(x, \lambda),$$

where $x = (x_1, x_2) \in R^2$, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in R^3$, $f = (f_1, f_2) \in C^\infty$,

(2) $$h(0, \lambda) = 0 \quad \text{for all} \quad \lambda,$$

(3) $$-h(-x, \lambda) = h(x, \lambda) \quad \text{for all} \quad \lambda, x$$

and $h(x, 0) = o(\|x\|^2)$. We assume that

$$A(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the matrix $A(\lambda)$ is a versal unfolding of $A(0)$. We may assume without loss of generality (see Arnold [1]) that

(H1) $$A(\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}.$$

Moreover, we may assume that the following hypotheses are satisfied:

(H2) $$\beta_1 := \frac{\partial^3 f_2(0, 0)}{\partial x_1^2\, \partial x_2} = 0,$$

(H3) $$\alpha := \frac{\partial^3 f_2(0, 0)}{\partial x_1^3} \neq 0,$$

(H4) $$\beta := \frac{\partial^5 f_2(0, 0)}{\partial x_1^4\, \partial x_2} \neq 0.$$

Under the hypotheses (H1)–(H4) the singular point $(0, 0)$ represents a singularity of codimension 3 in the space of all plane vector fields with Z_2 symmetry, and these hypotheses are generically satisfied in the space of all smooth families of the form (1) satisfying (2), (3) and endowed with the Whitney C^∞-topology. Moreover, the set of all such singularities generically consists of isolated points. If the hypotheses (H1)–(H4) are satisfied, except $\beta_1 = 0$, then the singular point $(0, 0)$ represents a singularity of codimension 2. Bifurcations near such a singularity are well-known (see Carr [5], Hale [18], Horozov [17] and Takens [27]). This singularity appears for the family (1) on a 1-dimensional submanifold of the parameter space.

The main results on codimension 2 bifurcations are well-known (see Arnold [1], Bogdanov [2], [3], Carr [5], Chow and Hale [7], Guckenheimer and Holmes [16], Horozov [17], Hale [18], Takens [26], [27] and Žoladek [28], [29]). Recently, multiparameter bifurcations of vector fields have been intensively studied (see Dangelmayr and Guckenheimer [10], Dumortier et al. [12], Medveď [21], [22], [24], Žoladek [30]). For applications of such results see e.g. Guckenheimer [14].

Very useful for applications are the results on bifurcations of equivariant vector fields. Many bifurcation problems concerning equivariant vector fields with higher dimensional state space are reducible to bifurcation problems of plane vector fields at least partially. As an example we mention the results concerning smooth parametrized vector fields on R^4 with non-zero nilpotent linear parts which are equivariant with respect to the diagonal action of $0(2)$, the group of orthogonal 2×2 matrices, on R^4 (see Guckenheimer [15] and Medveď [23]). A normal form of order ∞ of such vector fields derived in Medveď [23] has the form

$$(4) \qquad \begin{aligned} \dot{x}_1 &= x_3\,, \quad \dot{x}_2 = x_4\,, \\ \dot{x}_3 &= \lambda_1 x_1 + \lambda_2 x_3 + x_3 P(x_1^2 + x_2^2,\ x_1 x_4 - x_2 x_3, \lambda) + \\ &\quad + x_1 Q(x_1^2 + x_2^2, \lambda)\,, \\ \dot{x}_4 &= \lambda_1 x_2 + \lambda_2 x_4 + x_4 P(x_1^2 + x_2^2,\ x_1 x_4 - x_2 x_3, \lambda) + \\ &\quad + x_2 Q(x_1^2 + x_2^2, \lambda)\,, \end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in R^k$ is a parameter. This normal form has been obtained by the method presented in Elphick et al. [13] and it differs from that obtained by J. Guckenheimer [15], which is of order 3. It is also proved there that if the function P is bounded then the set $D = \{(x_1, x_2, x_3, x_4) \in R^4\colon x_1 x_4 - x_2 x_3 = 0\}$ is an invariant set of the family (4), and if $x = (x_1, x_2, x_3, x_4)$ is a solution of (4) with $x(0) \in D$ and $\alpha = x_1^2 + x_2^2$, $\beta = x_3^2 + x_4^2$ then $(X, Y) = (\sqrt{\alpha}, \sqrt{\beta})$ is a solution of the system

$$(5) \qquad \begin{aligned} \dot{X} &= Y\,, \\ \dot{Y} &= \lambda_1 X + \lambda_2 Y + Q(X^2, \lambda)\, X + P(X^2, \lambda)\, Y\,. \end{aligned}$$

This system satisfies the symmetry condition (3) and so our results concerning codimension 3 bifurcations of the family (1) can also be applied to this bifurcation problem.

The problem concerning the existence of homoclinic and heteroclinic trajectories as well as the number of simple limit cycles of plane vector fields can often be solved by using the properties of Abelian integrals (see Arnold [1], Ilyashenko [19], [20]).

This method has also been used in the study of codimension 2 bifurcations (see Bogdanov [2], [3], Carr [5], Drachman et al. [11], Dumortier et al. [12], Cushman and Sanders [9], Chow and Sanders [8], Carr et al. [6], Sanders and Cushman [25], Žoladek [28], [29], [30], Horozov [17]). We also use this method and the proofs of our results are close to the proofs of J. Carr [5] (see also Hale [18]).

One can check that the transformation $(x_1, x_2) \to |\alpha|^{-1/2}(x_1, x_2)$ transforms the family (1) into the same form with $\alpha = \pm 1$. We study the case $\alpha = -1$. Since our considerations are local, we may also assume without loss of generality that the family (1) is in the following normal form (see e.g. Elphic et al [13] and Guckenheimer and Holmes [16]):

(6) $$\dot{x} = y\,,$$
$$\dot{y} = (\varepsilon - x^2)\,x + (\mu_1 + \mu_2 x^2 + \beta x^4)\,y + R(x^2, \varepsilon, \mu)\,y\,,$$

where $(\varepsilon, \mu) = (\varepsilon, \mu_1, \mu_2) \in R^3$ is a parameter, $R \in C^\infty$, $R(x^2, 0, 0) = o(|x|^4)$, $\beta \neq 0$.

The introduction of the scaled variables

(7) $$x = \delta u\,, \quad y = \delta^2 v\,, \quad t = \delta^{-1}\tau\,, \quad \mu_1 = \delta^2 \nu_1\,, \quad \mu_2 = \nu_2$$

where $\delta = |\varepsilon|^{1/2}$ leads to new equations

(8) $$\dot{u} = v\,,$$
$$\dot{v} = (\varkappa - u^2)\,u + \delta(\nu_1 + \nu_2 u^2 + \delta^2 \beta u^4 + \delta^4 g(u^2, \delta, \nu_1, \nu_2))\,v\,,$$

where $\varkappa = \operatorname{sign} \varepsilon$, $g(x^2, 0, 0, 0) = o(|x|^4)$, $\delta = |\varepsilon|^{1/2}$.

2. HOMOCLINIC TRAJECTORIES

We study the system (6) with $\varepsilon > 0$. The family (8) has the form

(9) $$w^\delta\colon \dot{u} = v\,,$$
$$\dot{v} = (1 - u^2)\,u + \delta(\nu_1 + \nu_2 u^2 + \delta^2 \beta u^4 + \delta^4 g(u^2, \delta, \nu_1, \nu_2))\,v\,.$$

The family w^0 can be written in the form

(10) $$u = \frac{\partial H(u, v)}{\partial v}\,, \quad v = -\frac{\partial H(u, v)}{\partial u}\,,$$

where

$$H(u, v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4}\,.$$

The level set $H^{-1}(0)$ is composed of two homoclinic trajectories

(11) $$\Gamma^+\colon (u(t), v(t)) := (\sqrt{2}\operatorname{sech} t,\ -\sqrt{2}\operatorname{sech} t\,.\,\tanh t)\,,$$
$$\Gamma^-\colon (u^-(t), v^-(t)) := (-u(t), -v(t))\,,$$

where $\operatorname{sech} t = 2(e^t + e^{-t})^{-1}$, $\tanh t = (e^t - e^{-t})(e^t + e^{-t})^{-1}$. Using the same procedure as in Carr [5] one can derive an equation for the values of parameters δ, ν_1, ν_2 for which the equation (9) has homoclinic trajectories. This equation has the form

$$\nu_1 J_0 + \nu_2 J_2 + \delta^2 \beta J_4 + \omega(\delta, \nu_1, \nu_2) = 0\,, \tag{12}$$

where $\omega \in C^\infty$, $\omega(\delta, \nu_1, \nu_2) = O(\delta^4(\nu_1^2 + \nu_2^2))$,

$$J_{2k} = \int_{-\infty}^{\infty} v^2(t)\, u^{2k}(t)\, dt\,, \quad k = 0, 1, 2\,. \tag{13}$$

One can easily check that

$$Q_k := \int_{-\infty}^{\infty} \operatorname{sech}^2 t\,.\, \tanh^k t\, dt = [(k+1)^{-1} \tanh^{k+1} t]_{-\infty}^{\infty} = \frac{2}{k+1}\,,$$

$k = 0, 1, 2, \ldots$ and therefore we have

$$J_{2k} = 2^{k+1} \int_{-\infty}^{\infty} \operatorname{sech}^{2k+2} t\,.\, \tanh^2 t\, dt =$$

$$= 2^{k+1} \int_{-\infty}^{\infty} \operatorname{sech}^2 t (1 - \tanh^2 t)^k\,.\, \tanh^2 t\, dt\,, \quad \text{i.e.}$$

$$J_{2k} = 2^{k+2} \left(\frac{1}{3} - \binom{k}{1} \frac{1}{5} + \binom{k}{2} \frac{1}{7} - \ldots + (-1)^k \frac{1}{2k+3} \right), \tag{14}$$

$k = 0, 1, 2, \ldots.$

This yields

$$J_0 = 2\,, \quad J_2 = \tfrac{16}{15}\,, \quad J_4 = \tfrac{128}{105}\,. \tag{15}$$

The implicit function theorem implies that the equation (12) can be written in the same form, where the function ω is independent of ν_1, i.e. we obtain the equation

$$\nu_1 = -\frac{J_2}{J_0} \nu_2 - \beta \delta^2 \frac{J_4}{J_0} + \omega_1(\delta, \nu_2)\,, \tag{16}$$

where $\omega_1 \in C^\infty$, $\omega_1(\delta, \nu_2) = O(\delta^4 \nu_2)$. This equation written in the original coordinates $\mu_1, \mu_2, \varepsilon$ has the form

$$\mu_1 = \varphi_1(\mu_2, \varepsilon) := -\tfrac{8}{15}\varepsilon\mu_2 - \tfrac{64}{105}\beta\varepsilon^2 + O(\varepsilon^2 \mu_2^2)\,. \tag{17}$$

Let $L_1 := \operatorname{graph} \varphi_1$.

Theorem 2. 1. *There is an open neighbourhood $U \times V$ of the origin $(\mu_2, \varepsilon) = (0, 0)$ and a C^∞-function $\varphi_1 : U \times V \to R$ of the form (17) such that when $(\mu_1, \mu_2, \varepsilon) \in L_1 = \operatorname{graph} \varphi_1$, the equation (6) has a homoclinic trajectory* (see Fig. 3).

3. PERIODIC TRAJECTORIES SURROUNDING ALL THREE EQUILIBRIUM POINTS

Using the procedure as in Carr [5] one can obtain an equation for the values of parameters δ, ν_1, ν_2 for which the equation (9) has periodic trajectories surrounding all three equilibrium points and lying in a small neighbourhood of the homoclinic trajectory appearing for $(\mu, \varepsilon) \in L_1$. This equation has the form

(18) $$v_1 I_0 + v_2 I_1 + \delta^2 \beta I_2 + \tilde{\omega}(\delta, v_1, v_2) = 0\,,$$

where $\tilde{\omega} \in C^\infty$, $\tilde{\omega}(\delta, v_1, v_2) = O(\delta^4(v_1^2 + v_2^2))$,

(19) $$I_i = \int_0^c u^{2i}\, r(u)\, du\,, \quad i = 0, 1, 2\,,$$

$r(u) := (u^2 - \frac{1}{2}u^4 + 2b)^{1/2}$, $4b = c^4 - 2c^2$, $b > 0$. The implicit function theorem implies that the equation (18) can be written in the same form, where the function $\tilde{\omega}$ is independent of v_1, i.e. we obtain the equation

(20) $$v_1 = -\frac{I_1}{I_0}\, v_2 - \delta^2 \beta \frac{I_2}{I_0} + \tilde{\omega}_1(\delta, v_2)\,,$$

where $\tilde{\omega}_1 \in C^\infty$. This equation can be written in the original coordinates $\mu_1, \mu_2, \varepsilon$ and we obtain the equation

(21) $$\mu_1 = \varphi_2(\mu_2, \varepsilon) := -P(b)\, \varepsilon\mu_2 - \beta\, Q(b)\, \varepsilon^2 + O(\varepsilon^2 \mu_2^2)\,,$$

where $P(b) := I_1/I_0$, $Q(b) := I_2/I_0$. The properties of the function P are described in Carr [5] (see also Hale [18]) and we shall recall them later.

If $I_i' := dI_i/db$ then

(22) $$I_i' = \int_0^c \frac{u^{2i}}{r(u)}\, du\,.$$

Since $r(c) = 0$ we have

$$0 = \int_0^c \frac{d}{du}\,(u^{2i}\, r(u))\, du = (2i + 1)\int_0^c u^{2i}\,\frac{r^2(u)}{r(u)}\, du + \int_0^c u^{2i+1}\,\frac{u - u^3}{r(u)}\, du\,,$$

and using the definition of $r(u)$ and (22) we obtain

(23) $$(2i + 3)\, I_{i+2}' - 4(i + 1)\, I_{i+1}' - 4(2i + 1)\, b I_i' = 0\,, \quad i = 0, 1, 2, \dots.$$

Integration by parts yields

$$I_2 = \int_0^c u^4\, r(u)\, du = \frac{1}{5}\int_0^c (u^8 - u^6)\,(r(u))^{-1}\, du\,, \quad \text{i.e.}$$

(24) $$5I_2 = I_4' - I_3'\,.$$

From (24) we have

(25) $$I_2' = \frac{4}{3}(I_1' + bI_0')\,,$$
$$I_3' = \frac{8}{5}I_2' + \frac{12}{5}bI_1'\,,$$
$$I_4' = \frac{12}{7}I_3' + \frac{20}{7}bI_2'\,.$$

These equalities and (24) yield

(26) $$I_2 = \frac{4}{105}((8 + 29b)\, I_1' + 4b(2 + 5b)\, I_0')\,.$$

By Carr [5] and Hale [18]

(27) $$3I_0 = I_1' + 4bI_0'\,,$$
$$15I_1 = (4 + 12b)\, I_1' + 4bI_0'\,,$$

and from this system we obtain the following system of differential equations for I_0, I_1:

$$I_1' = \frac{1}{1+4b}(5I_1 - I_0),$$

$$I_0' = \frac{1}{4b(1+4b)}(4(1+3b)I_0 - 5I_1),$$

which are called the Picard-Fuchs equations for Abelian integrals. From (26) and (28) we have the formula

(29) $$I_2 = \tfrac{4}{7}(2I_1 + bI_0)$$

and thus the function $Q(b) = I_2/I_0$ has the form

(30) $$Q(b) = \tfrac{4}{7}(2P(b) + b).$$

Therefore the equation (21) has the form

(31) $$\mu_1 = \chi(b, \mu_2, \varepsilon) := -P(b)\,\varepsilon\mu_2 - \tfrac{8}{7}\beta\psi(b)\,\varepsilon^2 + O(\varepsilon^2\mu_2^2),$$

where

(32) $$\psi(b) := P(b) + b/2, \quad b > 0.$$

Let us recall the properties of the function $P(b)$ (see Carr [5], Guckenheimer and Holmes [16], Hale [18] and Fig. 1).

(I) $\lim_{b\to 0^+} P(b) = 1$, $\lim_{b\to 0^+} P'(b) = -\infty$;

(II) P has a unique singular point $b_1 > 0$, $\min_{b>0} P(b) = P(b_1) \doteq 0.752$;

(III) $P'(b) < 0$ for $b < b_1$ and $P'(b) > 0$ for $b > b_1$;

(IV) $\lim_{b\to\infty} P(b) = \infty$;

(33) $$P'(b) = \frac{1}{4b(1+4b)}(-4b + 4(2b-1)P(b) + 5(P(b))^2.$$

The equation (33) can be derived using (28). From (I)–(IV) and (32) it follows that

(a) $\lim_{b\to 0^+} \psi(b) = 1$, $\lim_{b\to 0^+} \psi'(b) = -\infty$, $\psi(b) > P(b)$ for $b > 0$;

(b) ψ has a unique singular point $c_1 > 0$, $c_1 < b_1$ (see (II)), $\min_{b>0} \psi(b) = \psi(c_1)$;

(c) $\psi'(b) < 0$ for $b < c_1$ and $\psi'(b) > 0$ for $b > c_1$;

(d) $\lim_{b\to\infty} \psi(b) = \infty$.

For the graph of ψ see Fig. 1.

Lemma 3.1. *If $\beta > 0$ ($\beta < 0$) then there exists a number $k > 0$ such that for all $(\mu_2, \varepsilon) \in [0, k) \times (0, k)$ ($(\mu_2, \varepsilon) \in (-k, 0] \times (0, k)$) the equality*

$$\frac{\partial\chi(b, \mu_2, \varepsilon)}{\partial b} = 0 \quad \textit{implies} \quad \frac{\partial^2\chi(b, \mu_2, \varepsilon)}{\partial b^2} < 0$$

$$\left(\frac{\partial^2\chi(b, \mu_2, \varepsilon)}{\partial b^2} > 0\right), \quad \textit{where the function } \chi(b, \mu_2, \varepsilon) \textit{ is defined by } (31).$$

Proof. If $\partial\chi(b, \mu_2, \varepsilon)/\partial b = 0$ then using (31) and (33) we obtain that

$$P'(b) = -\frac{28\beta b\varepsilon}{7\mu_2 + 8\beta\varepsilon} + \frac{O(\varepsilon\mu_2^2)}{7\mu_2 + 8\beta\varepsilon}$$

and therefore we obtain that if $\beta\mu_2 \geqq 0$, μ_2, ε sufficiently small then $P'(b) < 0$. Using the formula (33) one can show that

$$(34) \qquad 2b(1 + 4b)\, P''(b) = 4P'(b)\,(P(b) - 4b - 1) + 4P(b) - 2\,.$$

Since $P'(b) < 0$, the properties of the function $P(b)$ give that $0.7 < P(b) \leqq 1$. Therefore $4P(b) - 4b - 1 \leqq -4b < 0$, $4P(b) - 2 > 0$ and (34) implies that $P''(b) > 0$. We have obtained that if $k > 0$ is sufficiently small, $(\mu_2, \varepsilon) \in [0, k) \times (0, k)$, $\beta > 0$ $((\mu_2, \varepsilon) \in (-k, 0] \times (0, k)$, $\beta < 0)$ and $\partial\chi(b, \mu_2, \varepsilon)/\partial b = 0$ then $P''(b) > 0$ and (32) yields $\psi''(b) > 0$. Therefore from (31) we obtain that if $\beta > 0$, $\mu_2 \geqq 0$ $(\beta < 0$, $\mu_2 \leqq 0)$ then $\partial^2\chi(b, \mu_2, \varepsilon)/\partial b^2 < 0$ $(\partial^2\chi(b, \mu_2, \varepsilon)/\partial b^2 > 0)$.

As a consequence of Lemma 3.1 and the properties of the functions P and ψ we obtain

Lemma 3.2. *Let $\beta > 0$ and k be as in Lemma 3.1. Then for every $(\mu_2, \varepsilon) \in [0, k) \times (0, k)$ there is a unique point $b_1(\mu_2, \varepsilon)$ such that*

$$(35) \qquad \max_{b>0} \chi(b, \mu_2, \varepsilon) = \chi(b_1(\mu_2, \varepsilon), \mu_2, \varepsilon)$$

and a unique point $b_2 = b_2(\mu_2, \varepsilon)$ such that

$$(36) \qquad \chi(b_2(\mu_2, \varepsilon), \mu_2, \varepsilon) = \chi(0, \mu_2, \varepsilon)\,.$$

Moreover, $b_1(0, \varepsilon) = c_1$ and $b_2(0, \varepsilon) = c_2$, where $\psi(c_1) = \min_{b>0} \psi(b)$ and $\psi(c_2) = \psi(0)$, $c_2 > 0$. If $\beta < 0$ and $(\mu_2, \varepsilon) \in (-k, 0] \times (0, k)$ then the analogous assertion is valid but the equality (35) is with $\min_{b>0}$ on the left hand side.

Lemma 3.3. *The functions $b_1(\mu_2, \varepsilon)$, $b_2(\mu_2, \varepsilon)$ from Lemma 3.2 are smooth.*

Proof. Let us solve the equation

$$(37) \qquad \mathscr{F}(b, \mu_2, \varepsilon) := \frac{\partial\chi(b, \mu_2, \varepsilon)}{\partial b} = 0\,,$$

where χ is the function defined by (31) which is smooth and $\mathscr{F}(b, 0, \varepsilon) = -\frac{8}{7}\beta\psi'(b)\,\varepsilon^2$. Since $\psi'(c_1) = 0$ we have $\mathscr{F}(c_1, 0, 0) = 0$ for all $\varepsilon \in (0, k)$ and $\partial\mathscr{F}(c_1, 0, \varepsilon)/\partial b = -\frac{8}{7}\beta\,\psi''(c_1)\,\varepsilon^2 \neq 0$. The implicit function theorem implies that for every $\varepsilon_0 \in (0, k)$ there is a neighbourhood $U \times V$ of $(\mu_2, \varepsilon) = (0, \varepsilon_0)$ and a unique C^∞-function $d_1\colon U \times V \to R$ such that $d_1(0, \varepsilon_0) = c_1$, $\mathscr{F}(d_1(\mu_2, \varepsilon), \mu_2, \varepsilon) = 0$ for all $(\mu_2, \varepsilon) \in U \times V$. By Lemma 3.1 $d_1(\mu_2, \varepsilon) = b_1(\mu_2, \varepsilon)$ for all $(\mu_2, \varepsilon) \in U \times V$ and this means that $b_1 \in C^\infty$.

Let us solve the equation

$$G(b, \mu_2, \varepsilon) := \chi(b, \mu_2, \varepsilon) - \chi(0, \mu_2, 0) = 0 . \tag{38}$$

Obviously, $G \in C^\infty$, $G(c_2, 0, \varepsilon) = 0$ and $\partial\chi(c_2, 0, \varepsilon)/\partial b = -\frac{8}{7}\beta\, \psi'(c_2)\, \varepsilon^2 \neq 0$. Therefore the implicit function theorem implies that for every $\varepsilon_0 \in (0, k)$ there is a neighbourhood $U \times V$ of $(\mu_2, \varepsilon) = (0, \varepsilon_0)$ and a unique C^∞-function $d_2 : U \times V \to R$ such that $d_2(0, \varepsilon_0) = c_2$, $G(d_2(\mu_2, \varepsilon), \mu_2, \varepsilon) = 0$ for all $(\mu_2, \varepsilon) \in U \times V$. By Lemma 3.1 $d_2(\mu_2, \varepsilon) = b_2(\mu_2, \varepsilon)$ for all $(\mu_2, \varepsilon) \in U \times V$ and therefore $b_2 \in C^\infty$.

As a consequence of Lemma 3.2 and Lemma 3.3 we obtain

Theorem 3.4. *If $\beta > 0$ ($\beta < 0$) then there are numbers $k > 0$, $c_1 > 0$, $c_2 > 0$, C^∞-functions $\chi_{c_i}: [0, k) \times (0, k) \to R$, $i = 1, 2$ ($\chi_{c_i}: (-k, 0] \times (0, k) \to R$) of the form*

$$\chi_{c_1}(\mu_2, \varepsilon) := -P(c_1)\, \varepsilon\mu_2 - \tfrac{8}{7}\beta\, \psi(c_1)\, \varepsilon^2 + O(\varepsilon^2\mu_2^2) , \tag{39}$$

$$\chi_{c_2}(\mu_2, \varepsilon) := -P(c_2)\, \varepsilon\mu_2 - \tfrac{8}{7}\beta\, \psi(c_2)\, \varepsilon^2 + O(\varepsilon^2\mu_2^2) , \tag{40}$$

($\chi_{c_i}(\mu_2, \varepsilon) < k\varepsilon$ for all (μ_2, ε)) and a neighbourhood W of the origin $(x, y) = (0, 0)$ such that for $(\mu_2, \varepsilon) \in [0, k) \times (0, k)$ ($(\mu_2, \varepsilon) \in (-k, 0] \times (0, k)$) the following holds:

(1) *The set W contains all three equilibrium points of the system* (6).
(2) *If $\chi_{c_2}(\mu_2, \varepsilon) < \mu_1 < \chi_{c_1}(\mu_2, \varepsilon)$ ($\chi_{c_1}(\mu_2, \varepsilon) < \mu_1 < \chi_{c_2}(\mu_2, \varepsilon)$) then the system* (6) *has exactly two periodic trajectories surrounding all three equilibium points and lying in W.*
(3) *If $\mu_1 = \chi_{c_1}(\mu_2, \varepsilon)$ or $\mu_1 < \chi_{c_2}(\mu_2, \varepsilon)$ ($\mu_1 > \chi_{c_2}(\mu_2, \varepsilon)$) then the system* (6) *has exactly one periodic trajectory surrounding all three equilibrium points and lying in W.*
(4) *If $\mu_1 > \chi_{c_1}(\mu_2, \varepsilon)$ ($\mu_1 < \chi_{c_1}(\mu_2, \varepsilon)$) then the system* (6) *has no periodic trajectories surrounding all three equilibrium points and lying in W* (see Fig. 2, *where* $L_2 :=$ graph $\chi_{c_1} \subset E$, $L_1 :=$ graph $\chi_{c_2} \subset E$, $E := \{(\mu_1, \mu_2, \varepsilon): |\mu_2| \leqq k, 0 < \varepsilon < k, |\mu_1| < k\varepsilon\}$).

Remark. The case $\beta > 0$, $\mu_2 < 0$ can be obtained from the case $\beta < 0$, $\mu_2 < 0$ by using the transformations $t \to -t$, $y \to -y$, $\mu_1 \to -\mu_1$, $\mu_2 \to -\mu_2$.

Theorem 3.4 provides an information about the number of limit cycles surrounding all three equilibrium points and lying sufficiently close to the homoclinic trajectory appearing on L_1. However, there may exist other limit cycles surrounding all three equilibrium points lying outside the neighbourhood W which are not detectable in our coordinate system defined by (7).

Let us consider the system (6) with $\varepsilon < 0$, i.e.

$$\begin{aligned} \dot{x} &= y , \\ \dot{y} &= (\varepsilon - x^2)\, x + (\mu_1 + \mu_2 x^2 + \beta x^4)\, y + R(x^2, \varepsilon, \mu)\, y . \end{aligned} \tag{41}$$

The origin is a focus and $\mu_1 = 0$ is a bifurcation surface, where the Hopf bifurcation appears. Let us introduce the polar coordinates $x = \varrho \cos \varphi$, $y = \varrho \sin \varphi$. Then the differential equation for ϱ has the form

$$\dot{\varrho} = \varrho[(\varepsilon + 1) \cos \varphi \sin \varphi - \varrho^2 \cos^3 \varphi \sin \varphi + (\mu_1 \sin^2 \varphi + \\ + \varrho^2 \mu_2 \cos^2 \varphi \sin^2 \varphi + \beta \varrho^4 \cos^4 \varphi \sin^2 \varphi) + \tilde{R}(\varrho, \varphi, \varepsilon, \mu)] .$$

After averaging over the interval $(0, 2\pi)$ we obtain the equation

(42) $$\dot{\varrho} = \varrho(\tfrac{1}{2}\mu_1 + \tfrac{1}{8}\mu_2 \varrho^2 + \tfrac{1}{16}\beta \varrho^4) + Q(\varrho^2, \varepsilon, \mu)) ,$$

where $Q \in C^\infty$, $Q(\varrho^2, 0, 0) = o(\varrho^4)$. The number of periodic trajectories of (41) is the same as the number of positive solutions of the equation

(43) $$8\mu_1 + 2\mu_2 \varrho^2 + \beta \varrho^4 + 16 Q(\varrho^2, \varepsilon, \mu) = 0 .$$

One can check that if $\beta > 0$ then we have a bifurcation diagram as in Fig. 3. This result can also be obtained using the formulas for the first and second Lyapunov focus number (see Bautin and Leontovich [2] and also Medveď [22]). If $(\mu_1, \mu_2) \in D_k$ $(k = 0, 1, 2)$ and $\varepsilon < 0$ then the equation (41) has k periodic trajectories. This bifurcation is called the generalized Hopf bifurcation. Thus we obtain a surface G_0 as in Fig. 2 which corresponds to the curve $H := D_0 \cap D_2$. The form of the surface G_0 indicates that it should be possible to extend it to a surface G_1 lying in the half space $\varepsilon > 0$. We do not prove the existence of G_1. This surface is not detectable in the coordinate system defined by (7).

4. PERIODIC TRAJECTORIES SURROUNDING SINGLE EQUILIBRIUM POINTS

Similarly as in Carr [5] one can show that the equation for the values of parameters (δ, ν_1, ν_2) for which the equation (9) has periodic trajectories surrounding single equilibrium points has the form

(44) $$\nu_1 K_0 + \nu_2 K_1 + \delta^2 \beta K_2 + \tilde{W}(\delta, \nu_1, \nu_2) = 0 ,$$

where $\tilde{W} \in C$, $\tilde{W}(\delta, \nu_1, \nu_2) = 0(\delta^4(\nu_1^2 + \nu_2^2))$,

(45) $$K_i = K_i(c) := \int_c^d u^{2i}\, r(u)\, du , \quad i = 0, 1, 2 ,$$

$0 < c < 1$, $c < d$, $r(d) = 0$,

(46) $$r(u) := \left(u^2 - \frac{u^4}{2} + \frac{c^4}{2} - c^2\right)^{1/2} .$$

The implicit function theorem implies that (44) can be written in the same form, where the function $\tilde{W}$ is independent of ν_1, i.e. we obtain the equation

(47) $$\nu_1 = -\frac{K_1}{K_0}\nu_2 - \delta^2 \beta \frac{K_2}{K_0} + W(\delta, \nu_2) ,$$

where $W \in C^\infty$, $W(\delta, \nu_2) = O(\delta^4 \nu_2^2)$ and in the original coordinates μ_1, μ_2, ε we obtain the equation

$$(48)\qquad \mu_1 = \Phi_c(\mu_2, \varepsilon) := R(c)\,\mu_2\varepsilon + S(c)\,\varepsilon^2 + O(\varepsilon^2\mu_2^2),$$

where

$$(49)\qquad R(c) := -\frac{K_1(c)}{K_0(c)}, \quad S(c) := -\frac{K_2(c)}{K_0(c)}.$$

Since $r(d) = 0$ we have

$$0 = \int_c^d \frac{\mathrm{d}}{\mathrm{d}u}\left(u^{2i+1}\, r(u)\right) \mathrm{d}u = (2i+1)\int_c^d u^{2i}\, r(u)\, \mathrm{d}u +$$

$$+ \int_c^d \frac{u^{2i+1}(u - u^3)}{r(u)}\, \mathrm{d}u = (2i+1)\int_c^d u^{2i}\frac{r^2(u)}{r(u)}\, \mathrm{d}u +$$

$$+ \int_c^d \frac{u^{2i+2}}{r(u)}\, \mathrm{d}u - \int_c^d \frac{u^{2i+4}}{r(u)}\, \mathrm{d}u = (2i+1)\int_c^d \frac{u^{2i}}{r(u)}\left(u^2 - \frac{u^4}{2} + \frac{c^4}{2} - c^2\right) \mathrm{d}u +$$

$$+ \int_c^d \frac{u^{2i+2}}{r(u)}\, \mathrm{d}u - \int_c^d \frac{u^{2i+4}}{r(u)}\, \mathrm{d}u = (c^3 - c)^{-1}\Big(2(i+1)\, K'_{i+2} - \frac{2i+3}{2} K'_{i+2} +$$

$$+ \frac{c^4 - 2c^2}{2}(2i+1)\, K'_i\Big).$$

Thus we have obtained the equality

$$(50)\qquad (2i+3)\, K'_{i+2} - 4(i+1)\, K'_{i+1} - (2i+1)(c^4 - 2c^2)\, K'_i = 0, \quad i = 0, 1, \ldots.$$

Now we shall derive the relations between $K_0, K_1, K_2, K'_0, K'_1, K'_2$:

$$K_0 = \int_c^d \frac{r^2(u)}{r(u)}\, \mathrm{d}u = \int_c^d (r(u))^{-1}\left(u^2 - \frac{u^4}{2} + \frac{c^4}{2} - c^2\right) \mathrm{d}u =$$

$$= (c^3 - c)^{-1}\left(K'_1 - \tfrac{1}{2}K'_2 + \frac{c^4 - 2c^2}{2} K'_0\right), \quad \text{i.e.}$$

$$(51)\qquad K_0 = B^{-1}\left(\tfrac{1}{2}AK'_0 + K'_1 - \tfrac{1}{2}K'_2\right),$$

where

$$(52)\qquad A = c^4 - 2c^2, \quad B = c^3 - c.$$

$$K_1 = \textstyle\int_c^d (r(u))^{-1}\, u^2 (r(u))^2\, \mathrm{d}u =$$

$$= \int_c^d (r(u))^{-1} u^2\left(u^2 - \frac{u^4}{2} + \frac{c^4}{2} - c^2\right) \mathrm{d}u =$$

$$= B^{-1}\left(\tfrac{1}{2}AK'_1 + K'_2 - \tfrac{1}{2}K'_3\right), \quad \text{i.e.}$$

$$(53)\qquad K_1 = B^{-1}\left(\tfrac{1}{2}AK'_1 + K'_2 - \tfrac{1}{2}K'_3\right).$$

From (50) we obtain

$$(54)\qquad 5K_3' - 8K_2' - 3AK_1' = 0\,,$$

$$(55)\qquad 3K_2' - 4K_1' - AK_0' = 0\,,$$

and from these equations we have

$$(56)\qquad K_3' = \tfrac{1}{15}(8AK_0' + (32 + 9A)\,K_1')\,.$$

From (51), (54) and (56) we have

$$(57)\qquad 3BK_0 = AK_0' + K_1'$$

and from (53), (54) and (55) it follows that

$$(58)\qquad 15BK_1 = AK_0' + (3A + 4)\,K_1'\,.$$

From the system (57), (58) one can calculate that

$$(59)\qquad K_0' = \frac{B}{A(A+1)}((3A + 4)\,K_0 - 5K_1)\,,$$

$$(60)\qquad K_1' = \frac{B}{A+1}(5K_1 - K_0)\,.$$

Integration by parts yields

$$(61)\qquad 5BK_2 = K_4' - K_3'$$

and from (50) we obtain

$$(62)\qquad K_4' = \tfrac{12}{7}K_3' + \tfrac{5}{7}AK_2'\,,$$

$$K_3' = \tfrac{8}{7}K_2' + \tfrac{3}{5}AK_1'\,.$$

From these equation and (55), (56) we have

$$(63)\qquad K_2 = \frac{1}{105B}((8A + 5)\,AK_0' + (29A + 32)\,K_1')$$

and using the formulas (59), (60) we obtain

$$(64)\qquad K_2 = \frac{1}{35(A+1)}(2(4A^2 + 3A - 2)\,K_0 + 5(7A + 9)\,K_1)\,.$$

This yields

$$(65)\qquad \frac{K_2}{K_0} = \frac{1}{35(A+1)}(2(4A^2 + 3A - 2) + 5(7A + 9)\,Q_0(c))\,,$$

where

$$(66)\qquad Q_0(c) := \frac{K_1(c)}{K_0(c)}\,,\quad A = c^4 - 2c^2\,.$$

Lemma 4. 1 (Carr [5]).

$$Q_0'(c) := \frac{dQ_0(c)}{dc} > 0 \quad for \quad 0 < c < 1 \quad and \quad Q_0(0) = \tfrac{4}{5}.$$

Lemma 4.2.

$$\left(\frac{K_2(c)}{K_0(c)}\right)' > 0 \quad for \quad 0 < c < 1. \tag{67}$$

Proof.

$$\frac{K_2(c)}{K_0(c)} = G_1(A) + G_2(A)\,Q_0(c), \quad \text{where} \quad A = c^4 - 2c^2,$$

$$G_1(A) := \frac{2}{35}\frac{4A^2 + 3A - 2}{A + 1}, \quad G_2(A) := \frac{1}{7}\frac{7A + 9}{A + 1}.$$

$$\left(\frac{K_2(c)}{K_0(c)}\right)' = 4Y(c)\,c(c^2 - 1) + G_2(c^4 - 2c^2)\,Q_0'(c), \tag{68}$$

where

$$Y(c) := \left.\frac{dG_1(A)}{dA}\right|_{A = c^4 - 2c^2} + \left.\frac{dG_2(A)}{dA}\right|_{A = c^4 - 2c^2} Q_0(c). \tag{69}$$

Using Lemma 4.1 and (69) we obtain

$$Y(c) = \frac{2}{35}\frac{4A^2 + 8A + 5}{(A + 1)^2} - \frac{10}{7(A + 1)^2}\frac{4}{5} =$$

$$= \frac{2}{35(A + 1)^2}(4A^2 + 8A - 15) < 0 \quad \text{for all} \quad c \in (0, 1),$$

where $A = A(c) := c^4 - 2c^2$. Therefore $Y(c)\,c(c^2 - 1) > 0$ for all $c \in (0, 1)$. Since $-1 < A(c) < 0$ for $c \in (0, 1)$, $G_2(A(c))$ must be positive and Lemma 4.1 implies $Q_0'(c) > 0$. Therefore from (68) we obtain (67).

As a consequence of Lemma 4.1, Lemma 4.2 we obtain the following lemma.

Lemma 4.3. *Let the functions R and S be defined by* (49). *Then*
(1) $R'(c) < 0$ *for* $c \in (0, 1)$;
(2) *if* $\beta > 0$ $(\beta < 0)$ *then* $S'(c) < 0$ $(S'(c) > 0)$ *for* $c \in (0, 1)$.

Take the function

$$\mu_1 = \Phi_0(\mu, \varepsilon) := R(0)\,\varepsilon\mu_2 + S(0)\,\varepsilon^2 + O(\varepsilon^2\mu_2^2) \tag{70}$$

(see (47)), where $R(0) := -Q_0(0)$, $S(0) := -K_2(0)/K_1(0)$, $Q_0(0) := \lim_{c\to 0^+} Q(c)$ $K_i(c) := \lim_{c\to 0^+} K_i(c)$, $i = 1, 2$. As a consequence of Lemma 4.3 we obtain the following theorem.

Theorem 4.4. *If $\beta > 0$ ($\beta < 0$), $\varepsilon > 0$ then there is a number $k > 0$, a C^∞-function Φ_0: $[0, k) \times (0, k) \to R$ (Φ_0: $(-k, 0] \times (0, k) \to R$) of the form (70) and neighbourhoods W_1, W_2 of the points $(-1, 0)$ and $(0, 1)$, respectively, such that for $(\mu_2, \varepsilon) \in [0, k) \times (0, k)$ ($(\mu_2, \varepsilon) \in [-k, 0) \times (0, k)$) the following holds:*

(1) *The set $W_1 \times W_2$ contains both foci of the system (6).*

(2) *If $\mu_1 > \Phi_0(\mu_2, \varepsilon)$ ($\mu_1 < \Phi_0(\mu_2, \varepsilon)$) then there is no periodic trajectory of (6) surrounding the single foci and lying in $W_1 \times W_2$.*

(3) *If $\mu_1 \leqq \Phi_0(\mu_2, \varepsilon)$ ($\mu_1 \geqq \Phi_0(\mu_2, \varepsilon)$) then there is exactly one periodic trajectory of (6) surrounding each of the single foci and lying in $W_1 \times W_2$.*

(See Fig. 2, where $K :=$ graph Φ_0.)

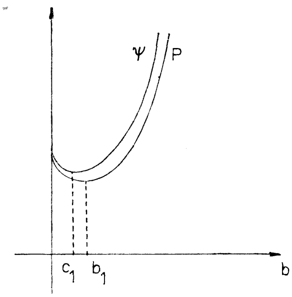

Fig. 1.

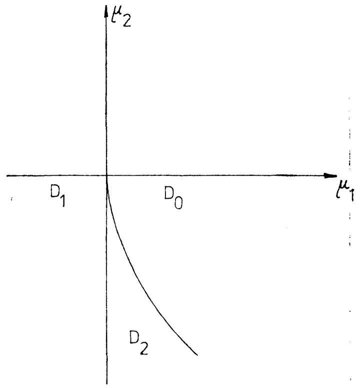

Fig. 2.

Remark. The case $\beta > 0$, $\mu_2 \leqq 0$ can be obtained from the case $\beta < 0$, $\mu_2 \leqq 0$ via the transformations $t \to -t$, $y \to -y$, $\mu_1 \to -\mu_1$, $\mu_2 \to -\mu_2$.

If $c = 0$ then $d = \sqrt{2}$ and thus

$$K_i(0) = \int_0^{\sqrt{2}} u^{2i} \left(u^2 - \frac{u^4}{2}\right)^{1/2} du\,, \quad i = 0, 1, 2\,;$$

$$K_0(0) = \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} u^5 (2 - u^2)^{1/2}\, du = 8\sqrt{2} \int_0^1 v^5 (1 - v^2)^{1/2}\, dv =$$

$$= \left[-8\sqrt{(2)}\,(1 - v^2)^{3/2} \left(\tfrac{1}{7}v^4 + \tfrac{4}{35}v^2 + \tfrac{8}{105}\right)\right]_0^1 = \tfrac{64}{105}\,;$$

$$K_1(0) = \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}} u^3 (2 - u^2)^{1/2}\, du =$$

$$= \frac{1}{\sqrt{2}} \left[\tfrac{1}{5}(2 - u^2)^{5/2} - \tfrac{2}{3}(2 - u^2)^{3/2}\right]_0^{\sqrt{2}} = \tfrac{6}{15}\,.$$

We obtain that

$$\frac{K_1(0)}{K_0(0)} = \frac{3}{5}, \quad \frac{K_2(0)}{K_0(0)} = \frac{32}{35}$$

and thus the function (70) has the form

$$\mu_1 = -\tfrac{3}{5}\varepsilon\mu_2 - \tfrac{32}{35}\beta\varepsilon^2 + O(\varepsilon^2\mu_2^2).$$

We conjecture that in the case $\beta > 0$ the complete bifurcation diagram and bifurcations look like in Fig. 3. We have proved that for $(\varepsilon, \mu_1, \mu_2)$ the system (6) has at

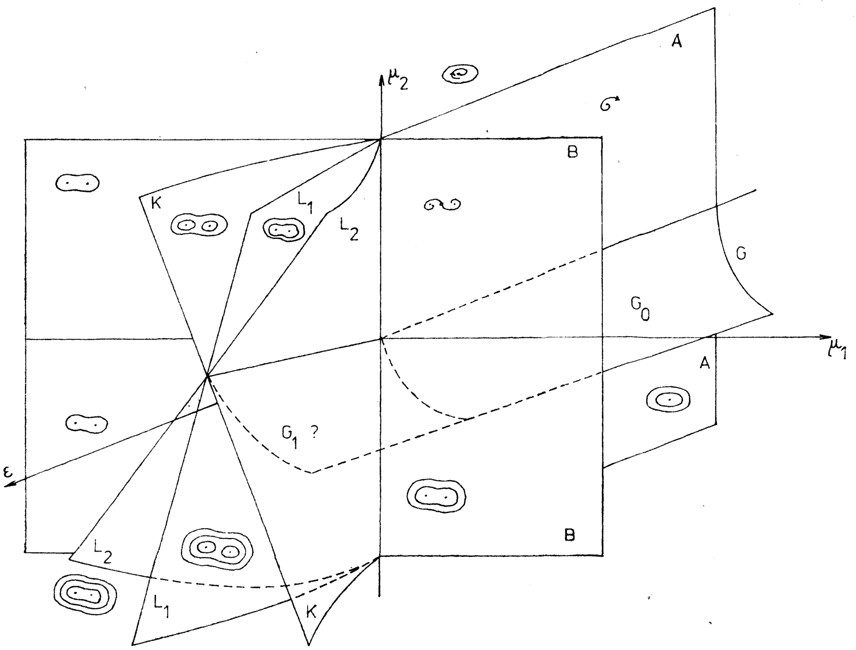

Fig. 3.

least two periodic trajectories between the surfaces L_1 and L_2 surrounding all three equilibrium points. By the hypothetical bifurcation diagram (Fig. 3) there are three such periodic trajectories. One of them should lie outside the set W (see Theorem 3.4). This trajectory should be the same which arises as a result of the generalized Hopf bifurcation, when the parameter crosses the surface A.

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