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RETRACT VARIETIES OF LATTICE ORDERED GROUPS

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Retracts of partially ordered sets were investigated in [1]–[4]. In [4], an order variety was defined as a nonempty class of partially ordered sets which is closed under direct product and retracts.

Retracts of abelian lattice ordered groups were dealt with in [5]. Let us define a nonempty class of abelian lattice ordered groups to be a *retract variety* if it is closed under direct products and retracts.

Let $\mathcal{R}$ be the collection of all retract varieties of abelian lattice ordered groups. The collection $\mathcal{R}$ is partially ordered by inclusion.

The present paper deals with the partially ordered collection $\mathcal{R}$. Sample results: A theorem proved in [5] concerning retracts of two-factor direct products is generalized to direct products with an arbitrary number of factors. The collection $\mathcal{R}$ is large in the sense that there exists an order-preserving injection of the class of all infinite cardinals into $\mathcal{R}$. Nevertheless, $\mathcal{R}$ behaves as a complete lattice; namely, if I is a nonempty class and $X \in \mathcal{R}$ for each $i \in I$, then $\bigwedge_{i\in I} X_i$ and $\bigwedge_{i\in I} X_i$ do exist in $\mathcal{R}$. Thus the terminology of lattice theory can be applied to $\mathcal{R}$. It will be shown that $\mathcal{R}$ is a Brouwer lattice. The collection of all principal retract varieties is an ideal of $\mathcal{R}$. Next, $\mathcal{R}$ has a large collection of atoms but no dual atom.

1. PRELIMINARIES

All lattice ordered groups dealt with in the present paper are assumed to be abelian.

1.1. Definition. Let H be a lattice ordered group and let G be an l-subgroup of H. Let f be a homomorphism of H onto G such that $f(x) = x$ for each $x \in H$. Then G and f are said to be a *retract* of H or a *retract mapping* of H, respectively.

Let H and H_i $(i \in I)$ be lattice ordered groups. The direct product $\prod_{i\in I} H_i$ is defined in the usual way. For $X \subseteq H$ we put

$$X^{\perp} = \{y \in H\colon |y| \wedge |x| \text{ for each } x \in X\} .$$

1.2. Definition. A convex l-subgroup A of H is said to be a *direct factor* of A if for each $h \in H$ there are elements $a \in A$ and $a' \in A^{\perp}$ such that $h = a + a'$.

It is easy to verify that if A is a direct factor of H and $h \in H$, then the elements a and a' from 1.2 are uniquely determined and the mapping $h \to (a, a')$ is an isomorphism of H onto the direct product $A \times A^{\perp}$. Under the above notation we put $h(A) = a$; $h(A)$ is the component of h in A.

1.3. Definition. Let $\{A_i \mid i \in I\}$ be a system of direct factors of H. Assume that the mapping $\varphi: H \to \prod_{i \in I} H_i$ defined by $\varphi(h) = (h(A_i))_{i \in I}$ is an isomorphism of H onto the direct product $\prod_{i \in I} H_i$. Then H is said to be an *internal direct product* of its l-subgroups H_i $(i \in I)$, and we denote this fact by writing $H = (i) \prod_{i \in I} A_i$.

It is easy to see that in the case $\operatorname{card} I = 2$ the above definition coincides with the definition of the internal direct product from [5], Section 1.

The verification of the following result consists of routine calculations.

1.4. Lemma. *Let $\{A_i \mid i \in I\}$ be a system of direct factors of H. Then $H = (i) \prod_{i \in I} A_i$ if and only if the following conditions are satisfied:*

(i) *$A_{i(1)} \cap A_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I.*

(ii) *If $0 \leqq x_i \in A_i$ for each $i \in I$, then $\bigvee_{i \in I} x_i$ exists in H.*

1.5. Definition. Let $H = (i) \prod_{i \in I} A_i$. Let H_1 be the l-subgroup of H consisting of all elements h of H such that the set $\{i \in I: h(A_i) \neq 0\}$ is finite. Then H_1 will be said to be an *internal direct sum* of its l-subgroups $A_i (i \in I)$, and we write $H_1 = (\mathrm{i}) \sum_{i \in I} A_i$.

The following lemma is easy to verify.

1.6. Lemma. *Let $H = (\mathrm{i}) \prod_{i \in I} A_i$ be such that for each $i \in I$ we have $A_i \neq \{0\}$.*

(i) *If $\operatorname{card} A_i \leqq \operatorname{card} I$ for each $i \in I$ and $H_1 \subseteq H$, $H_1 = (i) \sum_{i \in I} A_i$, then $\operatorname{card} H_1 = \operatorname{card} I$.*

(ii) *$\operatorname{card} H \geqq 2^{\operatorname{card} I}$.*

2. RETRACTS OF DIRECT PRODUCTS

In this section a result from [5] (concerning retracts of finite direct products of lattice ordered groups) will be generalized to the case of infinite direct products. (In fact, the results from [5] are essentially applied in the present proof.)

Again, let H be a lattice ordered group. Suppose that

(1) $$H = (\mathrm{i}) \prod_{i \in I} A_i .$$

Let $f: H \to G$ be a retract mapping of H.

2.1. Lemma. *Let $i \in I$. Then $f(A_i)$ is a direct factor of G.*

Proof. This is a consequence of Lemma 2.3 in [5].

2.2. Lemma. *Let $i(1), i(2) \in I$, $i(1) \neq i(2)$. Then $f(A_{i(1)}) \cap f(A_{i(2)}) = \{0\}$.*

Proof. From (1) and 1.4 we infer that $A_{i(1)} \cap A_{i(2)} = \{0\}$. Hence

(2) $$0 < x_1 \in A_{i(1)},\ 0 < x_2 \in A_{i(2)} \Rightarrow x_1 \wedge x_2 = 0 .$$

Thus, under the assumptions as in (2), the relation $f(x_1) \wedge f(x_2) = 0$ is valid. This implies that $f(A_1) \cap f(A_2) = \{0\}$.

2.3. Lemma. *Let $\{x_i\}_{i\in I} \subseteq H$ be such that for each $i \in I$ we have $0 \leqq x_i \in f(A_i)$. Then $\bigvee_{i\in I} x_i$ exists in H.*

Proof. For each $i \in I$ let $a = x_i(A_i)$. Then $a_i \leqq x_i$. In view of (1) there exists $\bigvee_{i\in I} a_i$ in H. According to 2.2, [5], the relation $f(a_i) = x_i$ holds for each $i \in I$. Next, for each $j \in I$,

(3) $\qquad f(a_j) \leqq f(\bigvee_{i\in I} a_i)\,.$

Let $v \in G$, $x_i \leqq v \leqq f(\bigvee_{i\in I} a_i)$ for each $i \in I$. Hence $a_i \leqq v$ for each $i \in I$ and thus $\bigvee_{i\in I} a_i = v$. We obtain

(4) $\qquad f(\bigvee_{i\in I} a_i) \leqq f(v) = v\,.$

The relations (3) and (4) yield that

$$\bigvee_{i\in I} x_i = f(\bigvee_{i\in I} a_i)$$

is valid.

From 1.4, 2.1, 2.2 and 2.3 we infer:

2.4. Theorem. *Let (1) be valid. Let $f: H \to G$ be a retract mapping of H. Then $G = (\mathrm{i}) \prod_{i\in I} f(A_i)$.*

2.5. Lemma. *Let the assumptions as in 2.4 hold. Let $i \in I$. Then $f(A_i)(A_i)$ is a retract of A_i. Moreover, $f(A_i)(A_i)$ is isomorphic to $f(A_i)$.*

Proof. Cf. 2.6 and 2.7 in [5].

2.6. Corollary. *Let (1) be valid and let G be a retract of H. Then G is isomorphic to a direct product of retracts of the lattice ordered groups A_i.*

3. BASIC PROPERTIES OF $\mathscr{R}$

Let $\mathscr{G}$ be the class of all lattice ordered groups. Let X be a subclass of $\mathscr{G}$ which is closed with respect to isomorphisms. (Whenever dealing with a subclass of $\mathscr{G}$, the closedness with respect to isomorphisms will be always assumed.)

We denote by

rX – the class of all retracts of lattice ordered groups belonging to X;

πX – the class of all direct products of lattice ordered groups belonging to X.

Clearly, we have $rrX = rX$ and $\pi\pi X = \pi X$.

From 2.6 we obtain:

3.1. Lemma. *Let $X \subseteq \mathscr{G}$. Then $r\pi rX = \pi rX$.*

3.2. Definition. A nonempty class X of lattice ordered groups will be said to be a *retract variety* if $X = rX = \pi X$. The collection of all retract varieties will be denoted by $\mathscr{R}$. The collection $\mathscr{R}$ is partially ordered by inclusion.

We denote by $\bar{0}$ the class of all one-element lattice ordered groups. Then $\bar{0}$ is the least element in $\mathscr{R}$; the class $\mathscr{G}$ is the greatest element in $\mathscr{R}$.

3.3. Lemma. *Let* $\emptyset \neq X \subseteq \mathscr{G}$. *Then* $\pi r X \in \mathscr{R}$. *If* $Y_1 \in R$ *such that* $X \subseteq Y_1$, *then* $\pi r X \subseteq Y_1$.

Proof. The first assertion follows from 3.1. The second assertion is obvious.

In view of 3.3, the retract variety $\pi r X$ will be said to be *generated* by X.

The symbols $\wedge$ and $\vee$ in the partially ordered collection $\mathscr{R}$ have the usual meaning.

3.4. Proposition. *Let* I *be a nonempty class and for each* $i \in I$ *let* $X_i \in \mathscr{R}$. *Then*

(i) $\bigcap_{i \in I} X_i = \bigwedge_{i \in I} X_i$.

(ii) $\pi \bigcup_{i \in I} X_i = \bigvee_{i \in I} X_i$.

Proof. The relation (i) is an immediate consequence of the definition of $\mathscr{R}$. The relation (ii) follows from 3.3 (since $r \bigcup_{i \in I} X_i = \bigcup_{i \in I} r X_i = \bigcup_{i \in I} X_i$).

In view of 3.4 we say that $\mathscr{R}$ is a complete lattice. (Let us remark that $\mathscr{R}$ is a proper collection in the sense that there exists an injective mapping of the class of all infinite cardinals into $\mathscr{R}$; cf. Proposition 4.7 below.)

3.5. Proposition. *Let* X_i $(i \in I)$ *be as in* 3.5 *and let* $X \in \mathscr{R}$. *Then*

$$X \wedge (\bigvee_{i \in I} X_i) = \bigvee_{i \in I} (X \wedge X_i) .$$

Proof. The relation $X \wedge (\bigvee_{i \in I} X_i) \geqq \bigvee_{i \in I} (X \wedge X_i)$ is obvious. Let $H \in X \wedge (\bigvee_{i \in I} X_i)$. In view of 3.5, $H \in X$ and $H \in \bigvee_{i \in I} X_i = \pi \bigcup_{i \in I} X_i$. Hence there are K_j $(j \in J)$ in $\bigcup_{i \in I} X_i$ such that $H = (i) \prod_{j \in J} K_j$. Thus each K_j is a retract in H and therefore $K_j \in X$ for each $j \in J$.

Let $j \in J$. There is $i(j) \in I$ with $K_j \in X_{i(j)}$. Hence

$$H \in \pi \bigcup_{j \in J} (X \wedge X_{i(j)}) \subseteq \pi \bigcup_{i \in I} (X \wedge X_i) = \bigvee_{i \in I} (X \wedge X_i) ,$$

which completes the proof.

3.6. Corollary. $\mathscr{R}$ *is a Brouwer lattice.*

Let $f_1 \colon H \to G_1$ and $f_2 \colon H \to G_2$ be retract mappings.

If the corresponding kernels $f_1^{-1}(0)$ and $f_2^{-1}(0)$ coincide, then G_1 and G_2 are isomorphic (since $G_1 \cong H/f_1^{-1}(0) = H/f_2^{-1}(0) \cong G_2$), but G_1 need not coincide with G_2. This can be verified by the following example:

3.7. Example. Let H be the set of all pairs (x, y) of reals with the operation $+$ defined componentswise; we put $(x, y) \geqq (0, 0)$ if either $x > 0$, or $x = 0$ and $y \geqq 0$. Let G_1 and G_2 be the sets of all $(x, y) \in H$ with $y = 0$ or $y = x$, respectively. For each $(x, y) \in H$ we put $f_1((x, y)) = (x, 0)$ and $f_2(x, y) = (x, x)$. Then $f_1 \colon H \to G_1$ and $f_2 \colon H \to G_2$ are retract mappings with $f_1^{-1}(0) = f_2^{-1}(0) = \{(0, y) \colon y \in R\}$, where R is the set of all reals.

4. PRINCIPAL RETRACT VARIETIES

A retract variety X will be said to be *small* if there is a set I and lattice ordered groups H_i $(i \in I)$ such that each element of X is isomorphic to a direct product of some H_i's.

Let $H \in \mathcal{G}$. The retract variety $\pi r\{H\}$ will be called *principal* (and generated by H).

4.1. Proposition. *Let $X \in \mathcal{R}$. The following conditions are equivalent:*

(i) *X is small.*

(ii) *X is principal.*

Proof. Let X be small and let H_i $(i \in I)$ be as above. Denote $H = \prod_{i \in I} H_i$. Then we have $H \in X$ and $X \subseteq \pi r\{H\}$, whence $X = \pi r\{H\}$. Thus X is principal.

Conversely, let X be principal, $X = \pi r\{H\}$. Let $\{H_i\}_{i \in I}$ be the set of all retracts of H. Then each element of X is isomorphic to a direct product of some H_i's, hence X is small.

4.2. Lemma. *Let $X \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *X is principal.*

(ii) *There is a cardinal α with the property that each $H \in X$ possesses an internal direct decomposition $H = (i) \prod_{i \in I} A_i$ such that* card $A_i \leqq \alpha$ *for each $i \in I$.*

Proof. This is an immediate consequence of 4.1.

If $H_1, H_2 \in \mathcal{G}$, then $\pi r\{H_1\} \vee \pi r\{H_2\} = \pi r\{H_1 \times H_2\}$, hence we obtain

4.3. Lemma. *If X_1 and X_2 are principal retract varieties, then $X_1 \vee X_2$ is principal as well.*

4.4. Lemma. *Let $X_1, X_2 \in \mathcal{R}$, $X_1 \leqq X_2$. Assume that X_2 is principal. Then X_1 is principal as well.*

Proof. In view of 4.1, X_2 is small. Thus from $X_1 \leqq X_2$ we infer that X_1 must be small. By applying 4.1 again we obtain that X_1 is principal.

Let P be the collection of all principal retract varieties. Lemmas 4.3 and 4.4 yield:

4.5. Proposition. *P is an ideal of the lattice $\mathcal{R}$.*

Let α be an infinite cardinal. We denote by X_α the class of all lattice ordered groups H which have an internal direct decomposition $H = (i) \prod_{i \in I} A_i$ such that card $A_i \leqq \alpha$ for each $i \in I$.

4.6. Lemma. *X_α is a principal retract variety.*

Proof It is obvious that X_α is closed with respect to direct products. Next, 2.6 yields that X_α is closed with respect to retracts. Hence X_α is a retract variety. Thus in view of 4.2, $X_\alpha \in P$.

Let N_0 be the linearly ordered group of all integers. Let I be a set, card $I = \alpha$, and for each $i \in I$ let $A_i = N_0$. Put

$$H_\alpha = N_0 \circ \sum_{i \in I} A_i\,,$$

where $\circ$ denotes the operation of lexicographic product and $\sum$ stands for direct sum (= restricted direct product). Then card $H_\alpha = \alpha$ (this is a consequence of 1.6 (i)) and H_α is directly indecomposamble.

4.7. Proposition. *There exists an injective order-preserving mapping of the class of all infinite cardinals into P.*

Proof. For each infinite cardinal α we put $\varphi(\alpha) = X_\alpha$. In view of 4.6, $X_\alpha \in P$. Let α and β be infinite cardinals, $\beta < \alpha$. Then $H_\alpha \in X_\alpha$. Since H_α is directly indecomposable, it does not belong to X_β. Hence $X_\alpha \neq X_\beta$. Therefore φ is injective. It is obvious that φ is order-preserving.

5. COVERING RELATIONS IN $\mathscr{R}$

If $X, Y \in \mathscr{R}, X < Y$ and if the interval $[X, Y]$ of $\mathscr{R}$ is prime, then Y is said to *cover* X and we denote this fact by writing $X \prec Y$.

Let $\mathscr{A}$ be the collection of all atoms of $\mathscr{R}$. The natural question arises whether $\mathscr{A}$ is nonempty; an analogous question concerning dual atoms of $\mathscr{R}$ can be proposed as well.

For an infinite cardinal α we denote by $\omega(\alpha)$ the first ordinal whose cardinality is α. Let $\omega'(\alpha)$ be the linearly ordered set dually isomorphic to $\omega(\alpha)$. For each $i \in \omega'(\alpha)$ let $A_i = N_0$. Next, let $H(\alpha)$ be the lexicographic product

$$H(\alpha) = \Gamma_{i \in \omega'(\alpha)} A_i .$$

For the notion of a large lexicographic factor of a linearly ordered group cf. [2]. From the definition of $H(\alpha)$ we immediately obtain that each large lexicographic factor of $H(\alpha)$ is isomorphic to $H(\alpha)$. Thus in view of Theorem 3.4, [5] we infer:

5.1. Lemma. *$r\{H(\alpha)\}$ consists of lattice ordered groups H such that either $H = \{0\}$ or H is isomorphic to $H(\alpha)$.*

5.2. Lemma. *$\pi r\{H(\alpha)\}$ consists of lattice ordered groups H such that either $H = \{0\}$ or H is isomorphic to a direct product of copies of $H(\alpha)$.*

Proof. This is a consequence of 5.1.

5.3. Proposition. *Let α, β be distinct infinite cardinals. Then $\pi r\{H(\alpha)\} \neq \pi r\{H(\beta)\} \in \mathscr{A}$.*

Proof. According to 5.2 we have $H(\beta) \notin \pi r\{H(\alpha)\}$, hence $\pi r\{H(\alpha)\} \neq \pi r\{H(\beta)\}$. Clearly $\pi r\{H(\beta)\} > \bar{0}$. Let $H \in \pi r\{H(\beta)\}, H \neq \{0\}$. In view of 5.2 there is a retract H_1 of H such that H_1 is isomorphic to $H(\beta)$. Hence $\pi r\{H\} = \pi r\{H(\beta)\}$. This shows that $\pi r\{H(\beta)\}$ is an atom in $\mathscr{R}$.

5.4. Corollary. *The mapping $\psi(\alpha) = \pi r\{H(\alpha)\}$ is an injection of the class of all infinite cardinals into $\mathscr{A}$.*

For $X \in \mathscr{R}$ we denote $\mathscr{A}(X) = \{Y \in \mathscr{R}: X \prec Y\}$.

If X and Y are distinct elements of $\mathscr{A}$, then in view of 3.6 we have $X \vee Y \succ Y$. Thus in view of 5.4 we obtain:

5.5. Corollary. *If X is an atom in $\mathscr{R}$, then $\mathscr{A}(X)$ is a proper collection.*

5.6. Lemma. *Let $X \in \mathscr{R}$, $H \in \mathscr{G} \setminus X$. Let $H_1 = (i) \sum_{i \in I} A_i$ be such that A_i is isomorphic to H for each $i \in I$ and* card $H <$ card I. *Then H_1 does not belong to $X \vee \pi r\{H\}$.*

Proof. By way of contradiction, assume that H_1 belongs to $X \vee \pi r\{H\}$. Hence in view of 3.4 there are $B \in X$ and $C \in \pi r\{H\}$ such that

(1) $$H_1 = (i)\, B \times C\,.$$

Next, there are $C_j (j \in J)$ in $r\{H\}$ with

(2) $$C = (i) \prod_{j \in J} C_j\,.$$

From (1) and from $H_1 = (i) \sum_{i \in I} A_i$ we obtain

(3) $$C = (i) \sum_{i \in I} (C \cap A_i)\,.$$

The relations (2) and (3) yield that for each $j \in J$ we have

(4) $$C_j = (i) \sum_{i \in I} (C_j \cap A_i)\,.$$

Let $I(j) = \{i \in I\colon C_j \cap A_i \neq \{0\}\}$. Since

$$\operatorname{card} C_j \leqq \operatorname{card} H < \operatorname{card} I\,,$$

in view of 1.6 (i) we must have

(5) $$\operatorname{card} I(j) < \operatorname{card} I \quad \text{for each} \quad j \in J\,.$$

If $C = \{0\}$, then $H_1 = B \in X$ and thus $A_i \in X$ for each $i \in I$, implying that $H \in X$, which is a contradiction. Hence $C \neq \{0\}$ and therefore without loss of generality we can assume that $C_j \neq \{0\}$ for each $j \in J$.

Let $i(0) \in I$. Since $A_{i(0)}$ is a convex l-subgroup of H_1, in view of (1) and (2) we get

(6) $$A_{i(0)} = (i)\,(A_{i(0)} \cap B) \times (i) \prod_{j \in J} (A_{i(0)} \cap C_j)\,.$$

If $A_{i(0)} \cap C_j = \{0\}$ for each $j \in J$, then according to (6) we have $A_{i(0)} = A_{i(0)} \cap B$. Also, in view of (1), $B = (i) \sum_{i \in I} (B \cap A_i)$, thus $A_{i(0)}$ is a retract of B and hence $A_{i(0)} \in X$, which implies $H \in X$, a contradiction. Hence there is $j \in J$ with $A_{i(0)} \cap$ $\cap C_j \neq \{0\}$.

Choose $i(1) \in I$. There is $j(1) \in J$ with $A_{i(1)} \cap C_{j(1)} \neq \{0\}$. Thus there is $0 < c_{j(1)} \in$ $\in A_{i(1)} \cap C_{j(1)}$.

In view of (5) there is $i(2) \in I$ such that

(7) $$i(2) \notin I(j(1))\,.$$

Hence, in particular, $i(2) \neq i(1)$. There exists $j(2)$ with $A_{i(2)} \cap C_{j(2)} \neq \{0\}$. Thus according to (7), $j(2) \neq j(1)$. There is $0 < c_{j(2)} \in A_{i(2)} \cap C_{j(2)}$.

Again, in view of (5) there is $i(3) \in I$ such that $i(3) \notin I(j(1) \cup I(j(2))$. We can find

$j(3) \in J$ in an analogous way as in the case of $j(2)$. In this manner we obtain distinct elements.

$$i(1), i(2), \ldots, i(n), \ldots \quad \text{in} \quad I,$$

distinct elements

$$j(1), j(2), j(3), \ldots, j(n), \ldots \quad \text{in} \quad J$$

and elements

$$(8) \qquad 0 < c_{j(n)} \in A_{i(n)} \cap C_{j(n)} \quad (n = 1, 2, \ldots).$$

In particular, we have verified that the set J must be infinite.

According to (2) there is $c \in C$ such that

$$c(C_{j(n)}) = c_{j(n)} \quad \text{for} \quad n = 1, 2, \ldots,$$

and $c(C_j) = 0$ whenever $j \notin \{j(1), j(2), \ldots\}$. Then $c \in H_1$ and in view of (8) we have

$$c(A_{i(n)}) \geqq c_{j(n)}(A_{i(n)}) = c_{j(n)} > 0 \quad \text{for} \quad n = 1, 2, \ldots,$$

which is a contradiction with respect to the relation $H_1 = (i) \sum_{i \in I} A_i$.

5.7. Lemma. *Let $\mathscr{G} \neq X \in \mathscr{R}$ and $H \in \mathscr{G}$. Then $X \vee \pi r\{H\} \neq \mathscr{G}$.*

Proof. This is an immediate consequence of 5.6.

5.8. Theorem. *The lattice $\mathscr{R}$ has no dual atom.*

Proof. By way of contradiction, assume that X is a dual atom in $\mathscr{R}$. Hence there is $H \in \mathscr{G}$ such that $H \notin X$. Thus $\pi r\{H\} \nleq X$. Since X is a dual atom in $\mathscr{R}$ we have $X \vee \pi r\{H\} = \mathscr{G}$, contradicting 5.7.

The class of all $X \in \mathscr{R}$, $\bar{0} \neq X$ such that there is no $X_1 \in \mathscr{A}$ with $X_1 \leqq X$ will be denoted by $\mathscr{A}'$; the elements of $\mathscr{A}'$ will be called *antiatoms*.

The following example shows that the class $\mathscr{A}'$ is nonempty.

5.9. Example. There exist archimedean linearly ordered groups A_n $(n \in N)$ such that $A_n \neq \{0\}$ for each $n \in N$ and $A_{n(1)}$ is not isomorphic to $A_{n(2)}$ whenever $n(1)$ and $n(2)$ are distinct positive integers. Thus there is a linearly ordered group H such that $H = (i)\, \Gamma_{n \in N} B_n$, where B_n is isomorphic to A_n for each $n \in N$.

Put $X = \pi r\{H\}$ and let $X_1 \leqq X$, $X_1 \neq \bar{0}$. In view of [5], Theorem 3.4 there is $m \in N$ such that $X_1 = \pi\{A_m, A_{m+1}, \ldots\}$. Choose $m(1) \in N$, $m(1) > m$. Then $\pi\{A_{m(1)}, A_{m(1)+1}, \ldots\} \in \mathscr{R}$ and $\bar{0} < \pi\{A_{m(1)}, A_{m(1)+1}, \ldots\} < X_1$.

For $X \in \mathscr{R}$ we denote by $\mathscr{A}'(X)$ the class of all $Y \in \mathscr{R}$ such that $X < Y$ and no element of the interval $[X, Y]$ covers X.

Put $X_a = \sup \mathscr{A}$, $X'_a = \sup \mathscr{A}'$.

5.10. Proposition. *$X_a \wedge X'_a = \bar{0}$, $\mathscr{A}(X'_a)$ is a proper collection and $A'(X_0)$ is nonempty.*

Proof. The relation $X_a \wedge X'_a = \bar{0}$ is a consequence of 3.6. Next, if $Z_1, Z_2 \in \mathscr{A}$, $Z_1 \neq Z_2$, then $Z_1 \vee X'_a$ and $Z_2 \vee X'_a$ are distinct elements of $\mathscr{A}(X'_a)$, hence according to 5.5, $\mathscr{A}(X'_a)$ is a proper collection. If $Z \in \mathscr{A}'$, then $X_a \vee Z \in \mathscr{A}'(X_a)$. Thus in view of $\mathscr{A}' \neq \emptyset$ we obtain $\mathscr{A}'(X_a) \neq \emptyset$.

The following two results will be just announced without proofs.

5.11. Proposition. *The collection $\mathscr{A}(X_a)$ is nonempty. $X_a \vee X'_a < \mathscr{G}$.*

5.12. Proposition. *If X is principal and $Y \in \mathscr{R}$, $X \prec Y$, then Y is principal as well.*

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