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ON COMPLETE α-IDEALS IN SEMIGROUPS

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By X^* we denote the free monoid over an alphabet X. Let S be a semigroup. By $\mathscr{P}(S)$ we denote the semigroup of all subset of S under set product with the unity $\emptyset$. For $\alpha \in \{0, 1\}^*$ we shall define $f_\alpha^S \colon \mathscr{P}(S) \to \mathscr{P}(S)$ as follows: $f_\alpha^S(A) = A_1 A_2 \ldots A_n$ if $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$, $\alpha_i \in \{0, 1\}$, where

$$A_i = \begin{cases} A & \text{fot} \quad \alpha_i = 1 \\ S & \text{for} \quad \alpha_i = 0 \end{cases}$$

We put $\Lambda = \{0, 1\}^* \setminus \{1\}^*$ and $f_\alpha := f_\alpha^S$. Let $\alpha \in \Lambda$. A subsemigroup M of a semigroup S is called an α-*ideal* of S if $f_\alpha(M) \subseteq M$ (see [6]). If, especially $f_\alpha(M) = M$, then we say that M is a *complete* α-ideal of S.

In this note we shall characterize the semigroups in which every α-ideal is complete.

For the undefined notions and notations we refer to [1].

It is well known (see [6]) the following

Lemma 1. *Let $\alpha, \beta \in \Lambda$ and A, B non-empty subsets of a semigroup S.*

(i) *If $A \subseteq B$ then $f_\alpha(A) \subseteq f_\alpha(B)$;*

(ii) $f_{\alpha\beta}(A) = f_\alpha(A) f_\beta(A)$;

(iii) *$A f_\alpha(A) \subseteq f_\alpha(A)$ and $f_\alpha(A) A \subseteq f_\alpha(A)$;*

(iv) $f_\alpha(A) f_\alpha(A) \subseteq f_\alpha(A)$;

(v) $f_\alpha(A \cup f_\alpha(A)) \subseteq f_\alpha(A)$.

The following lemma is an extension of Lemma 1 (v).

Lemma 2. *Let $\alpha \in \Lambda$ and k a positive integer. If A is a non-empty subset of a semigroup S, then*

$$f_\alpha(A \cup A^2 \cup \ldots \cup A^k \cup f_\alpha(A)) \subseteq f_\alpha(A)\,.$$

Proof. At first we prove that

$$(1) \qquad f_\alpha(A \cup A^2 \cup \ldots \cup A^k) \subseteq f_\alpha(A)$$

by induction on the length $l(\alpha)$ of α. It clear that the result is true for $l(\alpha) = 1$. Assume now that $l(\alpha) \geqq 2$ and the result holds for every word β such that $l(\beta) =$

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$= l(\alpha) - 1$.

Case 1: $\alpha = 1\beta$. Then $\beta \in \Lambda$ and $f_\beta(A \cup \ldots \cup A^k) \subseteq f_\beta(A)$. Using Lemma 1 (ii), (iii)

$$\begin{aligned} f_\alpha(A \cup A^2 \cup \ldots \cup A^k) &= (A \cup A^2 \cup \ldots \cup A^k) f_\beta(A \cup A^2 \cup \ldots \cup A^k) \subseteq \\ &\subseteq (A \cup A^2 \cup \ldots \cup A^k) f_\beta(A) \subseteq A f_\beta(A) \cup A^2 f_\beta(A) \cup \ldots \cup A^k f_\beta(A) = \\ &= A f_\beta(A) \cup A(A f_\beta(A)) \cup \ldots \cup A^{k-1}(A f_\beta(A)) \subseteq \ldots \subseteq A f_\beta(A) = \\ &= f_{1\beta}(A) = f_\alpha(A) . \end{aligned}$$

Case 2: $\alpha = 0\beta$. If $\beta \in \Lambda$,

$$f_\alpha(A \cup A^2 \cup \ldots \cup A^k) = S f_\beta(A \cup A^2 \cup \ldots \cup A^k) \subseteq S f_\alpha(A) = f_\alpha(A) .$$

If $\beta \notin \Lambda$, $\alpha = \gamma 1$ with $\gamma \in \Lambda$. This is dual to case 1.

Now, by Lemma 1(i), (v) and by (1),

$$\begin{aligned} & f_\alpha(A \cup A^2 \cup \ldots \cup A^k \cup f_\alpha(A)) \subseteq \\ & \subseteq f_\alpha(A \cup A^2 \cup \ldots \cup A^k \cup f_\alpha(A \cup A^2 \cup \ldots \cup A^k)) \subseteq \\ & \subseteq f_\alpha(A \cup A^2 \cup \ldots \cup A^k) \subseteq f(A) . \end{aligned}$$

Theorem 3. *Let A be a non-empty subset of a semigroup S. Then $A \cup A^2 \cup \ldots \cup A^{l(\alpha)-1} \cup f_\alpha(A)$ is the smallest α-ideal of S containing A, where $l(\alpha)$ is the length of α.*

Proof. Since $(A \cup A^2 \cup \ldots \cup A^{l(\alpha)-1} \cup f_\alpha(A))\ (A \cup A^2 \cup \ldots \cup A^{l(\alpha)-1} f_\alpha(A)) \subseteq$
$\subseteq A^2 \cup A^3 \cup \ldots \cup A^{l(\alpha)-1} \cup A^{l(\alpha)} \cup A f_\alpha(A) \cup (A^{l(\alpha)+1} \cup A^2 f(A)) \cup \ldots$
$\ldots \cup (A^{2l(\alpha)-2} \cup A^{l(\alpha)-1} f_\alpha(A)) \cup (f_\alpha(A) A \cup \ldots \cup f_\alpha(A) A^{l(\alpha)-1}) \cup f_\alpha(A) f_\alpha(A) \subseteq$
$\subseteq A^2 \cup A^3 \cup \ldots \cup A^{l(\alpha)-1} \cup \ldots \cup A^{2l(\alpha)-2} \cup f_\alpha(A) \subseteq A^2 \cup \ldots \cup A^{l(\alpha)-1} \cup f_\alpha(A)$
we see that $A \cup A^2 \cup \ldots \cup A^{l(\alpha)-1} \cup f_\alpha(A)$ is a subsemigroup of S containing A. By Lemma 2 it is an α-ideal of S. Moreover, let B be an α-ideal of S containing A. Then $A \cup \ldots \cup A^{l(\alpha)-1} \cup f_\alpha(A) \subseteq B \cup f_\alpha(B) \subseteq B$.

If A is a non-empty subset of a semigroup S then $A \cup A^2 \cup \ldots \cup A^{l(\alpha)-1} \cup f_\alpha(A)$ is called the *α-ideal generated by A* and denoted by $(A)_\alpha$, where $l(\alpha)$ is the length of α.

Let $\alpha \in \Lambda$. A semigroup S is called *α-semigroup* if every α-ideal of S is complete.

Theorem 4. *Let $\alpha \in \Lambda$. A semigroup S is an α-semigroup if and only if $a \in f_\alpha(\{a\})$ for all $a \in S$.*

Proof. Suppose that every α-ideal of S is complete. Let $a \in S$. Then, by Theorem 3 and Lemma 2, $a \in \{a\} \cup \ldots \cup \{a^{l(\alpha)-1}\} \cup f_\alpha(\{a\}) = f_\alpha(\{a\} \cup \ldots \cup \{a^{l(\alpha)-1}\} \cup f_\alpha(\{a\})) \subseteq$ $\subseteq f_\alpha(\{a\})$.

Conversely, if A is an α-ideal of S. Then, by Lemma 1(i), $a \in f_\alpha(\{a\}) \subseteq f_\alpha(A)$ for all $a \in A$. Hence A is a complete α-ideal of S.

Remarks. 1) If $\alpha = 10$ then any α-ideal of S is a left ideal of S. By Theorem 4, every left ideal of S is complete iff $a \in aS$ for all $a \in S$ (see [2]).

2) If $\alpha = 101$ then any α-ideal of S is a bi-ideal of S. By Theorem 4, every bi-ideal of S is complete iff S is a regular semigroup (see [7]).

3) If $\alpha = 1011$ then any α-ideal of S is a (1,2)-ideal of S (see [3]). By Theorem 4, every (1,2)-ideal of S is complete iff S is a completely regular semigroup (see [4]).

Now, we remark that in general the intersection of complete α-ideals of a semigroup S need not be complete (see [2]).

Theorem 5. *Let $\alpha \in \Lambda$. If A, B are complete α-ideals of a semigroup S, then $(A \cup B)_\alpha$ is a complete α-ideal of S.*

Proof. By Lemma 1(i), $A \cup B = f_\alpha(A) \cup f_\beta(B) \subseteq f_\alpha(A \cup B)$. Thus, by Lemma 1(iii), $(A \cup B)^k \subseteq f_\alpha(A \cup B)$ for all positive integers k. Hence $(A \cup B)_\alpha = (A \cup B) \cup \cup (A \cup B)^2 \cup \ldots \cup (A \cup B)^{l(\alpha)-1} \cup f_\alpha(A \cup B) \subseteq f_\alpha(A \cup B) \subseteq f_\alpha((A \cup B)_\alpha)$.

It is clear that the product of two α-ideals of a semigroup, in general, will not be an α-ideal (see [6]). We will prove that the product of two complete α-ideals of a semigroup is an α-ideal when $\alpha \in \Lambda$ and α containing *a* 1. But we first need some auxiliary statements.

Let $\alpha \in \Lambda$. If $\alpha = 0^n$ for some n positive integer, we put $\alpha^* = 0$. If $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ and there exists $j > 1$ such that $\alpha_j \neq \alpha_1$ we put

$$\alpha^* = \begin{cases} \alpha_1\alpha_j\alpha_n & \text{if} \quad \alpha_n \neq \alpha_j \\ \alpha_1\alpha_j & \text{if} \quad \alpha_n = \alpha_j \end{cases}$$

We remark that $\alpha^* \in \{0, 10, 01, 101, 010\}$.

Lemma 6. *Let $\alpha \in \Lambda$ and let A, B be complete α-ideals of a semigroup S. Then $f_\alpha(AB) \subseteq f_{\alpha^*}(AB)$.*

Proof. Let $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$. If $\alpha_1 = 1$ then there exists $j > 1$ such that $\alpha_j = 0$. Thus $f_\alpha(AB) \subseteq ABf_{\alpha_2 \ldots \alpha_{j-1}}(AB)\, S f_{\alpha_{j+1} \ldots \alpha_n}(AB) \subseteq AB\, S f_{\alpha_n}(AB)$.

If $\alpha_n = 0$ then $f_\alpha(AB) \subseteq ABS = f_{\alpha^*}(AB)$. If $\alpha_n = 1$ then $f_\alpha(AB) \subseteq ABSAB = = f_{\alpha^*}(AB)$.

Now, we suppose that $\alpha_1 = 0$. If $\alpha_j = 0$ for all $j \in \{2, \ldots, n\}$ then evidently $f_\alpha(AB) \subseteq S = f_{\alpha^*}(AB)$. If $\alpha_n = 1$ for some $h > 1$, then $f_\alpha(AB) \subseteq SABf_{\alpha_{h+1} \ldots \alpha_n}(AB) \subseteq \subseteq SABSf_{\alpha_n}(AB)$ if there is r such that $h + 1 \leqq r < n$ and $\alpha_r = 0$ or $f_\alpha(AB) \subseteq \subseteq SABf_{\alpha_n}(AB)$. Thus $f_\alpha(AB) \subseteq SABS = f_{\alpha^*}(AB)$ if $\alpha_n = 0$ and $f_\alpha(AB) \subseteq SABAB \subseteq \subseteq SAB = f_{\alpha^*}(AB)$ if $\alpha_n = 1$.

Lemma 7. *Let $\alpha \in \Lambda$, $\alpha^* \neq 0$ and let A, B be complete α-ideals of a semigroup S. Then $f_{\alpha^*}(AB) \subseteq AB$.*

Proof. If $\alpha^* = 01$ then $\alpha = 0\alpha'$ with $\alpha' \in \{0, 1\}^*$. Thus $f_{\alpha^*}(AB) = SAB = = S f_\alpha(A)\, B = SS f_{\alpha'}(A)\, B \subseteq S f_{\alpha'}(A)\, B \subseteq f_\alpha(A)\, B = AB$. The case $\alpha^* = 10$ is dual to the preceding case. If $\alpha^* = 010$, then $f_{\alpha^*}(AB) = SABS = S f_\alpha(A) f_\alpha(B)\, S \subseteq \subseteq f_\alpha(A) f_\alpha(B) = AB$. If $\alpha^* = 101$, then $f_{\alpha^*}(AB) = ABSAB \subseteq ABSB =$
$= A f_\alpha(B)\, S f_\alpha(B) \subseteq A f_{\alpha 0 \alpha}(B) \subseteq A f_{\alpha_1}(B) f_{\alpha_2}(B) \ldots f_{\alpha_j}(B) \ldots f_{\alpha_n}(B)\, S f_{\alpha_1}(B) \ldots$
$\ldots f_{\alpha_j}(B) \ldots f_{\alpha_n}(B) \subseteq A f_\alpha(B) = AB$, where $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$.

Lemma 8. *Let $\alpha \in \Lambda$, $\alpha^* \neq 0$ and let A be a complete α-ideal of a semigroup S. Then $f_{\alpha^*}(A) = A$.*

Proof. If $\alpha^* = 010$ then $f_{\alpha^*}(A) = SAS = Sf_\alpha(A)S \subseteq f_\alpha(A)$. The cases $\alpha^* \in \{01, 10\}$ are similar. If $\alpha^* = 101$ then $f_{\alpha^*}(A) = ASA = f_\alpha(A)Sf_\alpha(A) = f_{\alpha 0 \alpha}(A) \subseteq f_\alpha(A)$.

Moreover, by the proof of Lemma 6, $f_\alpha(A) \subseteq f_{\alpha^*}(A)$. Thus $A = f_{\alpha^*}(A)$.

Theorem 9. *Let $\alpha \in \Lambda$, $\alpha^* \neq 0$. The product of two complete α-ideals A, B of a semigroup S is an α-ideal.*

Proof. We prove that AB is a subsemigroup of S. If $\alpha^* = 010$ then, by Lemma 8, $ABAB = ABASBS \subseteq ASASBS \subseteq ASBS \subseteq AB$. We proceed similarly if $\alpha^* \in \{01, 10, 101\}$.

Furthermore, by Lemma 6 and Lemma 7, $f_\alpha(AB) \subseteq AB$. Thus AB is an α-ideal of S.

We remark that if $\alpha \in \Lambda$ and $\alpha^* \neq 0$, in general, the product of two complete α-ideals will not be complete and if $\alpha^* = 0$, the product of two complete α-ideals need not be an α-ideal.

Let $\alpha \in \Lambda$, $\alpha^* \neq 0$ and let S be an α-semigroup. Let F_α be the set of all α-ideals of S. By Theorem 9 F_α is a semigroup respect to the set product.

Theorem 10. *Let $\alpha \in \Lambda$, $\alpha^* \neq 0$. A semigroup S is an α-semigroup if and only if F_α is an α-semigroup.*

Proof. Let S be an α-semigroup and let A be an element of F_α. Then $A = f_\alpha(A) \in f_\alpha^{F_\alpha}(\{A\})$. Thus, by Theorem 4, F_α is an α-semigroup. Conversely, if $A \in F_\alpha$, by Theorem 4, $A \in f_\alpha^{F_\alpha}(\{A\})$. Hence $A \subseteq f_\alpha(A)$.

Remarks. 1) If $\alpha = 101$, then by Theorem 10, S is regular iff the semigroup of all bi-ideal of S is regular (see [7]).

2) If $\alpha = 1011$, then by Theorem 10, S is completely regular iff the semigroup of all (1,2)-ideals of S is completely regular (see [5]).

References

[1] *Clifford, A. H.* and *G. B. Preston:* The algebraic theory of semigroups, vol. I, Amer. Math. Soc., Providence, 1961.

[2] *Fabrici, I.:* On complete ideals in semigroups, Mat. Časopis Sloven. Akad. Vied *18* (1968), 34—39.

[3] *Lajos, S.:* Generalized ideals in semigroups, Acta Sci. Math., *22* (1961), 217—222.

[4] *Lajos, S.:* On the characterization of completely regular semigroups, Math. Japonicae (Special Issue) *20* (1975), 33—35.

[5] *Lajos, S.:* Some characterizations of completely regular semigroups, Kyungpook Math. J. *19* (1979), 213—214.

[6] *Miccoli M. M.* and *B. Pondělíček:* On α-ideals and generalized α-ideals in semigroups (to appear).

[7] *Steinfeld, O.:* Quasi-ideals in rings and semigroups, Akadémiai Kiadó, Budapest, 1978.

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