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# ANALYTIC CONTINUATION BY MEANS OF THE METHODS OF DIVERGENT SERIES

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## 1. INTRODUCTION

Let $f(z)$ be the principal branch of an analytic function regular at the origin. Then its Taylor series

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} f^{(n)}(0)\, z/n! = \sum_{n=0}^{\infty} a_n z^n$$

has nonzero radius of convergence $R_0$ and for $|z| > R_0$ diverges. However very often an summability method $\sigma$ exists yielding a finite sum of this series even for $|z| > R_0$. We say that a summability method $\sigma$ is regular if

$$(2) \qquad \sigma\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n$$

whenver the r.h.s. of (2) converges in the usual sense. In the paper only such regular summability methods are studied which in addition give an analytic continuation of the r.h.s. of (1). This means that whenever the $\sigma$-sum exists then

$$\sigma\left(\sum_{n=0}^{\infty} a_n z^n\right) = f(z)\,.$$

The class of the analytic regular summability methods is quite large and includes well-known summability methods as Lindelöf's, Mittag-Leffler's and Borel's one. For details see [1].

The Lindelöf theorem deals only with tbe series $\sum_{n=0}^{\infty} z^n$, giving its analytic continuation as a limit of the entire functions

$$(3) \qquad \sum_{n=0}^{\infty} A_n(\delta)\, z^n = 1 + \sum_{n=1}^{\infty} z^n \exp\left(-\delta n \ln n\right)$$

for $\delta \to 0_+$ in the whole Mittag-Leffler star of the function $1/(1 - z)$ (see theorem 32 of [1]). The general case is then dealt with the Mittag-Leffler theorem ([1], theorem 135). For the proof of the Mittag-Leffler theorem is decisive uniform convergence of the series (3) for $\delta \to 0_+$ in some star-like regions $\Delta(\eta, R)$ (see also the next section).

Our basic observation is that uniform convergence for $\delta \to 0_+$ holds for some unbounded regions $\Delta(\eta, \infty) = \lim_{R\to\infty} \Delta(\eta, R)$. This fact enables us to generalize the Mittag-Leffler theorem. We cannot prove uniform convergence in unbounded regions in general. We can however obtain an estimation of the approximation of $f(z)$ due to the Mittag-Leffler sum of its Taylor series at the origin when $z$ tends to infinity, ($f(z)$ being from subclass of functions regular at the origin, which are regular in some sector-like domain). This is done in the section 3. The results obtained have very useful applications to the perturbation methods. They can be applied to the resolvent operator in the Rayleigh-Schrödinger perturbation theory [2]. Some results in this direction have been obtained by Reeken using the Borel summability method [3]. In quantum field theory the perturbative series in a coupling constant have mostly zero radius of convergence [4]. However in some other perturbative schemes it is not the case [5] and the results of our paper can be used. Very important example of the perturbation series with nonzero radius of convergence provides so-called planar field theory [6].

## 2. LINDELÖF THEOREM

Denote like [1]

$$
\begin{aligned}
(4) \qquad & g_\delta(z) = \sum_{n=1}^{\infty} z^n \exp\left(-\delta n \ln n\right), \quad \delta > 0\,, \\
& g(z) = z/(1 - z)\,, \\
& \Delta(\eta, R) := \{z = r \exp(\mathrm{i}\theta) \mid r \leqq R,\ 0 < \eta \leqq \theta \leqq 2\pi - \eta\}\,, \\
& I_\delta(z) = \int_C z^u \exp\left(-\delta u \ln u\right)/(\mathrm{e}2^{\pi \mathrm{i} u} - 1)\, \mathrm{d}u\,,
\end{aligned}
$$

where the integral is taken along the contour $C$ depending on an angle $\varphi_0$, $0 < \varphi_0 < \pi/2$, and which is formed by the circular arc

$$(4a) \qquad \{u = \varrho \exp(\mathrm{i}\varphi) \mid \varrho = 1/2,\ |\varphi| \leqq \varphi_0 < \pi/2\}\,,$$

and the two rays

$$(4b) \qquad \{u = \varrho \exp(\mathrm{i}\varphi) \mid \varrho > 1/2,\ \varphi = \pm\varphi_0\}\,.$$

*The function $\ln u$ is taken to be the principal branch of $\mathrm{Ln}\, u$ cut along the negative real axis, i.e. $-\pi < \mathrm{Im} \ln u < \pi$.*

*Let us formulate now the Lindelöf theorem [1].*

**Theorem 1.** *Let $g_\delta(z)$ and $g(z)$ be defined by (4). Then for $\delta \to 0_+$*

$$g_\delta(z) \rightrightarrows g(z)$$

*in an arbitrary compact set not containing any point of the interval $\langle 1, \infty)$.*

Our generalization is given by the following theorem.

**Theorem 2.** *Let* $g_\delta(z)$ *and* $g(z)$ *be as in Theorem* 1. *Then for* $\delta \to 0_+$

$$g_\delta(z) \to g(z)$$

*uniformly in the unbounded region* $\Delta(\eta, \infty) := \lim_{R\to\infty} \Delta(\eta, R)$.

The proof of Theorem 2 is based on the following idea. Let us assume for a moment we know that

$$\lim_{\substack{z\to\infty \\ z\in\Delta(\eta,\infty)}} g_\delta(z) = \lim_{\substack{z\to\infty \\ z\in\Delta(\eta,\infty)}} g(z) = -1\,, \tag{5}$$

uniformly in $\delta$, $0 < \delta < \delta_0$, where $\delta_0$ is sufficiently small. On the other words it means that for each $\varepsilon > 0$ exists such a $R(\varepsilon)$ that

$$|g_\delta(z) - g(z)| < \varepsilon\,, \tag{6}$$

for $z \in \Delta(\eta, \infty) \setminus \Delta(\eta, R(\varepsilon))$, uniformly in $\delta$, $0 < \delta < \delta_0$. On the other hand the Lindelöf theorem ensures existence of such $\delta(\varepsilon)$ that for $0 < \delta < \delta(\varepsilon)$

$$|g_\delta(z) - g(z)| < \varepsilon\,, \tag{7}$$

uniformly in $z \in \Delta(\eta, R(\varepsilon))$. Giving (6) and (7) together one easily obtains the required inequality

$$|g_\delta(z) - g(z)| < \varepsilon$$

uniformly in $z \in \Delta(\eta, \infty)$ for $0 < \delta < \min(\delta_0, \delta(\varepsilon))$. In this manner the proof of Theorem 2 is reduced to the problem of finding the asymptotic behaviour of $g_\delta(z)$ for $z$ tending to infinity in some region $\Delta(\eta, \infty)$, when $\delta$ is sufficiently small. The clue to the problem of finding the asymptotic behaviour of $g_\delta(z)$ consists in the observation that the value of the parameters $\varphi_0$ in (4a, b) can be taken to be $\pi/2$. This is an subject of Lemma 1. In the next the contour of integration $C$ (4a, b) is deformed in order to apply standart methods of asymptotic analysis to particular parts of the so deformed contour. This is done in Lemmas 2–4. Eventually the relation (5) is proved what completes the proof of Theorem 2.

Let us turn to the Lemma 1 now.

**Lemma 1.** *For all* $z \in \Delta(\eta, \infty)$, *where* $\eta$ *is fixed and* $\delta$ *such that* $0 < \delta \leqq \delta_0 < \eta/\pi$

$$g_\delta(z) = \int_C \exp(u \ln z - \delta u \ln u)/(e^{2\pi i u} - 1)\,du\,.$$

*The contour* $C$ *is given by* (4a, b) *where* $\varphi_0$ *is taken to be* $\pi/2$.

Proof. Expressing the integrand of $I_\delta(z)$ (4) in polar coordinates $(\varrho, \varphi)$ and $(r, \theta)$ one obtains

$$\begin{aligned} z^u &= \exp\left[\varrho(\cos\varphi \ln r - \theta \sin\varphi) + i\varrho(\sin\varphi \ln r + \theta\cos\varphi)\right], \\ \exp(-\delta u \ln u) &= \\ &= \exp\left[\delta\varrho(\varphi\sin\varphi - \cos\varphi\ln\varrho) - i\delta\varrho(\sin\varphi\ln\varrho + \varphi\cos\varphi)\right], \\ 1/(e^{2\pi i u} - 1) &= 1/(\exp(2\pi i\varrho\cos\varphi - 2\pi\varrho\sin\varphi) - 1)\,. \end{aligned} \tag{8}$$

Giving all these factors together one easily findes that the integrand of $I_\delta(z)$ is for $\varrho$ tending to infinity and $|\varphi| \leqq \varphi_0 < \pi/2$ uniformly majorised by the term $\exp(-\delta\varrho \ln \varrho \cos \varphi_0)$. Using the Cauchy theorem,

$$g_\delta(z) = I_\delta(z)$$

for all $z$ in the cut complex plane $z$, because the integrand of $I_\delta(z)$ is a meromorphic function $u$ in the whole complex right halfplane with simple poles at the points $u = 1, 2, \ldots$ of the real positive axis. Let us investigate a behaviour of the integrand of $I_\delta(z)$ on the rays (4b) when $\varrho$ tends to infinity. One can always choose such $\varphi_0$ and $\delta_0$ that

$$\sin \varphi_0 > 1/2\,, \quad \operatorname{tg} \varphi_0 > 4 \ln R/\eta\,, \quad \delta_0\varphi_0 < \eta/2\,, \tag{9}$$

where $\eta$ and $R$ are some constant (4). For such $\varphi_0$ and $\delta_0$ we get

$$|z^u \exp(-\delta u \ln u)| = \exp[\varrho(\cos \varphi_0 \ln r - \theta \sin \varphi_0) + \tag{10a}$$
$$+ \delta\varrho(\varphi_0 \sin \varphi_0 - \cos \varphi_0 \ln \varrho) < \exp(\delta_0 \ln 2/2 - \varrho\eta/8)\,;$$

$$|e^{2\pi i u} - 1| < 1\,, \tag{10b}$$

uniformly for $z \in \Delta(\eta, R)$. Thus the integrand of $I_\delta(z)$ is shown to be majorised by the term

$$\exp(\delta_0 \ln 2/2 - \varrho\eta/8) \tag{11}$$

for $\varrho$ tending to infinity and $\varphi = \varphi_0$. Note that the bound remains to be valid also for $\varphi_0 = \pi/2$, because the relations (9) under which is derived remain unchanged for such $\varphi_0$ if $\delta_0$ satisfies the relation

$$0 < \delta_0 < \eta/\pi\,. \tag{12}$$

However, for our purposes this restriction on $\delta_0$ is unimportant. Completely analogously one can treat the case of $\varphi = -\varphi_0$. The estimations like (10a, b) are slightly different in this case but the resulting bound on the behaviour of the integrand of $I_\delta(z)$ when $\varrho$ tends to infinity is the same. Let us take now some $z \in \Delta(\eta, \infty)$. It is always possible to choose constants $R$ and $\varphi_0 = \varphi_0(\eta, R)$ such that $|z| < R$ and $\varphi_0$ satisfies (9). For all $|\varphi|$ from the interval $\langle \varphi_0, \pi/2 \rangle$ the integrand of $I_\delta(z)$ has been shown to be uniformly majorised by (11) for $\varrho$ tending to infinity. Using the Cauchy theorem,

$$\int_C \exp(u \ln z - \delta u \ln u)/(e^{2\pi i u} - 1)\, du = I_\delta(z)\,,$$

and the proof is over.

We emphasize that Lemma 1 enables us to work with a fixed contour $C$ independently on the regions $\Delta(\eta, R)$ where $R$ tends to infinity and $\eta$ is fixed. We can investigate the asymptotic behaviour of $I_\delta(z)$ without any difficulties now. In the next it will be convenient to introduce a function $h(u)$,

$$h(u) = \exp(iu\,\theta - \delta u \ln u)/(e^{2\pi i u} - 1)\,, \tag{13}$$

so that

$$(14)\qquad I_\delta(z) = \int_C e^{u\ln r}\, h(u)\, du\,.$$

According to (13) $h(u)$ has simply poles at the points $u = 1, 2, \ldots$ of the positive real axis. The dependence of $h(u)$ on the parameter $\delta$ will not be written down. The parts of the contour $C$ given by the relation (4b) are now lying on the imaginary axis. It is useful to consider the two parts (4a) and (4b) of the contour $C$ separately. We shall write $I_\delta(z)$ in the form

$$(15)\qquad I_\delta(z) = -\mathrm{i}\int_{1/2}^{\infty} e^{\mathrm{i}\varrho\ln r}\, h[u(\varrho, \pi/2)]\, d\varrho - \\ - \mathrm{i}\int_{1/2}^{\infty} e^{-\mathrm{i}\varrho\ln r}\, h[u(\varrho, -\pi/2)]\, d\varrho + K_\delta(z)\,,$$

where

$$(16)\qquad K_\delta(z) = \int e^{u\ln r}\, h(u)\, du$$

and the integral is taken in positive sense along the semicircle (4a) ($\varphi_0 = \pi/2$). The first two integrals on the r.h.s. of (15) are of the Fourier type and their asymptotic behaviour for $r$ tending to infinity can be easily evaluated, although the Riemann-Lebesgue lemma can not be used directly (the integration runs over the unbounded interval).

**Lemma 2.** *Let $h(u)$ be defined by (13). Then*

$$-\mathrm{i}\int_{1/2}^{\infty} e^{\mathrm{i}\varrho\ln r}\, h[u(\varrho, \pi/2)]\, d\varrho \sim -h(\mathrm{i}/2)\, e^{\mathrm{i}\ln r/2}/\ln r + o(1/\ln r) \quad (r\to\infty)\,;$$
$$-\mathrm{i}\int_{1/2}^{\infty} e^{-\mathrm{i}\varrho\ln r}\, h[u(\varrho, -\pi/2)\, d\varrho \sim -h(-\mathrm{i}/2)\, e^{-\mathrm{i}\ln r/2}/\ln r + o(1/\ln r) \\ (r\to\infty)\,,$$

*uniformly in $\Delta(\eta, \infty)$.*

Proof. In virtue of the theorems on the asymptotic behaviour of the Fourier integrals [7, 8] it suffices only to show that functions $h(\pm\mathrm{i}\varrho)$ are integrabile on the interval $(1/2, \infty)$. We have

$$h[u(\varrho, \pi/2)] = \{\exp(\varrho(\pi\delta/2 - \theta) - \mathrm{i}\delta\varrho\ln\varrho)\}/(\exp(2\pi\varrho) - 1)\,,$$
$$h[u(\varrho, -\pi/2)] = \\ = \{\exp[\varrho(-2\pi + \theta + \pi\delta/2) + \mathrm{i}\delta\varrho\ln\varrho]\}/(1 - \exp(-2\pi\varrho))\,,$$

so that the functions $h(\pm\mathrm{i}\varrho)$ are exponentially decreasing when $\varrho$ tends to infinity, uniformly for $z\in\Delta(\eta, \infty)$, since $\delta$ satisfies the bound (12). On the other hand $h(u)$ is regular on the rays (4b) with $\varphi_0 = \pi/2$, so that integrability of $h(u)$ on theses rays is obvious.

Now it remains to find an asymptotic behaviour of the part $K_\delta(z)$ of $I_\delta(z)$ (15). The method of stationary phase is unapplicable to $K_\delta(z)$ (no stationary point exists for large $r$), so that we will proceed other way. Because of the term $\exp(\varrho\ln r\cos\varphi)$ it is advantageous to deform the semicircle (4a) to the left halfplane $u$. An obstacle appears at the origin due to the branch point of the integrand here. Hence the resulting integration contour is taken as on the figure 1 ($\zeta$ fixed). We denote by $K_l$, where $1 \leqq l \leqq 7$, the integral taken along the part $l$ of the resulting contour. One gets

(17) $$K_{1(7)} = \pm \exp(\pm \mathrm{i} \ln r/2) \int_0^\zeta \mathrm{e}^{-\mathrm{i} l \ln r} h(\pm \mathrm{i}/2 - t)\, \mathrm{d}t\,,$$

(18) $$K_{2(6)} = \mp \mathrm{i} \exp(-\zeta \ln r) \int_0^{1/2} \mathrm{e}^{\pm \mathrm{i} s \ln r} h(\pm \mathrm{i}s - \zeta)\, \mathrm{d}s\,,$$

(19) $$K_4(\varepsilon) = \int_\varepsilon \mathrm{e}^{u \ln r} h(u)\, \mathrm{d}u\,,$$

(20) $$K_3(\varepsilon) + K_5(\varepsilon) = \int_\varepsilon^\zeta \mathrm{e}^{-t \ln r}[h^-(-t) - h^+(-t)]\, \mathrm{d}t\,,$$

where $u = t + \mathrm{i}s$ and $h^+(-t)$ or $h^-(-t)$ are the values of $h(u)$ on the upper or down side of the cut respectively. Now we are in a position to prove the following Lemma.

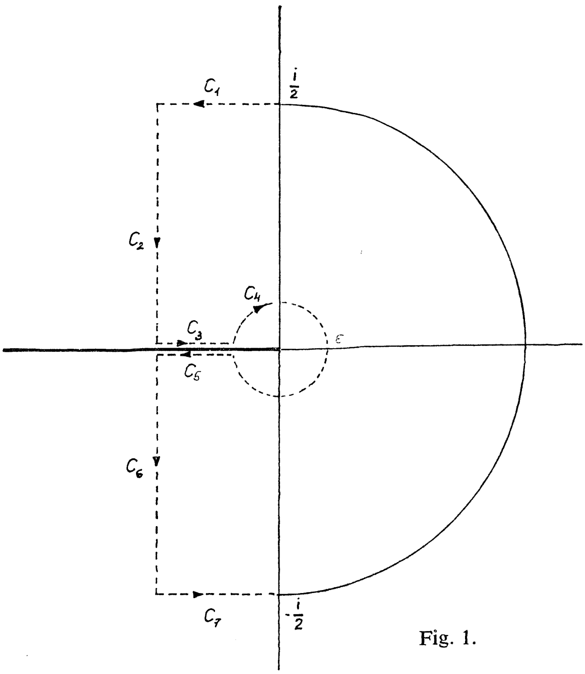


Fig. 1.

**Lemma 3.** $\forall r$ *and* $\forall \delta$ *from the interval* $(0, \delta_0\rangle$, *where* $\delta_0 < \eta/\pi$,

$$\lim_{\varepsilon \to 0} K_3(\varepsilon) + K_5(\varepsilon)$$

*exists and*

$$\lim_{\varepsilon \to 0} (K_3 + K_5)(\varepsilon) = 2\mathrm{i} \int_0^\zeta \{\exp(-tz - \delta t \ln t)\} \sin(\delta t \pi)\, \mathrm{d}t/(1 - \mathrm{e}^{-2\pi \mathrm{i} t})\,.$$

Proof. Firstly

$$h^-(-t) - h^+(-t) = \exp(-\mathrm{i}t\theta)\{\exp[\delta t(\ln t - \mathrm{i}\pi)] -$$
$$- \exp[\delta t(\ln t + \mathrm{i}\pi)]\}/(\exp(-2\pi \mathrm{i}t) - 1) =$$
$$= 2\mathrm{i}[\exp(-\mathrm{i}t\theta - \delta t \ln t)] \sin(\delta t\pi)/(1 - \exp(-2\pi \mathrm{i}t))\,,$$

so that the integrand on the r.h.s. of (20) is a regular function on the whole interval $(0, \zeta)$. Therefore this interval is finite the integral

$$\int_0^\zeta e^{-t\ln r}[h^-(-t) - h^+(-t)]\,dt$$

exists. Hence

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\zeta e^{-t\ln r}[h^-(-t) - h^+(-t)]\,dt = \int_0^\zeta e^{-t\ln r}[h^-(-t) - h^+(-t)]\,dt\,.$$

Let us now estimate the integrals (17, 18, 20) when $r$ tends to infinity. As we have mentioned above we expect the leading term of the asymptotic expansion of $I_\delta(z)$ for $z$ tending to infinity to be of order $O(1)$, so that the following roughly estimates are sufficiently for us.

(21) $$|K_{1(7)}| \leqq \text{cont} \int_0^\zeta e^{-t\ln r}\,dt < \text{const}/\ln r \quad (r \to \infty)\,,$$

(see for instance [9]),

(22) $$K_2 + K_6 = O([\exp(-\zeta \ln r)]/\ln r) \quad (r \to \infty)\,,$$

because these integrals are transparently of the Fourier type. Eventually

(23) $$|K_3 + K_5| \leqq \text{const} \int_0^\zeta e^{-t\ln r}\,dt < \text{const}/\ln r \quad (r \to \infty)\,.$$

All the integrals have been just shown tend to zero for $z$ tending to infinity. The last integral (19) is dealt with the following Lemma.

**Lemma 4.** *For all $z$ in the cut complex plane*

$$\lim_{\varepsilon\to 0} K_4(\varepsilon) = -1\,,$$

*uniformly in $\delta$ such that* $0 < \delta \leqq \delta_0 < \eta/\pi$.

Proof. $K_4(\varepsilon) = \int_4 \exp(u \ln z - \delta u \ln u)/(\exp(2\pi i u) - 1)\,du$. Let us expand $\exp(-\delta u \ln u)$ into the power series

(24) $$\exp(-\delta u \ln u) = \sum_{n=0}^{\infty} (-\delta u \ln u)^n/n!\,.$$

Obviously $\lim_{|u|\to 0} u \ln u = 0$, so that (24) converges for all $u$ from the cut complex plane. This convergence is uniform on each compact set of this plane. Specially for any finite positive $M > 0$ and $|u| \leqq M$ the relation

(25) $$K_4(\varepsilon) = \sum_{n=0}^{\infty} (-\delta)^n/n! \int_4 e^{u\ln z}(u \ln u)^n/(e^{2\pi i u} - 1)\,du$$

holds. Now we wish to prove uniform convergence of the r.h.s. of (25) in $\varepsilon$, for $0 < \varepsilon < M$ and $M$ sufficiently small. In other words we want to prove that

$$\lim_{\varepsilon\to 0} K_4(\varepsilon) = \sum_{n=0}^{\infty} (-\delta)^n/n! \lim_{\varepsilon\to 0} \int_4 e^{u\ln z}(u \ln u)^n/(e^{2\pi i u} - 1)\,du\,.$$

Let us consider the individual integrals on the r.h.s. of (25).

a) $n = 0$: In this case the integral can be immediately performed and the result is

$$\int_4 \exp(u \ln z)/(e^{2\pi i u} - 1)\, du = -2\pi i \operatorname{Res}(e^{u \ln z}/(e^{2\pi i u} - 1); 0) = -1,$$

because of the simple pole of the integrand at the origin.

b) $n = 1$:

$$\int_4 e^{u \ln z} u \ln u/(e^{2\pi i u} - 1)\, du = 1/(2\pi i) \int_4 \ln u\, du + \tag{26}$$
$$+ \int_4 [e^{u \ln z}/(e^{2\pi i u} - 1) - 1/(2\pi i u)]\, u \ln u\, du.$$

The first integral on the r.h.s. of (26) can be also performed and the result is

$$(1/(2\pi i)) \int_4 \ln u\, du = (1/(2\pi i))(u \ln u - u)\Big|_{u = e^{i}}^{u = e^{-i}} = -\varepsilon.$$

The second one can be majorised in the following manner for $M < 1/e$:

$$|u \ln u| \leqq M(|\ln M| + \pi),$$
$$|e^{u \ln z}/(e^{2\pi i u} - 1) - 1/(2\pi i u)| \leqq N(z),$$

where $N(z)$ depends on $z$ only, so that

$$\int_4 [e^{u \ln z}/(e^{2iu} - 1) - 1/(2\pi i u)]\, u \ln u\, du \leqq 2\pi\varepsilon\, N(z)\, M(|\ln M| + \pi).$$

This means that

1. $\lim_{\varepsilon \to 0} \int_4 e^{u \ln z}\, u \ln u/(e^{2\pi i u} - 1)\, du = 0,$
2. $|\int_4 e^{u \ln z} u \ln u/(e^{2\pi i u} - 1)\, du| \leqq 2\pi M^2 N(z)(|\ln M| + \pi) + M <$
   $< M(|\ln M| + \pi)\, \tilde{N}(z).$

c) $n > 1$: Persuing the case b) of $n = 1$ step by step one obtains

$$\int_4 e^{u \ln z}(u \ln u)^n/(e^{2\pi i u} - 1)\, du = (1/(2\pi i)) \int_4 u^{n-1} \ln u\, du +$$
$$+ \int_4 [e^{u \ln z}/(e^{2\pi i u} - 1) - 1/(2\pi i u)]\, (u \ln u)^n\, du,$$
$$|1/(2\pi i) \int_4 u^{n-1} \ln u\, du| \leqq \varepsilon M^{n-1}(|\ln M| + \pi)^n,$$
$$|\int_4 [e^{u \ln z}/(e^{2\pi i u} - 1) - 1/(2\pi i u)]\, (u \ln u)^n\, du| \leqq$$
$$\leqq 2\pi\varepsilon\, N(z)\, M^n(|\ln M| + \pi)^n,$$

i.e.

1) $\lim_{\varepsilon \to 0} \int_4 [e^{u \ln z}/(e^{2\pi i u} - 1)]\, (u \ln u)^n\, du = 0,$

2. $|\int_4 e^{u \ln z}(u \ln u)^n/(e^{2\pi i u} - 1)\, du| \leqq M^n(|\ln M| + \pi)^n (2\pi M\, N(z) + 1) =$
   $= \tilde{N}(z)\, M^n(|\ln M| + \pi)^n$

uniformly in $\varepsilon \in (0, M\rangle$.

Giving all the cases $n = 0$, $n = 1$ and $n > 1$ together we are led to the following assertions:

1. The series on the r.h.s. of (25) is absolutely majorised by the convergent series

$$\tilde{N}(z) \sum_{n=0}^{\infty} \delta^n/n! \left(|\ln M| + \pi\right)^n M^n ,$$

uniformly for $\varepsilon \in (0, M\rangle$, where $M < 1/e$;

2. $\lim_{\varepsilon \to 0} K_4(\varepsilon) = \sum_{n=0}^{\infty} (-\delta)^n/n! \lim_{\varepsilon \to 0} \int_4 e^{u \ln z} (u \ln u)^n / (e^{2\pi i u} - 1) \, du = -1$,

q.e.d.

Lemma 4 complete the proof of Theorem 2, because from Lemmas 1–3 we already know that

$$g_\delta(z) = K_4(\varepsilon) + O(1/\ln r) \quad (z \to \infty, \; z \in \Delta(\eta, \infty)) .$$

Using Lemma 4 we have that

$$g(z) = -1 + O(1/\ln r) \quad (z \to \infty, \; z/\Delta(\eta, \infty)) ,$$

what justifies the relation (5).

## 3. MITTAG-LEFFLER THEOREM

As it has been mentioned above the Mittag-Leffler theorem is a version of the Lindelöf theorem for the general case [1].

**Theorem 3** (Mittag-Leffler). *Suppose that*

1. $\sum_{n=0}^{\infty} A_n(\delta) z^n$ *is an entire function of z for every positive* $\delta$;
2. *that* $\varphi_\delta(z) := \sum_{n=0}^{\infty} A_n(\delta) z^n \to 1/(1 - z)$ *when* $\delta \to 0_+$, *uniformly in any closed and bounded domain containing no point of the line* $\langle 1, \infty)$;
3. *that* $f(z)$ *is the principal branch of an analytic function regular at the origin and represented there by the Taylor series*

$$\sum_{n=0}^{\infty} a_n z^n , \tag{27}$$

*with nonzero radius of convergence* $R_0$.

*Then*

$$\sum_{n=0}^{\infty} A_n(\delta) a_n z^n \to f(z) \tag{28}$$

*when* $\delta \to 0_+$, *uniformly in an arbitrary compact set* $\Delta$ *interior to the Mittag-Leffler star of* $f(z)$.

We remember that the Mittag-Leffler star of $f(z)$ means the domain obtaining by drawing rays through zero to every singular point of $f(z)$ and removing from the plane the parts of the rays beyond the singular points. In the next we will deal only with representation of $A_n(\delta)$ as given by the Lindelöf theorem. Another possible representations of $A_n(\delta)$ are given in [1] and [10].

Using Theorem 2 we are able to give the following generalization of Theorem 3:

**Theorem 4.** *Let $f(z)$ satisfies the hypothese 3 of Theorem 3 and, in additon, is regular in a region $Q$ (see fig. 2) with the boundary given by*

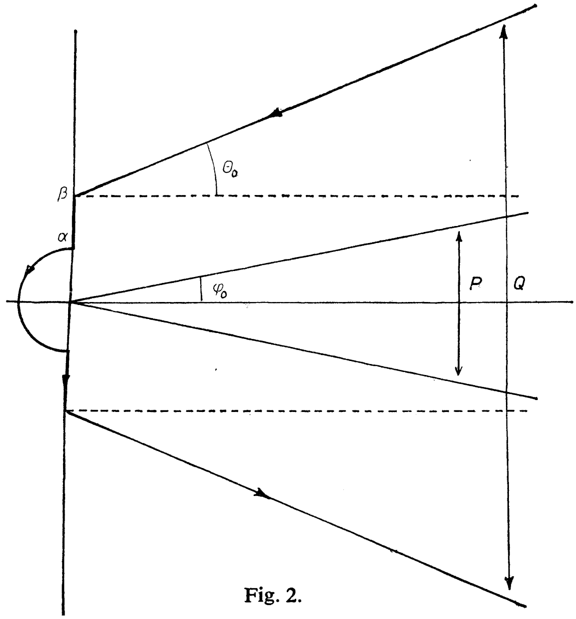


Fig. 2.

i) *two abcissas* $\mathrm{i}(\alpha, \beta)$ *and* $-\mathrm{i}(\alpha, \beta)$, *where* $0 < \alpha < R_0$ *and* $\alpha \leqq \beta$;
ii) *the semicircle* $\{z = \alpha \exp(\mathrm{i}\theta) \mid \pi/2 \leqq \theta \leqq 3\pi/2\}$;
iii) *two rays* $\{z = x + \mathrm{i}y \mid y = \pm\beta \pm (\mathrm{tg}\,\theta_0) \,.\, x,\ 0 < \theta_0 < \pi/2\}$.

*Let*

$$P = \{z = r\exp(\mathrm{i}\varphi) \mid |\varphi| \leqq \varphi_0 < \theta_0\}\,,$$

$$P(r) := P \cap \{z \mid |z| \leqq r\}\,, \quad Q(r) := Q \cap \{z \mid |z| \leqq r\}\,.$$

*Then for each $\varepsilon > 0$ exists such a $\delta(\varepsilon)$ that for $0 < \delta \leqq \delta(\varepsilon)$*

$$|f(z) - \sum_{n=0}^{\infty} A_n(\delta)\, a_n z^n| \leqq \varepsilon/2\pi \int_{\partial Q(\gamma r)} |f(u)/u|\, |\mathrm{d}u|\,, \tag{29}$$

*uniformly in $z \in P(r)$, where $\gamma$ is some constant, $\gamma > 1$.*

Before the proof of the theorem let us note the following.

Remark 1. Under the assumption of the theorem the integral on the r.h.s. of (29) converges for every finite $r$, what means that the series on the l.h.s. of (28) converges to $f(z)$ uniformly in the region $P(r)$ as should be due to the Mittag-Leffler theorem.

Nontrivial case is whenever the integral on the r.h.s. of (29) converges when taken along the boundary of the region $Q$, because its convergence justifies uniform convergence of the l.h.s. of (28) to $f(z)$ in the unbounded region $P$.

Proof. Under the hypotheses of the theorem for $z$ interior to $P(r)$ holds

$$f(z) = 1/(2\pi i) \int_{\partial Q(\gamma r)} [f(u)/u] [1/(1 - z/u)] \, du \,. \tag{30}$$

Let $\delta > 0$ and denote by $\omega(u)$ the function

$$\omega(u) := 1 + \sum_{n=1}^{\infty} \exp(-\delta n \ln n) \, u^n - 1/(1 - u) \,. \tag{31}$$

This function is regular in complex plane $u$ except for the point $u = 1$, where it has a simple pole, and $u = \infty$, where it has an essential singularity. For each $r > 0$ and $u \in \partial Q(\gamma r)$, $z' \in P(r)$

$$z'/u = r'/\varrho \exp i(\theta - \varphi) \in \tilde{\Delta} := \Delta(\eta, \infty) \cup \{z | |z| \leqq 1/\gamma\} \,,$$

where $\eta = \theta_0 - \varphi_0$ and $\Delta(\eta, \infty)$ is defined in the same way as in the section 2. In other words for each $\varepsilon > 0$ one can choose such a $\delta(\varepsilon)$ that for any $r > 0$ and $u \in \partial Q(\gamma r)$, $z' \in P(r)$

$$0 \leqq |\omega(z'/u)| < \varepsilon \,, \tag{32}$$

for all $\delta$ such that $0 < \delta \leqq \delta(\varepsilon)$. On the other hand from (30–31) if follows that

$$\begin{aligned} f(z) = {} & (1/(2\pi i)) \int_{\partial Q(\gamma r)} (f(u)/u) \, du \, + \\ & + (1/(2\pi i)) \sum_{n=1}^{\infty} \exp(-\delta n \ln n) \, z^n \int_{\partial Q(\gamma r)} (f(u)/u^{n+1}) \, du \, - \\ & - (1/(2\pi i)) \int_{\partial Q(\gamma r)} (\omega(z/u) f(u)/u) \, du \,. \end{aligned} \tag{33}$$

Performing the first two integrals one obtains

$$\begin{aligned} f(z) = f(0) & + \sum_{n=1}^{\infty} \exp(-\delta n \ln n) \, a_n z^n \, - \\ & - (1/(2\pi i)) \int_{\partial Q(\gamma r)} (\omega(z/u) f(u)/u) \, du \,, \end{aligned}$$

or

$$f(z) - \sum_{n=0}^{\infty} A_n(\tilde{\varphi}) \, a_n z^n = (1/(2\pi i)) \int_{\partial Q(\gamma r)} (\omega(z/u) f(u)/u) \, du$$

respectively. From above we know that (32) holds uniformly for $z \in P(r)$ if $u \in \partial Q(\gamma r)$, so that

$$\left| f(z) - \sum_{n=0}^{\infty} A_n(\tilde{\varphi}) \, a_n z^n \right| \leqq (\varepsilon/2\pi) \int_{\partial Q(\gamma r)} (f(u)/u) \, du$$

q.e.d.

Note that the contour along $Q(\gamma r)$ can not be deformed without change of (32).

## 4. CONCLUSION

As we have said above the results obtained can be applied for example to the Rayleigh-Schrödinger perturbation theory. Let $H_0$ be a closed linear operator acting in a Banach space and let $V$ be another closed linear operator acting in this space, which is relatively bounded with respect to $H_0$, with relative bound equal to zero. We will be interested in the resolvent operator $R_\lambda(z) = (H_0 + \lambda V - z)^{-1}$ defined for $z$ not in the spectrum of $H = H_0 + \lambda V$. Usual perturbation theory starts from the identity

$$R_\lambda(z) = R_0(z) \left[1 + \lambda V R_0(z)\right]^{-1} .$$

By our assumptions $V R_0(z)$ is a bounded operator, so that the series

$$[1 + \lambda V R_0(z)]^{-1} = \sum_{n=0}^{\infty} (-\lambda)^n \left[V R_0(z)\right]^n \tag{34}$$

will converge in norm for $\|\lambda V R_0(z)\| < 1$. The role of our results and in general of analytic regular summability methods is now obvious. They can immediately provide analytic continuation of the r.h.s. of (34) and in such way enable us to study the resolvent operator $R_\lambda(z)$ for larger domain of $z$ (see also [11]).

Having at our disposal an information about the position of singularities of $f(z)$ (27) more strong results could be achieved using a conformal mapping. This depends, however, on a particular case. Another problem is to compare the numerical efficiency of the methods described here with the methods of Padé approximants becoming very popular in physics [12]. As it has been proved the Mittag-Leffler summability method can at best sum the series on the r.h.s. of (27) uniformly on some unbounded ray only if $f(z)$ is regular in a larger sector-like region. In [13] the analytic momentum summability methods are studied. A summability method have been constructed giving analytic continuation of the r.h.s. of (27) due an absolute convergent integral on each ray lying in the Mittag-Leffler star of $f(z)$. It is shown that such methods are deeply conected with entire functions tending to zero in every radial direction except for one.

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