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CONTRIBUTIONS TO THE ASYMPTOTIC BEHAVIOUR OF THE EQUATION $\dot{z} = f(t, z)$ WITH A COMPLEX-VALUED FUNCTION f

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1. INTRODUCTION

This paper deals with the asymptotic properties of the equation

$$\dot{z} = f(t, z)\,, \tag{1.1}$$

where f is a continuous complex-valued function of a real variable t and a complex variable z. It is convenient to write the equation (1.1) in the form

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)]\,, \tag{1.2}$$

where G is a real-valued function and h, g are complex-valued functions, t or z being a real or complex variable, respectively. The function h is assumed to be holomorphic in a simply connected region Ω containing zero, and to satisfy the conditions $h(z) = 0 \Leftrightarrow z = 0$, $h^{(j)}(0) = 0$ $(j = 1, 2, \ldots, n - 1)$, $h^{(n)}(0) \neq 0$, where $n \geqq 2$ is an integer. The technique of the proofs of the results is based on the Liapunov function method with the „Liapunov-like" function $W(z)$ defined in [1]. Several results of this type were proved in [2], [3]. The assumptions of these results imply that $z(t) \equiv 0$ is a solution of (1.2). In the present paper, we attempt to remove this restriction. The last section deals with the application of the results to equations

$$\dot{z} = q(t, z) - p(t)\, z^2 \tag{1.3}$$

and

$$\ddot{x} = x\psi(t, \dot{x}x^{-1})\,.$$

The asymptotic behaviour of solutions of the Riccati differential equation, which is a special case of (1.3), was investigated e.g. in [6], [7], [8], [9]. For completeness notice that the case $n = 1$ was studied in several previous papers such as [4], [5].

Throughout the paper we use the following notation:

$\mathbb{C}$ – Set of all complex numbers
$\mathbb{N}$ – Set of all positive integers
$\mathbb{R}$ – Set of all real numbers

I — Interval $[t_0, \infty)$
Ω — Simply connected region in $\mathbb{C}$ such that $0 \in \Omega$
$S(a, \varrho)$ — Set $\{z \in \mathbb{C}: |z - a| = \varrho\}$
$\bar{b}$ — Conjugate of a complex number b
Re b — Real part of a complex number b
Arg z — Principal value of the multivalued function arg z
$C(\Gamma)$ — Class of all continuous real-valued functions defined on the set Γ
$\tilde{C}(\Gamma)$ — Class of all continuous complex-valued functions defined on the set Γ
$\mathscr{H}(\Omega)$ — Class of all complex-valued functions holomorphic in the region Ω
Cl Γ — Closure of a set $\Gamma \subset \mathbb{C}$
Bd Γ — Boundary of a set $\Gamma \subset \mathbb{C}$
$\tilde{C}^1(I)$ — Class of all continuously differentiable complex-valued functions defined on I
k, $W(z)$ — see [1, pp. 66–67]
$\lambda_+, \lambda_-, \mathscr{T}^+, \mathscr{T}^-, \varphi$ — see [1, pp. 73–74]
Int Γ — Interior of a Jordan curve with the geometric image Γ.

Let $\mathscr{S}^+ \in \mathscr{T}^+/\varphi$ and $\mathscr{S}^- \in \mathscr{T}^-/\varphi$ be fixed. Then $\mathscr{S}^+ = \{\hat{K}(\lambda): 0 < \lambda < \lambda_+\}$, $\mathscr{S}^- = \{\hat{K}(\lambda): \lambda_- < \lambda < \infty\}$, where $\hat{K}(\lambda)$ are the geometric images of Jordan curves such that $0 \in \hat{K}(\lambda)$, the equality $W(z) = \lambda$ holds for $z \in \hat{K}(\lambda) \setminus \{0\}$ and $\hat{K}(\lambda_1) \setminus \{0\} \subset \subset \text{Int}\, \hat{K}(\lambda_2)$ for $0 < \lambda_1 < \lambda_2 < \lambda_+$ or $\hat{K}(\lambda_2) \setminus \{0\} \subset \text{Int}\, \hat{K}(\lambda)$ for $\lambda_- < \lambda_1 < \lambda_2 < < \infty$. Define

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \hat{K}(\mu) \setminus \{0\} \quad \text{for} \quad 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_+$$

and

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_2 < \mu < \lambda_1} \hat{K}(\mu) \setminus \{0\} \quad \text{for} \quad \lambda_- \leqq \lambda_2 < \lambda_1 \leqq \infty .$$

2. MAIN RESULTS

Consider the equation

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)]\,, \tag{2.1}$$

where $G(t, z)\,[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$, $G \in C(I \times (\Omega \setminus \{0\}))$, $h \in \mathscr{H}(\Omega)$, $g \in \in \tilde{C}(I \times (\Omega \setminus \{0\}))$. Assume that $h(z) = 0 \Leftrightarrow z = 0$ and $h^{(j)}(0) = 0$ $(j = 1, 2, \dots \dots, n - 1)$, $h^{(n)}(0) \neq 0$, where $n \geqq 2$ is an integer.

Theorem 1. *Let $0 < \vartheta \leqq \lambda_+$. Suppose that $s_0 \in I$ and that for any $T > s_0$ there are $\delta_T \geqq 0$ and $E_T(t) \in C[s_0, T)$ such that*

(i) $\inf_{z \in \text{Bd}\Omega} |z| > \delta_T$ *for any* $T > s_0$,

(ii) $\vartheta < \lambda_+$ *or* $E_T(t) \leqq 0$ *for* $t \in [s_0, T)$, $T > s_0$,

and

(iii) *the inequality*

$$G(t, z)\,\mathrm{Re}\left\{k\,h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_T(t) \tag{2.2}$$

is fulfilled for $t \in [s_0, T)$, $z \in K(0, \vartheta)$, $|z| > \delta_T$.

If a solution $z(t)$ *of* (2.1) *satisfies*

$$z(t) \in K(0, \vartheta) \cup \{0\}$$

for $t \in (t_1, \omega)$, *where* $[t_1, \omega)$ *is the right maximal interval of existence of* $z(t)$ *and* $t_1 \geqq s_0$, *then* $\omega = \infty$.

Proof. Suppose $\omega < \infty$. Then $\vartheta = \lambda_+$ and there is $t^* \in (t_1, \omega)$ such that $|z(t)| > \delta_\omega$ for $t \in [t^*, \omega)$. For $t \in [t^*, \omega)$ we have

$$\dot{W}(z) = G(t, z)\,W(z)\,\mathrm{Re}\left\{k\,h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\},$$

where $z = z(t)$. Using (2.2) we get

$$\dot{W}(z(t)) \leqq E_\omega(t)\,W(z(t))$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp\left[-\textstyle\int_{t^*}^{t} E_\omega(s)\,\mathrm{d}s\right]\} \leqq 0\,. \tag{2.3}$$

Integrating (2.3) over $[t^*, t] \subset [t^*, \omega)$ we have

$$W(z(t)) \exp\left[-\textstyle\int_{t^*}^{t} E_\omega(s)\,\mathrm{d}s\right] - W(z(t^*)) \leqq 0\,,$$

whence

$$W(z(t)) \leqq W(z(t^*)) \exp\left[\textstyle\int_{t^*}^{t} E_\omega(s)\,\mathrm{d}s\right] \leqq W(z(t^*)) = \vartheta^* < \vartheta\,.$$

Thus $z(t) \in \mathrm{Cl}\,K(\vartheta^*) \subset K(0, \vartheta) \cup \{0\}$, which is a contradiction with the supposition $\omega < \infty$. Therefore $\omega = \infty$.

Theorem 2. *Let* $0 < \vartheta \leqq \lambda_+$. *Assume that* $s_j \in I$, $\delta_j \geqq 0$ *for* $j \in \mathbb{N}$. *Suppose there are functions* $E_j(t) \in C[t_0, \infty)$ *such that*

(i) *for* $j \in \mathbb{N}$

$$\liminf_{t \to \infty} \textstyle\int_{t_0}^{t} E_j(s)\,\mathrm{d}s = -\infty \tag{2.4}$$

holds;

(ii) *the inequality*

$$G(t, z)\,\mathrm{Re}\left\{k\,h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t) \tag{2.5}$$

is fulfilled for $t \geqq s_j$, $z \in K(0, \vartheta)$, $|z| > \delta_j$, $j \in \mathbb{N}$. *Define*

$$\delta = \inf_{j \in N} \delta_j\,.$$

If a solution $z(t)$ *of* (2.1) *satisfies*

$$z(t) \in K(0, \vartheta) \cup \{0\}$$

for $t > t_1$, *where* $t_1 \geqq t_0$, *then*

$$\liminf_{t\to\infty} |z(t)| \leqq \delta . \tag{2.6}$$

Proof. Put $\mathscr{M}_j = \{t \geqq s_j: z(t) \in K(0, \vartheta), |z(t)| > \delta_j\}$. For $t \in \mathscr{M}_j$ we have

$$\dot{W}(z) = G(t, z)\, W(z) \operatorname{Re}\left\{k\, h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\},$$

where $z = z(t)$. By virtue of (2.5) we get

$$\dot{W}(z(t)) \leqq E_j(t)\, W(z(t))$$

for $t \in \mathscr{M}_j$. This inequality is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp\left[-\textstyle\int_{t_1}^t E_j(s)\, \mathrm{d}s\right]\} \leqq 0 . \tag{2.7}$$

If (2.6) is not true, there exist $\varepsilon_0 > \delta$ and $\tau > t_1$ such that $|z(t)| \geqq \varepsilon_0$ for $t \geqq \tau$. Choosing $j \in \mathbb{N}$ so that $\delta_j < \varepsilon_0$ and integrating (2.7) over $[T, t]$, where $t \geqq T = \max(\tau, s_j)$, we obtain

$$W(z(t)) \exp\left[-\textstyle\int_{t_1}^t E_j(s)\, \mathrm{d}s\right] - W(z(T)) \exp\left[-\textstyle\int_{t_1}^T E_j(s)\, \mathrm{d}s\right] \leqq 0 .$$

Hence

$$W(z(t)) \leqq W(z(T)) \exp\left[\textstyle\int_T^t E_j(s)\, \mathrm{d}s\right]$$

for $t \geqq T$. From (2.4) it follows that

$$\liminf_{t\to\infty} W(z(t)) = \liminf_{t\to\infty} |z(t)| = 0 ,$$

which is impossible. Thus we have proved (2.6).

Analogously we can prove the following two theorems:

Theorem 1′. *Let* $\lambda_- \leqq \vartheta < \infty$. *Assume that* $s_0 \in I$ *and that for any* $T > s_0$ *there are* $\delta_T \geqq 0$ *and* $E_T(t) \in C[s_0, T)$ *such that*

$$\inf_{z \in \mathrm{Bd}\,\Omega} |z| > \delta_T \quad \textit{for any} \quad T > s_0 ,$$

$\vartheta > \lambda_-$ *or* $E_T(t) \leqq 0$ *for* $t \in [s_0, T)$, $T > s_0$, *and the inequality*

$$-G(t, z) \operatorname{Re}\left\{k\, h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_T(t) \tag{2.2′}$$

is fulfilled for $t \in [s_0, T)$, $z \in K(\infty, \vartheta)$, $|z| > \delta_T$. *If a solution* $z(t)$ *of* (2.1) *satisfies*

$$z(t) \in K(\infty, \vartheta) \cup \{0\}$$

tor $t \in (t_1, \omega)$, *where* $[t_1, \omega)$ *is the right maximal interval of existence of* $z(t)$ *and* $f_1 \geqq s_0$, *then* $\omega = \infty$.

Theorem 2′. *Let* $\lambda_- \leqq \vartheta < \infty$. *Assume that* $s_j \in I$, $\delta_j \geqq 0$ *for* $j \in \mathbb{N}$. *Suppose there are* $E_j(t) \in C[t_0, \infty)$ *such that*

(i) *for* $j \in \mathbb{N}$

$$\liminf_{t\to\infty} \int_{t_0}^{t} E_j(s)\,ds = -\infty \tag{2.4'}$$

holds;

(ii) *the inequality*

$$-G(t, z)\,\mathrm{Re}\left\{k\, h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t) \tag{2.5'}$$

is fulfilled for $t \geqq s_j$, $z \in K(\infty, \vartheta)$, $|z| > \delta_j$, $j \in \mathbb{N}$. *Define*

$$\delta = \inf_{j\in N} \delta_j .$$

If a solution $z(t)$ *of* (2.1) *satisfies*

$$z(t) \in K(\infty, \vartheta) \cup \{0\}$$

for $t > t_1$, *where* $t_1 \geqq t_0$, *then*

$$\liminf_{t\to\infty} |z(t)| \leqq \delta . \tag{2.6'}$$

Theorem 3. *Suppose there exist a region* $\Omega_1 \subset \Omega$, *an* $R > 0$ *and a nonnegative function* $B(t) \in C[t_0, \infty)$ *such that* $G \in C(I \times \Omega_1)$, $g \in \tilde{C}(I \times \Omega_1)$,

$$\int_{t_0}^{\infty} B(s)\,ds < \infty$$

and

$$G(t, z)\,\mathrm{Re}\{\bar{z}[h(z) + g(t, z)]\} \leqq |z|\, B(t) \tag{2.8}$$

for $t \geqq t_0$, $z \in \Omega_1$, $|z| < R$. *If a solution* $z(t)$ *of* (2.1) *satisfies*

$$\liminf_{t\to\infty} |z(t)| \leqq \delta < R \tag{2.9}$$

and $z(t) \in \Omega_1 \cup \{0\}$ *for* $t > t_1$, *where* $t_1 \geqq t_0$, *then*

$$\limsup_{t\to\infty} |z(t)| \leqq \delta .$$

Proof. It can be easily derived that

$$\frac{d}{dt}|z(t)| = G(t, z(t))\,|z(t)|^{-1}\,\mathrm{Re}\{\bar{z}(t)\,[h(z(t)) + g(t, z(t))]\} \tag{2.10}$$

holds for $t \in \mathcal{M} = \{t > t_1 : z(t) \neq 0, |z(t)| < R\}$. Let $\tau > t_1$ be such that $z(\tau) = 0$. Then

$$\lim_{t\to\tau+} \frac{|z(t)| - |z(\tau)|}{t - \tau} = \lim_{t\to\tau+} \frac{|z(t)|}{t - \tau} = |\dot{z}(\tau)| = |G(\tau, 0)\, g(\tau, 0)| .$$

Similarly

$$\lim_{t\to\tau-} \frac{|z(t)| - |z(\tau)|}{t - \tau} = \lim_{t\to\tau-} \frac{|z(t)|}{t - \tau} = -|\dot{z}(\tau)| = -|G(\tau, 0)\, g(\tau, 0)| .$$

Therefore $\mathrm{d}|z(\tau)|/\mathrm{d}t$ exists if and only if $G(\tau, 0)\, g(\tau, 0) = 0$. In this case $\mathrm{d}|z(\tau)|/\mathrm{d}t = 0$.

Put $\mathcal{M}_1 = \{t > t_1 \colon z(t) = 0\}$, $\mathcal{M}_0 = \{t > t_1 \colon G(t, 0)\, g(t, 0) = 0\}$. It is known that the set $\mathcal{M}_1 \setminus \mathcal{M}_0$ is at most countable. Using (2.10) and (2.8), we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|z(t)|\right| \leqq |G(t, z(t))\, [h(z(t)) + g(t, z(t))]|\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|z(t)| \leqq B(t)$$

for $t \in \mathcal{M}$. Define

$$B^*(t) = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}|z(t)| & \text{whenever} \quad t \in \mathcal{M}\,, \\ 0 & \text{whenever} \quad t \in \mathcal{M}_1\,. \end{cases}$$

It is clear that

$$(2.11) \qquad |B^*(t)| \leqq |G(t, z(t))\, [h(z(t)) + g(t, z(t))]|\,,$$

$$(2.12) \qquad B^*(t) \leqq B(t)$$

for $t > t_1$ such that $|z(t)| < R$. By (2.10) and (2.11), the function $B^*(t)$ is continuous on $\mathcal{M} \cup \mathcal{M}_0$. Any set $\mathcal{M}_2 \subset \mathcal{M}_1 \setminus \mathcal{M}_0$ is at most countable. Moreover, $B^*(t)$ is bounded on any compact subinterval of $\mathcal{M} \cup \mathcal{M}_1 = \{t > t_1 \colon |z(t)| < R\}$.

Hence, taking (2.12) into account, we get

$$(2.13) \qquad |z(t)| - |z(\sigma)| = \textstyle\int_\sigma^t B^*(s)\, \mathrm{d}s \leqq \int_\sigma^t B(s)\, \mathrm{d}s$$

for $t > \sigma > t_1$ provided $\sigma, t \in \mathcal{M} \cup \mathcal{M}_1$.

Choose ε, $0 < \varepsilon < R - \delta$. Let $T > t_1$ be such that $T \leqq t_2 \leqq t_3$ implies

$$\textstyle\int_{t_2}^{t_3} B(s)\, \mathrm{d}s < \varepsilon/2\,.$$

In view of (2.9), there is $\sigma_1 \geqq T$ such that

$$|z(\sigma_1)| < \delta + \varepsilon/2\,.$$

Suppose there is $t^* > \sigma_1$ such that $|z(t^*)| = \delta + \varepsilon$, $|z(t)| < \delta + \varepsilon$ for $t \in [\sigma, t^*]$. By (2.13) we have

$$|z(t^*)| \leqq |z(\sigma_1)| + \textstyle\int_{\sigma_1}^t B(s)\, \mathrm{d}s < \delta + \varepsilon/2 + \varepsilon/2 = \delta + \varepsilon\,,$$

a contradiction. Therefore $|z(t)| \leqq \delta + \varepsilon$ for $t \geqq \sigma_1$ and

$$\limsup_{t \to \infty} |z(t)| \leqq \delta\,.$$

Theorem 4. *Let $a_j \in \mathbb{C}$, α_j, β_j, $\delta \in \mathbb{R}$ be such that $\beta_j \geqq t_0$, $0 \leqq \delta < \alpha_j - |a_j|$ for $j \in \mathbb{N}$, $\alpha_j \to \delta$ as $j \to \infty$. Suppose there is a region $\Omega_1 \subset \Omega$ such that*

$$(2.14) \qquad G(t, z)\, \mathrm{Re}\, \{(\bar{z} - \bar{a}_j)\, [h(z) + g(t, z)]\} < 0$$

is fulfilled for $t > \beta_j$ and $z \in \Omega_1 \cap S(a_j, \alpha_j)$, $j \in \mathbb{N}$. If a solution $z(t)$ of (2.1) satisfies

$$(2.15) \qquad \liminf_{t \to \infty} |z(t)| \leqq \delta$$

and $z(t) \in \Omega_1 \cup \{0\}$ *for* $t > t_1$, *where* $t_1 \geqq t_0$, *then*

$$\limsup_{t \to \infty} |z(t)| \leqq \delta .$$

Proof. Clearly $a_j \to 0$ as $j \to \infty$. Choose $\varepsilon > 0$. Pick $j \in \mathbb{N}$ such that $|a_j| + \alpha_j < \delta + \varepsilon$. Let $\gamma_j \in \mathbb{R}$ be such that $\delta < \gamma_j < \alpha_j - |a_j|$. From (2.15) it follows that there is $\sigma > \max(t_1, \beta_j)$ for which $|z(\sigma)| < \gamma_j$. Now we have $|z(\sigma) - a_j| \leqq |z(\sigma)| + |a_j| < \gamma_j + |a_j| < \alpha_j$. Since (2.14) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} |z(t) - a_j| = \alpha_j^{-1} G(t, z(t)) \operatorname{Re} \{(\overline{z(t)} - \bar{a}_j) [h(z(t)) + g(t, z(t))]\} < 0$$

for all $t \geqq \sigma$ such that $|z(t) - a_j| = \alpha_j$, we infer that $|z(t) - a_j| < \alpha_j$ for $t \geqq \sigma$, whence

$$|z(t)| \leqq |a_j| + \alpha_j < \delta + \varepsilon$$

for $t \geqq \sigma$. Thus

$$\limsup_{t \to \infty} |z(t)| \leqq \delta .$$

3. APPLICATION TO EQUATIONS $\dot{z} = q(t, z) - p(t) z^2$ AND $\ddot{x} = x \psi(t, \dot{x}x^{-1})$

In this section we shall consider the equation

$$(3.1) \qquad \dot{z} = q(t, z) - p(t) z^2 ,$$

where $q \in \tilde{C}(I \times \mathbb{C})$, $p \in \tilde{C}(I)$ and

$$(3.2) \qquad \ddot{x} = x \psi(t, \dot{x}x^{-1}) ,$$

where $\psi \in \tilde{C}(I \times \mathbb{C})$. Notice that the choice $\psi(t, z) = -P(t) z - Q(t)$ leads to a linear equation $\ddot{x} + P(t) \dot{x} + Q(t) x = 0$. Supposing $\alpha, \beta \in \tilde{C}^1(I)$, $\varrho \in \tilde{C}(I)$ and $\beta(t) \neq 0$ for $t \in I$, we can easily verify the following lemma:

Lemma 1. *Put*

$$p(t) = \beta^{-1}(t) + \varrho(t) ,$$
$$q(t, z) = \beta \psi(t, (z + \alpha) \beta^{-1}) + \varrho z^2 + (\dot{\beta} - 2\alpha) \beta^{-1} z + $$
$$+ (\dot{\beta} - \alpha) \alpha \beta^{-1} - \dot{\alpha} .$$

(i) *A function* $z(t)$ *is a solution of* (3.1) *defined on an interval* $J \subset I$ *if and only if*

$$z(t) = \beta(t) \dot{x}(t) x^{-1}(t) - \alpha(t) ,$$

where $x(t)$ *is a solution of* (3.2) *on* J.

(ii) *A function* $x(t)$ *is a solution of* (3.2) *defined on* $J \subset I$ *if and only if*

$$x(t) = \Theta \exp \left[\int_\omega^t [z(s) + \alpha(s)] \beta^{-1}(s) \, \mathrm{d}s\right] ,$$

where Θ *is a constant different from zero,* $\omega \in J$, *and* $z(t)$ *is a solution of* (3.1) *on* J.

In view of Lemma 1 we shall obtain the results concerning the asymptotic behaviour of the solutions of (3.2) as immediate consequences of the results concerning the solutions of the equation (3.1). If $a \in C$, $a \neq 0$, then (3.1) may be written in the form

(3.3) $$\dot{z} = G(t, z)\,[h(z) + g(t, z)]\,,$$

where $h(z) = -az^2$, $G(t, z) \equiv 1$ and $g(t, z) = q(t, z) + az^2 - p(t)\,z^2$. From [1, Example 1], where $\Omega = C$, $b = -a$, we have $h'(z) = -2az$, $h''(z) = -2a$, $n = 2$, $W(z) = \exp[\operatorname{Re}(2\bar{a}z^{-1})]$, $\lambda_+ = \lambda_- = 1$, $k = -\bar{a}$. The sets $\hat{K}(\lambda)$, where $0 < \lambda < \lambda_+ = 1$ or $1 = \lambda_- < \lambda < \infty$, are circles with centres $\bar{a}(\ln \lambda)^{-1}$ and radii $|a|\,|\ln \lambda|^{-1}$, $K(0, 1) = \{z \in C\colon \operatorname{Re}(az) < 0\}$, $K(\infty, 1) = \{z \in C\colon \operatorname{Re}(az) > 0\}$.

For $a \in C, a \neq 0, A > 0, B > 0, \delta \in (0, \pi/4]$ denote

$$\Omega_{A,B}(a) = \{z \in C\colon -A \operatorname{Re}[a^2z^2] - B|\operatorname{Im}[a^2z^2]| > 0\}\,,$$

$\Omega_\delta(a) = \{z = \mu e^{i\vartheta}\colon \mu \in R \setminus \{0\},\ \operatorname{Arg}\bar{a} + \pi/2 - \delta < \vartheta < \operatorname{Arg}\bar{a} + \pi/2 + \delta\}$. It can be easily verified that

$$\Omega_{A,B}(a) \subset \Omega_{\pi/4}(a) = \{z \in C\colon \operatorname{Re}(a^2z^2) < 0\}$$

for any $A, B > 0$, and for any $A, B > 0$ there exists $\delta_0 \in (0, \pi/4)$ such that

(3.4) $$\Omega_\delta(a) \subset \Omega_{A,B}(a) \quad \text{for} \quad \delta \in (0, \delta_0]\,.$$

The following lemma will be useful in our further considerations.

Lemma 2. *Suppose there are $a \in C$ and $C \geqq 0$ such that*

(3.5) $$\operatorname{Re}[\bar{a}\,p(t)] > 0 \quad \textit{for} \quad t \in I\,,$$

(3.6) $$\liminf_{t\to\infty} \operatorname{Re}[\bar{a}\,p(t)] > 0\,, \quad \limsup_{t\to\infty} |\operatorname{Im}[\bar{a}\,p(t)]| < \infty\,,$$

(3.7) $$\operatorname{Re}[a\,q(t, z)] \geqq -C|\operatorname{Im}[a^2z^2]| \quad \textit{for} \quad t \in I\,, \quad z \in \Omega_{\pi/4}(a)$$

and

(3.8) $$q(t, 0) \neq 0 \quad \textit{for} \quad t \in I\,.$$

Then every solution $z(t)$ of (3.1) satisfying at $t_1 \geqq t_0$ the condition $\operatorname{Re}[a\,z(t_1)] \geqq 0$ *fulfils* $\operatorname{Re}[a\,z(t)] \geqq 0$ *for all $t > t_1$ for which $z(t)$ exists.*

Moreover, $\operatorname{Re}[a\,z(t)] > 0$ *provided $z(t) \neq 0$.*

Proof. Let $A, B > 0$ be such that

$$\operatorname{Re}[\bar{a}\,p(t)] \geqq |a|^2 A\,, \quad |\operatorname{Im}[\bar{a}\,p(t)]| \leqq |a|^2 (B - C)$$

for $t \geqq t_1$. There exists a $\delta_0 \in (0, \pi/4)$ with the property $\Omega_{\delta_0}(a) \subset \Omega_{A,B}(a)$. For $t \geqq t_1$ such that $z = z(t) \in \Omega_{\delta_0}(a)$ we obtain

$$\frac{d}{dt}\operatorname{Re}[a\,z(t)] = \operatorname{Re}[a\,\dot{z}(t)] = \operatorname{Re}[a\,q(t, z)] - \operatorname{Re}[a\,p(t)\,z^2] =$$

$$= \operatorname{Re}[a\,q(t, z)] - |a|^{-2}\operatorname{Re}[\bar{a}\,p(t)\,a^2z^2] =$$

$$= \operatorname{Re}[a\,q(t, z)] - |a|^{-2}\{\operatorname{Re}[\bar{a}\,p(t)]\operatorname{Re}[a^2z^2] -$$

$$- \operatorname{Im} [\bar{a}\, p(t)] \operatorname{Im} [a^2 z^2]\} \geqq -C |\operatorname{Im} [a^2 z^2]| - A \operatorname{Re} [a^2 z^2] -$$
$$- (B - C) |\operatorname{Im} [a^2 z^2]| \geqq -A \operatorname{Re} [a^2 z^2] - B |\operatorname{Im} [a^2 z^2]| > 0 .$$

If $z(t) = 0$ we have

$$\frac{d}{dt} \operatorname{Re} [a\, z(t)] = \operatorname{Re} [a\, q(t, 0)] > 0 \tag{3.9}$$

or

$$\frac{d}{dt} \operatorname{Re} [a\, z(t)] = \operatorname{Re} [a\, q(t, 0)] = 0 . \tag{3.10}$$

With respect to (3.8) we infer that

$$\frac{d}{dt} \operatorname{Im} [a\, z(t)] = \operatorname{Im} [a\, q(t, 0)] \neq 0$$

in the case (3.10). Taking into account that $\operatorname{Re} [az] = 0$ implies $z \in \Omega_{\delta_0}(a) \cup \{0\}$, we get $\operatorname{Re} [a\, z(t)] \geqq 0$ for all $t \geqq t_1$ for which $z(t)$ is defined. Clearly, $\operatorname{Re} [a\, z(t)] > 0$ if $z(t) \neq 0$.

Remark. If the condition (3.8) of Lemma 2 is replaced by $\operatorname{Re} [a\, q(t, 0)] > 0$, we get the assertion $\operatorname{Re} [a\, z(t)] > 0$ for all $t > t_1$ for which $z(t)$ exists.

Combining Lemma 2, Theorem 1' and Theorem 2', we obtain the following generalization of Theorem 1 of [7]:

Theorem 5. *Let the assumptions* (3.5), (3.6), (3.8) *and*

$$\operatorname{Re} [a\, q(t, z)] \geqq 0 \quad \text{for} \quad t \in I\,, \quad z \in \mathbb{C} \tag{3.11}$$

be satisfied. Suppose there exist $D(t) \in C(I)$ and $\delta \geqq 0$ such that

$$|q(t, z)| \leqq D(t) \quad \text{for} \quad t \in I\,, \quad z \in \mathbb{C}\,, \tag{3.12}$$
$$|a| \limsup_{t \to \infty} D(t) \leqq \delta^2 \liminf_{t \to \infty} \operatorname{Re} [\bar{a}\, p(t)] . \tag{3.13}$$

Then any solution $z(t)$ of (3.1) *satisfying* $\operatorname{Re} [a\, z(t_1)] \geqq 0$, *where $t_1 \geqq t_0$, satisfies the condition*

$$\liminf_{t \to \infty} |z(t)| \leqq \delta$$

and $\operatorname{Re} [a\, z(t)] \geqq 0$ *for $t \geqq t_1$.*

Proof. From Lemma 2 it follows that $\operatorname{Re} [a\, z(t)] \geqq 0$ for all $t \geqq t_1$ for which $z(t)$ exists. It is sufficient to prove that $z(t)$ exists for all $t \geqq t_1$ and that

$$\liminf_{t \to \infty} |z(t)| \leqq \delta^*$$

for any $\delta^* > \delta$· Choose $\delta_T > 0$ such that

$$|a|\, \delta_T^{-2}\, D(t) < \inf_{t \geqq t_0} \operatorname{Re} [\bar{a}\, p(t)] \quad \text{for} \quad t \geqq t_0$$

and put $\vartheta = \lambda_- = 1, s_j = t_0\,(j = 0, 1, 2, \ldots), E_T(t) = 2[|a|\,\delta_T^{-2}\,D(t) - \operatorname{Re}[\bar{a}\,p(t)]]$. Then

$$-G(t, z)\operatorname{Re}\left\{k\,h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} = 2\operatorname{Re}[\bar{a}z^{-2}\,q(t, z)] - 2\operatorname{Re}[\bar{a}\,p(t)] \leqq$$

$$\leqq 2|a|\,|z|^{-2}\,D(t) - 2\operatorname{Re}[\bar{a}\,p(t)]$$

and hence

$$-G(t, z)\operatorname{Re}\left\{k\,h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq 2|a|\delta_T^{-2}\,D(t) - 2\operatorname{Re}[\bar{a}\,p(t)] = E_T(t)$$

for $t \geqq t_0$, $z \in K(\infty, 1)$, $|z| > \delta_T$. In view of Lemma 2 we have $z(t) \in K(\infty, 1) \cup \{0\}$ for $t \in (t_1, \omega)$, where $[t_1, \omega)$ is the right maximal interval of existence of $z(t)$. Using Theorem 1′ we obtain $\omega = \infty$.

Put now $\delta_j = \delta^*$, $E_j(t) = 2[|a|\,\delta^{*-2}\,D(t) - \operatorname{Re}[\bar{a}\,p(t)]]$. For $t \geqq t_0$, $z \in K(\infty, 1)$, $|z| > \delta^*$ we have

$$-G(t, z)\operatorname{Re}\left\{k\,h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq 2[|a|\,\delta^{*-2}\,D(t) - \operatorname{Re}[\bar{a}\,p(t)]] = E_j(t).$$

Since

$$|a|\limsup_{t\to\infty} D(t) < \delta^{*2}\liminf_{t\to\infty}\operatorname{Re}[\bar{a}\,p(t)],$$

we have

$$\liminf_{t\to\infty}\int_{t_0}^{t} E_j(s)\,ds = -\infty.$$

By Theorem 2′ we get

$$\liminf_{t\to\infty}|z(t)| \leqq \delta^*.$$

Theorem 6. *Let the assumptions* (3.5), (3.6), (3.8) *and* (3.11) *be satisfied. Suppose there exist* $D(t) \in C(I)$ *and* $\delta \geqq 0$ *such that*

$$(3.14)\qquad |q(t, z)| \leqq D(t) \quad \text{for} \quad t \in I,\quad z \in \mathbb{C},$$

$$(3.15)\qquad \int_{t_0}^{\infty} D(t)\,dt < \infty.$$

Then any solution $z(t)$ *of* (3.1) *satisfying* $\operatorname{Re}[a\,z(t_1)] \geqq 0$, *where* $t_1 \geqq t_0$, *satisfies the condition*

$$\liminf_{t\to\infty}|z(t)| = 0$$

and $\operatorname{Re}[a\,z(t)] \geqq 0$ *for* $t \geqq t_1$.

Proof. Let $\delta > 0$ be arbitrary. For any $T > t_0$ choose $\delta_T > 0$ such that

$$|a|\,D(t) < \delta_T^2\inf_{t\geqq t_0}\operatorname{Re}[\bar{a}\,p(t)] \quad \text{for} \quad t \in [t_0, T),$$

and put $\vartheta = \lambda_- = 1, s_j = t_0\,(j = 0, 1, 2, \ldots), E_T(t) = 2[|a|\,\delta_T^{-2}\,D(t) - \operatorname{Re}[\bar{a}\,p(t)]]$.

Then

$$-G(t, z)\,\mathrm{Re}\left\{k\, h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq 2|a|\, \delta_T^{-2}\, D(t) - 2\,\mathrm{Re}\,[\bar{a}\, p(t)] = E_T(t)$$

for $t \geqq t_0$, $z \in K(\infty, 1)$, $|z| > \delta_T$, and $E_T(t) \leqq 0$ for $t \in [t_0, T)$. Because of Lemma 2 we have $z(t) \in K(\infty, 1) \cup \{0\}$ for $t \in (t_1, \omega)$, where $[t_1, \omega)$ is the right maximal interval of existence of $z(t)$. Making use of Theorem 1′ we get $\omega = \infty$.

Put now $\delta_j = \delta$, $E_j(t) = 2[|a|\, \delta^{-2}\, D(t) - \mathrm{Re}\,[\bar{a}\, p(t)]]$. As

$$-G(t, z)\,\mathrm{Re}\left\{k\, h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t)$$

for $t \geqq t_0$, $z \in K(\infty, 1)$, $|z| > \delta_j$ and

$$\liminf_{t \to \infty} \int_{t_0}^{t} E_j(s)\, \mathrm{d}s = -\infty$$

we obtain

$$\liminf_{t \to \infty} |z(t)| \leqq \delta$$

by Theorem 2′. Since $\delta > 0$ was chosen arbitrarily,

$$\liminf_{t \to \infty} |z(t)| = 0\,.$$

By virtue of Theorem we get 3

Theorem 7. *Let the assumptions of Theorem* 6 *be fulfilled and let*

$$\int_{t_0}^{\infty} |p(t) - a|\, \mathrm{d}t < \infty\,.$$

Then any solution $z(t)$ *of* (3.1) *satisfying* $\mathrm{Re}\,[a\, z(t_1)] \geqq 0$, *where* $t_1 \geqq t_0$, *satisfies the condition*

$$\lim_{t \to \infty} z(t) = 0\,.$$

Proof. Choose $R > 0$ and put $\Omega_1 = K(\infty, 1)$, $B(t) = D(t) + |p(t) - a|\, R^2$. Obviously

$$G(t, z)\,\mathrm{Re}\,\{\bar{z}[h(z) + g(t, z)]\} = \mathrm{Re}\,\{\bar{z}[q(t, z) - p(t)\, z^2]\} =$$
$$= -|z|^2\,\mathrm{Re}\,[az] + \mathrm{Re}\,\{\bar{z}[q(t, z) - (p(t) - a)\, z^2]\} \leqq$$
$$\leqq |z|\, |q(t, z) - (p(t) - a)\, z^2| \leqq$$
$$\leqq |z|\, [D(t) + |p(t) - a|\, R^2] = |z|\, B(t)$$

for $t \geqq t_0$, $z \in \Omega_1$, $|z| < R$. With respect to Theorem 6 and Lemma 2 the assumptions of Theorem 3 are satisfied with $\delta = 0$ and therefore

$$\lim_{t \to \infty} z(t) = 0\,.$$

Similarly we obtain the following generalization of Theorem 2 of [9]:

Theorem 8. *Let the assumptions of Theorem* 6 *be fulfilled and let* $\mathrm{Im}\,[\bar{a}\, p(t)] = 0$

for $t \geqq t_0$. *Then any solution* $z(t)$ *of* (3.1) *satisfying* $\operatorname{Re}[a\, z(t_1)] \geqq 0$, *where* $t_1 \geqq t_0$, *fulfils*

$$\lim_{t\to\infty} z(t) = 0\,.$$

Proof. Choose $R > 0$ and put $\Omega_1 = K(\infty, 1)$, $B(t) = D(t)$. It is clear that

$$G(t, z) \operatorname{Re}\{\bar{z}[h(z) + g(t, z)]\} = \operatorname{Re}\{\bar{z}[q(t, z) - p(t)\, z^2]\} =$$
$$= \operatorname{Re}[\bar{z}\, q(t, z)] - |z|^2 \operatorname{Re}[a^{-1}\, p(t)\, az] \leqq$$
$$\leqq |z|\, |q(t, z)| - |z|^2\, |a|^{-2} \operatorname{Re}[\bar{a}\, p(t)] \operatorname{Re}[az] \leqq |z|\, B(t)$$

for $t \geqq t_0$, $z \in \Omega_1$, $|z| < R$. In view of Theorem 6 and Lemma 2 the assumptions of Theorem 3 are satisfied with $\delta = 0$ and hence

$$\lim_{t\to\infty} z(t) = 0\,.$$

Using Theorem 4, we can generalize Theorem 1 of [9]:

Theorem 9. *Let the assumptions* (3.5), (3.8) *and* (3.11) *be satisfied. Assume there exists* $D(t) \in C(I)$ *such that*

$$|q(t, z)| \leqq D(t) \quad for \quad t \in I\,, \quad z \in \mathbb{C}\,,$$
$$\lim_{t\to\infty} D(t) = 0$$

and suppose

$$\lim_{t\to\infty} p(t) = a\,. \tag{3.16}$$

Then

$$\lim_{t\to\infty} z(t) = 0$$

for any solution $z(t)$ *of* (3.1) *satisfying* $\operatorname{Re}[a\, z(t_1)] \geqq 0$, *where* $t_1 \geqq t_0$.

Proof. Choose $R > 0$ and put $\Omega_1 = K(\infty, 1)$, $B(t) = D(t) + R^2|p(t) - a|$, $\delta = 0$. Let $j_0 \in \mathbb{N}$ be such that $j_0 > 3R^{-1}$. Set

$$a_j = |a|\, a^{-1}(j + j_0)^{-1}\,, \quad \alpha_j = 2(j + j_0)^{-1}\,.$$

In view of Theorem 5 and Lemma 2 we have $z(t) \in \Omega_1 \cup \{0\}$ for $t > t_1$ and

$$\liminf_{t\to\infty} |z(t)| = 0\,.$$

Putting $z = a_j + \alpha_j e^{i\vartheta}$, where $\vartheta \in \mathbb{R}$, we obtain

$$G(t, z) \operatorname{Re}\{(\bar{z} - \bar{a}_j)\,[h(z) + g(t, z)]\} =$$
$$= \operatorname{Re}\{(\bar{z} - \bar{a}_j)\,[-az^2 + g(t, z)]\} \leqq$$
$$\leqq \operatorname{Re}\{-\alpha_j e^{-i\vartheta}\, a(a_j + \alpha_j e^{i\vartheta})^2\} + |z - a_j|\, |g(t, z)| =$$
$$= \alpha_j\{\operatorname{Re}[-aa_j^2 e^{-i\vartheta} - 2a\alpha_j a_j - a\alpha_j^2 e^{i\vartheta}] + |g(t, z)|\}\,.$$

For $t > t_1$, $z \in K(\infty, 1) \cap \{z \in \mathbb{C}: |z - a_j| = \alpha_j\}$ we have $|z| \leqq |a_j| + \alpha_j \leqq$

$\leqq 3(j + j_0)^{-1} < R$ and therefore, using the inequality $\cos(\vartheta + \operatorname{Arg} a) \geqq -\cos\omega \geqq \geqq -|a_j|\, \alpha_j^{-1}$ (see Fig. 1), we get

$$\begin{aligned} &\operatorname{Re}\left[-a a_j^2 e^{-i\vartheta} - 2a\alpha_j a_j - a\alpha_j^2 e^{i\vartheta}\right] = \\ &= -|a|\, |a_j|^2 \cos(\vartheta + \operatorname{Arg} a) - 2\,\alpha_j |a|\, |a_j| - |a|\, \alpha_j^2 \cos(\vartheta + \operatorname{Arg} a) \leqq \\ &\leqq |a|\, |a_j|^3\, \alpha_j^{-1} - \alpha_j |a|\, |a_j| \end{aligned}$$

and

$$\begin{aligned} &G(t, z) \operatorname{Re}\{(\bar{z} - \bar{a}_j)\,[h(z) + g(t, z)]\} \leqq \\ &\leqq \alpha_j\big[|a|\, |a_j|^3\, \alpha_j^{-1} - \alpha_j |a|\, |a_j| + |q(t, z) + (a - p(t))\, z^2|\big] \leqq \\ &\leqq \alpha_j\big[|a|\, |a_j|\, \alpha_j^{-1}(|a_j|^2 - \alpha_j^2) + B(t)\big]\,. \end{aligned}$$

Since $|a_j| < \alpha_j$ and $B(t) \to 0$ as $t \to \infty$, it is clear that for any $j \in \mathbb{N}$ there is $\beta_j > t_1$ such that

$$G(t, z) \operatorname{Re}\{(\bar{z} - \bar{a}_j)\,[h(z) + g(t, z)]\} < 0$$

for $t > \beta_j$ and $z \in \Omega_1 \cap S(a_j, \alpha_j)$, $j \in \mathbb{N}$. Now all assumptions of Theorem 4 are fulfilled and the assertion follows from Theorem 4.

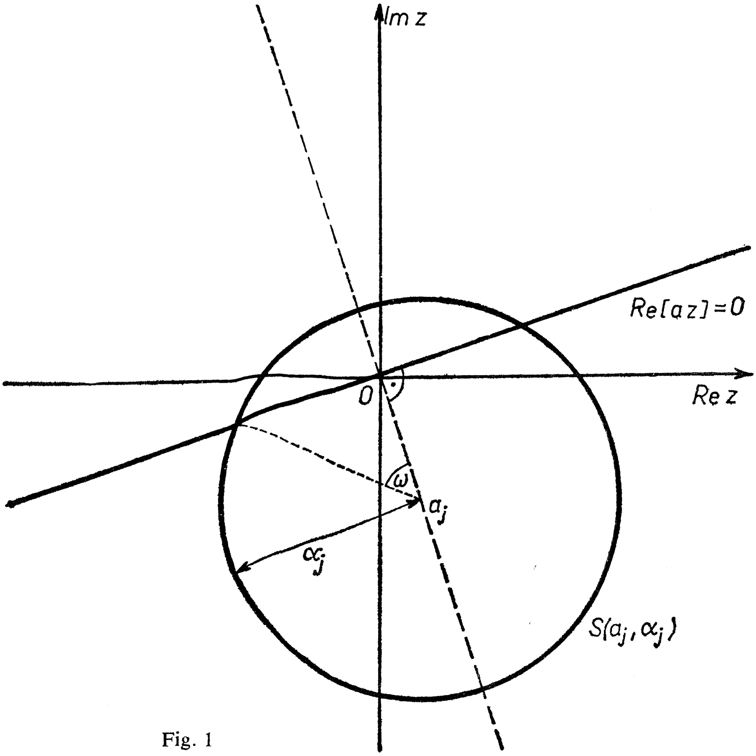

Fig. 1

Let $\alpha, \beta \in \tilde{C}^1(I)$, $\varrho \in \tilde{C}(I)$ and $\beta(t) \neq 0$ for $t \in I$. Defining functions $p(t)$, $q(t, z)$ as in Lemma 1 and combining Lemma 1 with Theorems 5–9, we obtain the following results concerning the equation (3.2):

Corollary 1. *Let the assumptions* (3.5), (3.6), (3.8) *and* (3.11) *be fulfilled. If there exist* $D(t) \in C(I)$ *and* $\delta \geqq 0$ *such that the conditions* (3.12) *and* (3.13) *hold, then any solution* $x(t)$ *of* (3.2) *satisfying*

$$(3.17) \qquad \operatorname{Re}\left[a(\beta(t_1)\,\dot{x}(t_1)\,x^{-1}(t_1) - \alpha(t_1))\right] \geqq 0\,,$$

where $t_1 \geqq t_0$, *fulfils the conditions*

$$\operatorname{Re}\left[a(\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t))\right] \geqq 0 \quad \textit{for} \quad t \geqq t_1\,,$$

$$\liminf_{t\to\infty} \left|\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)\right| \leqq \delta\,.$$

Corollary 2. *Let the assumptions* (3.5), (3.6), (3.8) *and* (3.11) *be fulfilled. Suppose there exist* $D(t) \in C(I)$ *and* $\delta \geqq 0$ *such that the conditions* (3.14) *and* (3.15) *hold. Then any solution* $x(t)$ *of* (3.2) *satisfying* (3.17), *where* $t_1 \geqq t_0$, *fulfils the conditions*

$$\operatorname{Re}\left[a(\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t))\right] \geqq 0 \quad \textit{for} \quad t \geqq t_1\,,$$

$$\liminf_{t\to\infty} \left|\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)\right| = 0\,.$$

Corollary 3. *Let the assumptions of Corollary* 2 *be fulfilled and let*

$$\int_{t_0}^{\infty} \left|p(t) - a\right| \mathrm{d}t < \infty\,.$$

Then any solution $x(t)$ *of* (3.2) *satisfying* (3.17), *where* $t_1 \geqq t_0$, *fulfils*

$$\lim_{t\to\infty} \left[\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)\right] = 0\,.$$

Corollary 4. *Let the assumptions of Corollary* 2 *be fulfilled and let* $\operatorname{Im}\left[\bar{a}\,p(t)\right] = 0$ *for* $t \geqq t_0$. *Then any solution* $x(t)$ *of* (3.2) *satisfying* (3.17), *where* $t_1 \geqq t_0$, *fulfils*

$$\lim_{t\to\infty} \left[\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)\right] = 0\,.$$

Corollary 5. *Let the assumptions* (3.5), (3.8), (3.11) *and* (3.16) *be satisfied. Assume there is* $D(t) \in C(I)$ *such that* (3.14) *and*

$$\lim_{t\to\infty} D(t) = 0$$

hold. Then

$$\lim_{t\to\infty} \left[\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)\right] = 0$$

for any solution $x(t)$ *of* (3.2) *satisfying* (3.17), *where* $t_1 \geqq t_0$.

Remark. Putting $\beta(t) \equiv 1$, $\alpha(t) = -\frac{1}{2}P(t)$, $\varrho(t) \equiv 0$, $a = 1$, $\psi(t, z) = -P(t)\,z - Q(t)$, where $P \in \tilde{C}^1(I)$, $Q \in \tilde{C}(I)$, we obtain several results from [9].

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