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ON INTEGRATION IN BANACH SPACES, XI (INTEGRATION WITH RESPECT TO POLYMEASURES)

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INTRODUCTION

In the case of integration with respect to an operator valued measure $m: \mathscr{P} \to \to L(X, Y)$ countably additive in the strong operator topology, $\mathscr{P}$-measurable functions $f: T \to X$ with continuous L_1-pseudonorm $\hat{m}(f, \cdot): \sigma(\mathscr{P}) \to [, +\infty]$ form a complete pseudonormed linear space $\mathscr{L}_1(m)$, which shares many important properties of the classical $\mathscr{L}_1(\mu)$ spaces, see Parts II – VII. In particular, the Lebesgue Dominated Convergence Theorem (LDCT) holds in $\mathscr{L}_1(m)$, see Theorem II.17.

Concerning integration with respect to a d-polymeasure $\Gamma: \mathsf{X}\mathscr{P}_i \to L^{(d)}(X_i; Y)$ separately countably additive in the strong operator topology, in Theorem IX.7 we extended the LDCT to the class $\hat{\mathscr{L}}_1(\Gamma)$ of integrable d-tuples of functions $(f_i) = = (f_1, \ldots, f_d)$ whose multiple L_1-gauge $\hat{\Gamma}_1$-gauge $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ is separately cpntinuous, see Definition 2 below. If $c_0 \not\subset Y$, then $f \in \mathscr{L}_1(m)$ if and only if f is $\mathscr{P}$-measurable and $\hat{m}(f, T) < +\infty$. For $d > 1$ the analog is not true for the class $\hat{\mathscr{L}}_1(\Gamma)$ Nonetheless, it is true for the greater class $\mathscr{L}_1(\Gamma)$ introduced by Definition 3. Namely, in Theorem 5 we prove the „if" part, and, postponing the case of dimensions $d > 2$ to the forthcoming Part XIII, in Theorem 8 we prove the implication $(f_i) \in \mathscr{L}_1(\Gamma) \Rightarrow \hat{\Gamma}[(f_i), (T_i)] < +\infty$ for $d = 2$.

That $\mathscr{L}_1(\Gamma)$ is the „right" class is confirmed by Theorem 6 (Fubini theorem in $\mathscr{L}_1(\Gamma)$) and Theorem 10 (LDCT in $\mathscr{L}_1(\Gamma)$). As a byproduct we explain why a third of the definition of strict MT integrability in [1] is enough, see the paragraph after Corollary 3 of Theorem 6.

We shall use freely the notation and concepts of the previous parts, treated as chapters, particularly the abbreviated notation from Part VIII.

THE CLASSES $\hat{\mathscr{L}}_1(\Gamma)$ AND $\mathscr{L}_1(\Gamma)$

Throughout this paper, if not specified otherwise, we assume that $\Gamma: \mathsf{X}\mathscr{P}_i \to \to L^{(d)}(X_i; Y)$ is a given operator valued d-polymeasure separately countably additive in the strong operator topology with locally σ-finite semivariation $\hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}_i)$, see the beginning of Part IX = [13].

Let us first introduce a useful notion.

Definition 1. For each $i = 1, \ldots, d$ let $f_i, f_{i,n}\colon T_i \to X_i$, $n = 1, 2, \ldots$ be $\mathscr{P}_i$-measurable functions. We say that the sequence of d-tuples $(f_{i,n})$, $n = 1, 2, \ldots$ *converges Γ-almost everywhere*, shortly Γ-a.e., to the d-tuple (f_i) if there are sets $N_i \in \sigma(\mathscr{P}_i)$ $i = 1, \ldots, d$ such that $\hat{\Gamma}(N_1, T_2, \ldots, T_d) = \ldots = \hat{\Gamma}(T_1, \ldots, T_{d-1}, N_d) = 0$, and $f_{i,n}(t_i) \to f_i(t_i)$ for each $t_i \in T_i - N_i$, $i = 1, \ldots, d$.

Obviously, in our previous theorems we may replace convergence everywhere of d-tuples of measurable or integrable functions by convergence Γ-a.e.

The title of this section indicates that there are two worthwhile generalizations of the space $\mathscr{L}_1(m)$. Having in mind the notions from Parts I and II let us recall that a function $g\colon T \to X$ belongs to $\mathscr{L}_1(m)$ if g is $\mathscr{P}$-measurable and its L_1-pseudonorm $\hat{m}(g, \cdot)\colon \sigma(\mathscr{P}) \to [0, +\infty]$ is continuous (equivalently, exhaustive). By Corollary of Theorem II.5 then $\hat{m}(g, T) < +\infty$. (Here is an elementary proof of this fact: Put $G = \{t \in T, g(t) \neq 0\}$, take $G_k \in \mathscr{P}$, $k = 1, 2, \ldots$ such that $G_k \nearrow G$ and $\hat{m}(G_k) < +\infty$ for each $k = 1, 2, \ldots$. Define $G'_k = G_k \cap \{t \in T, |g(t)| \leqq k\}$, $k = 1, 2, \ldots$. Then $G - G'_k \searrow \emptyset$, hence there is a k_1 such that $\hat{m}(g, G - G'_{k_1}) < 1$. But then $\hat{m}(g, T) = \hat{m}(g, G) \leqq \hat{m}(g, G'_{k_1}) + 1 \leqq k_1\, \hat{m}(G'_{k_1}) + 1 < +\infty$.) This suggests the following, as we shall see, „strong" generalization of $\mathscr{L}_1(m)$.

Definition 2. Let $g_i\colon T_i \to X_i$, $i = 1, \ldots, d$. We say that the d-tuple (g_i) belongs to $\hat{\mathscr{L}}_1(\Gamma)$ if g_i is $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$, and the L_1-gauge $\hat{\Gamma}[(g_i), (\cdot)]\colon \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ is separately continuous (equivalently, separately exhaustive). By Theorem VIII.6 then $\hat{\Gamma}[(g_i), (T_i)] < +\infty$.

The last fact may be again proved in an elementary way. The following lemma is also immediate.

Lemma 1. *Let $(g_i) \in \hat{\mathscr{L}}_1(\Gamma)$. Then:*

1) *If $(f_1, g_2, \ldots, g_d) \in \hat{\mathscr{L}}_1(\Gamma)$, then $(f_1 + g_1, g_2, \ldots, g_d) \in \hat{\mathscr{L}}_1(\Gamma)$. The analogs hold for the coordinates $i = 2, \ldots, d$.*

2) *If $f_i\colon T_i \to X_i$, $i = 1, \ldots, d$, are $\mathscr{P}_i$-measurable and $|f_i(t_i)| \leqq |g_i(t_i)|$ for Γ-almost every $(t_i) \in \mathsf{X}T_i$, then $(f_i) \in \hat{\mathscr{L}}_1(\Gamma)$.*

3) *If $\varphi_i\colon T_i \to K$, $i = 1, \ldots, d$, are bounded scalar valued $\mathscr{P}_i$-measurable functions, then $(\varphi_i g_i) \in \hat{\mathscr{L}}_1(\Gamma)$. Particularly $(a_i g_i) \in \hat{\mathscr{L}}_1(\Gamma)$ for any scalars a_i, $i = 1, \ldots, d$.*

4) *If $U\colon Y \to Z$ is a bounded linear operator, then $(g_i) \in \hat{\mathscr{L}}_1(U\Gamma)$.*

It is easy to see that Theorem IX.7, with the convergence Γ-almost everywhere, is a generalization to $\hat{\mathscr{L}}_1(\Gamma)$ of the Lebesgue Dominated Convergence Theorem in $\mathscr{L}_1(m)$, i.e., of Theorem II.17. By this theorem $\hat{\mathscr{L}}_1(\Gamma) \subset \mathscr{I}_1(\Gamma)$. The next theorem is a generalization of Theorems II.16 and V.1, i.e., of the Vitali Convergence Theorem in $\mathscr{L}_1(m)$.

Theorem 1. *For each $i = 1, \ldots, d$ let $f_i, f_{i,n}\colon T_i \to X_i$ be $\mathscr{P}_i$-measurable, let*

$(f_{i,n}) \in \hat{\mathscr{L}}_1(\Gamma)$ *for each* $n = 1, 2, \ldots$, *and let* $(f_{i,n}) \to (f_i)$ Γ*-almost everywhere. Then the following conditions are equivalent*:

a) $(f_i) \in \hat{\mathscr{L}}_1(\Gamma)$ *and* $\hat{\Gamma}[(f_{i,n}), (A_i)] \to \hat{\Gamma}[(f_i), (A_i)]$ *for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$;

b) *the* L_1*-gauges* $\hat{\Gamma}[(f_{i,n}), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$, $n = 1, 2, \ldots$ *are separately uniformly continuous (equivalently, separately uniformly exhaustive on* $\mathsf{X}\mathscr{P}_i$), *and*

c) $\hat{\Gamma}[(f_{i,n}), (A_i)] \to \hat{\Gamma}[(f_i), (A_i)]$ *uniformly with respect to* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$;

and if they hold, then

$$\lim_{n\to\infty} \int_{(A_i)} (f_{i,n})\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$$

uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Clearly a) $\Rightarrow$ b) by separate monotonocity and separate continuity of the L_1-gauge $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty)$. The equivalence in b) is a consequence of the Fatou property of the L_1-gauge, see Theorem VIII.4 and also Theorem 11 in [22].

b) $\Rightarrow$ c). For each $i = 1, \ldots, d$ put $F_i = \bigcup_{n=0}^{\infty} \{t_i \in T_i, f_{i,n}(t_i) \neq 0\}$, where $f_{i,0} = f_i$. By local σ-finiteness of the semivariation $\hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}_i)$, see the beginning of Part IX, there is a sequence of d-tuples of sets $(F_{i,k}) \in \mathsf{X}P_i$, $k = 1, 2, \ldots$ such that $F_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(F_{i,k}) < +\infty$ for each $k = 1, 2, \ldots$.

Owing to the Fatou property of the multiple L_1-gauge, see Theorem VIII.4, we have

$$\hat{\Gamma}[(f_i), (A_i)] = \hat{\Gamma}[(|f_i|), (A_i)] \leqq \liminf_n \hat{\Gamma}[(|f_{i,n}|), (A_i)]$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Hence $(f_i) \in \hat{\mathscr{L}}_1(\Gamma)$ by b). Thus $\hat{\Gamma}[(f_{i,n}), (T_i)] < +\infty$ for each $n = 0, 1, 2, \ldots$, where $(f_{i,0}) = (f_i)$.

For $A_1 \in \sigma(\mathscr{P}_1)$ put $\mu_1(A_1) = \sup \hat{\Gamma}[(f_{1,n}, f_{2,n}, \ldots, f_{d,n}), (A_1, T_2, \ldots, T_d)]$, $n \in \{0, 1, \ldots\}$. Similarly we define $\mu_i: \sigma(\mathscr{P}_i) \to [0, +\infty]$ for $i = 2, \ldots, d$. b) implies that each μ_i, $i = 1, \ldots, d$, is a subadditive semimeasure in the sense of Definition 1 in [22]. Since the Egoroff-Lusin theorem, see Section 1.4 in Part I, still holds if μ is a semimeasure, for each $i = 1, \ldots, d$ there are sets $N_i \in \sigma(\mathscr{P}_i)$ and $F'_{i,k} \in \mathscr{P}_i$, $k = 1, 2, \ldots$ such that $\mu_i(N_i) = 0$, $F'_{i,k} \nearrow F_i - N_i$, and on each $F'_{i,k}$, $k = 1, 2, \ldots$ the sequence $f_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to the function f_i.

Finaly, put $F^*_{i,k} = F_{i,k} \cap F'_{i,k} \cap \{t_i \in T_i, |f_i(t_i)| \leqq k\}$, $i = 1, \ldots, d$, and $k = 1, 2, \ldots$. Without loss of generality we may suppose that $|f_{i,n}(t_i)| \leqq 2k$ for each $t_i \in F^*_{i,k}$, $i = 1, \ldots, d$, and $k = 1, 2, \ldots$.

Let $\varepsilon > 0$. Since $F^*_{i,k} \nearrow F_i - N_i$ for each $i = 1, \ldots, d$, by b) there is a positive integer k_ε such that

$$\hat{\Gamma}[(f_i), (\ldots, T_{j-1}, F_j - N_j - F^*_{j,k_\varepsilon}, T_{j+1}, \ldots)] < \varepsilon/2d$$

for each $j = 1, \ldots, d$. Hence

$$\hat{\Gamma}[(f_i), (F^*_{i,k_\varepsilon})] \leqq \hat{\Gamma}[(f_i), (T_i)] = \hat{\Gamma}[(f_i), (F_i - N_i)] \leqq$$
$$\leqq \hat{\Gamma}[(f_i), (F^*_{i,k_\varepsilon})] + \varepsilon/2\,.$$

Since the uniformly bounded sequence $f_{i,n} \cdot \chi_{F^*_{i,k\varepsilon}}$, $n = 1, 2, \ldots$ converges uniformly to the function $f_i \cdot \chi_{F^*_{i,k\varepsilon}}$ for each $i = 1, \ldots, d$, and since $\hat{\Gamma}(F^*_{i,k_\varepsilon}) < +\infty$, there is a positive integer n_0 such that

$$\left|\hat{\Gamma}[(f_{i,n}), (F^*_{i,k_\varepsilon})] - \hat{\Gamma}[(f_i), (F^*_{i,k_\varepsilon})]\right| < \varepsilon/2$$

for $n \geqq n_0$. Hence b) $\Rightarrow$ c).

Trivially c) $\Rightarrow$ a).

The last assertion of the theorem is a consequence of Theorem X.11. The theorem is proved.

The following theorem is a generalization of Theorem II.5.

Theorem 2. *Let $(g_i) \in \hat{\mathscr{L}}_1(\Gamma)$. Then there are countably additive measures $\lambda_i\colon \sigma(\mathscr{P}_i) \to [0, 1]$, $i = 1, \ldots, d$, such that $\hat{\Gamma}[(g_i), (\ldots, T_{j-1}, \cdot, T_{j+1}, \ldots)]\colon \sigma(\mathscr{P}_j) \to [0, +\infty)$ is $(\delta - \varepsilon)$ (equivalently $(0 - 0)$) absolutely λ_j-continuous for each $j = 1, \ldots, d$.*

Proof. Let $\mathscr{S}\{(g_i)\} = \{(f_i) \in \mathsf{X}S(\mathscr{P}_i, X_i),\ |f_i| \leqq |g_i| \text{ for each } i = 1, \ldots, d\}$ and let $\mathscr{M}\{(g_i)\} = \{\omega_{(f_i)}, \omega_{(f_i)}(A_i) = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma,\ (f_i) \in \mathscr{S}\{(g_i)\}\}$. Since $\bar{\omega}_{(f_i)}(A_i) \leqq \hat{\Gamma}[(f_i), (A_i)] \leqq \hat{\Gamma}[(g_i), (A_i)]$ for each $(f_i) \in \mathscr{S}\{(g_i)\}$ and each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, the family $\mathscr{M}\{(g_i)\}$ of vector d-polymeasures on $\mathsf{X}\sigma(\mathscr{P}_i)$ is separately uniformly countably additive. Now the assertion of the theorem immediately follows from the well known result of Bartle, Dunford and Schwartz, see Theorem I.2.4 in [3] ($\hat{\Gamma}[(g_i), (T_i)] < +\infty$). The theorem is proved.

The next corollary is a generalization of the second part of the $*$-Theorem from Part I. Its validitiy is obvious.

Corollary. Let the semivariation $\hat{\Gamma}\colon \mathsf{X}\mathscr{P}_i \to [0, +\infty]$ be separately continuous. Then for each $(A_i) \in \mathsf{X}\mathscr{P}_i$ there are countably additive measures $\lambda_{j,(A_i)}\colon A_j \cap \mathscr{P}_j \to [0, 1]$, $j = 1, \ldots, d$, such that $\hat{\Gamma}(\ldots, A_{j-1}, \cdot, A_{j+1}, \ldots)\colon A_j \cap \mathscr{P}_j \to [0, +\infty)$ is $(\delta - \varepsilon)$ absolutely $\lambda_{j,(A_i)}$-continuous for each $j = 1, \ldots, d$. The analog holds if each $\mathscr{P}_i$, $i = 1, \ldots, d$ is replaced by $\sigma(\mathscr{P}_i)$, and in both cases the semivariation $\hat{\Gamma}$ has locally control d-polymeasure on $\mathsf{X}\sigma(\mathscr{P}_i)$.

We are now ready to prove the following generalization of Theorem V.4:

Theorem 3. *Let $(g_i) \in \hat{\mathscr{L}}_1(\Gamma)$. Then for each $\varepsilon > 0$ there is a positive integer N_ε such that whenever $i \in \{1, \ldots, d\}$, $f_{i,j}\colon T_i \to X_i$, $j = 1, \ldots, N_\varepsilon$ are $\mathscr{P}_i$-measurable and $\sum_{j=1}^{N_\varepsilon} |f_{i,j}| \leqq |g_i|$, then $\hat{\Gamma}[(\ldots, g_{i-1}, f_{i,j}, g_{i+1}, \ldots), (T_i)] < \varepsilon$ for at least one $j \in \{1, \ldots, N_\varepsilon\}$.*

Proof. Let $\varepsilon > 0$. Using theorem 2 and its Corollary, similarly as in the proof of Theorem V.4 we obtain positive integers $N_{i,\varepsilon}$, $i = 1, \ldots, d$ with the corresponding properties. Clearly $N_\varepsilon = \max\{N_{i,\varepsilon},\ i \in \{1, \ldots, d\}\}$ has the required property. The theorem is proved.

Let us again have the setting of Parts I and II. If Y does not contain a subspace isomorphic to the space c_0, shortly if $c_0 \not\subset Y$, then by Theorem II.5 each $\mathscr{P}$-measurable function $g: T \to X$ with finite L_1-pseudonorm $\hat{m}(g, T) < +\infty$ belongs to $\mathscr{L}_1(m)$. Since, due to the nice example of Hans Weber, see Part VIII, there are Hilbert space valued bimeasures defined on the Cartesian product of two σ-rings which are not uniform bimeasures, the analog of Theorem II.5 for $\hat{\mathscr{L}}_1(\Gamma)$ for $d > 1$ does not holds. The idea how to define the „right" $\mathscr{L}_1(\Gamma)$ came from the following simple characterization of elements of $\mathscr{L}_1(m)$. This theorem may be proved similarly as Theorem 1 in [15].

Theorem 4. *Let $g: T \to X$. Then $g \in \mathscr{L}_1(m)$ if and only if g is $\mathscr{P}$-measurable and each $\mathscr{P}$-measurable function $f: T \to X$ with $|f| \leqq |g|$ is integrable.*

Definition 3. Let $g_i: T_i \to X_i$, $i = 1, \ldots, d$. We say that (g_i) *belongs to* $\mathscr{L}_1(\Gamma)$ if g_i is $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$, and for any $\mathscr{P}_i$-measurable functions $f_i: T_i \to X_i$, $i = 1, \ldots, d$, the inequalities $|f_i| \leqq |g_i|$ for each $i = 1, \ldots, d$ imply that (f_i) is an integrable d-tuple, i.e., $(f_i) \in \mathscr{I}(\Gamma)$.

Obviously $\hat{\mathscr{L}}_1(\Gamma) \subset \mathscr{L}_1(\Gamma)$. Further, we immediately obtain

Lemma 2. *The assertions of Lemma 1 still hold if $\hat{\mathscr{L}}_1(\Gamma)$ is replaced by $\mathscr{L}_1(\Gamma)$.*

For a $\mathscr{P}_i$-measurable function $g_i: T_i \to X_i$, $i \in \{1, \ldots, d\}$ and $k = 1, 2, \ldots$ put

$$\mathscr{P}_{g_i,k} = \{t_i \in T_i,\ g_i(t_i)| \geqq 1/k\} \cap \mathscr{P}_i$$

and let $\mathscr{P}_{g_i} = \bigcup_{k=1}^{\infty} \mathscr{P}_{g_i,k}$. Then $\mathscr{P}_{g_i}$ is evidently a δ-ring and g_i is $\mathscr{P}_{g_i}$-measurable. We shall use this notation as well as the following fact.

Theorem 5. *Let $c_0 \not\subset Y$, let $g_i: T_i \to X_i$ be $\mathscr{P}_i$-measurable, $i = 1, \ldots, d$, and let $\hat{\Gamma}[(g_i), (T_i)] < +\infty$. Then $(g_i) \in \mathscr{L}_1(\Gamma)$.*

Proof. For every $i = 1, \ldots, d$ take a sequence $g_{i,n} \in S(\mathscr{P}_i, X_i)$ such that $g_{i,n}(t_i) \to g_i(t_i)$ and $|g_{i,n}(t_i)| \nearrow |g_i(t_i)|$ for each $t_i \in T_i$. Clearly $g_{i,n} \in S(\mathscr{P}_{g_i}, X_i)$ for each $n = 1, 2, \ldots$. Let $\Gamma' = \Gamma: \mathsf{X}\mathscr{P}_{g_i} \to L^{(d)}(X_i; Y)$.

For any given $n_2, \ldots, n_d \in \{1, 2, \ldots\}$ and $A_i \in \sigma(\mathscr{P}_{g_i})$, $i = 2, \ldots, d$, the mapping $(A_1, x_1) \to \int_{(A_i)} (x_1 \cdot \chi_{A_1}, g_{2,n_2}, \ldots, g_{d,n_d}) \, d\Gamma'$, $A_1 \in \mathscr{P}_{g_1}$, $x_1 \in X_1$, defines a measure $m_{1,(A_2,\ldots,A_d),(n_2,\ldots,n_d)}: \mathscr{P}_{g_1} \to L(X_1, Y)$ countably additive in the strong operator topology, whose semivariation on a set $A_1 \in \mathscr{P}_{g_1,k}$ is bounded by $k \cdot \hat{\Gamma}[(g_i, (T_i)]$ for all $n_2, \ldots, n_d = 1, 2, \ldots$, and all $(A_2, \ldots, A_d) \in \sigma(\mathscr{P}_{g_2}) \times \ldots \times \sigma(\mathscr{P}_{g_d})$. By symmetry in coordinates the analogs hold for the coordinates $i = 2, \ldots, d$.

Since obviously $\hat{m}_{1,(\cdot),(\cdot)}(g_1, T_1) \leqq \hat{\Gamma}[(g_i), (T_i)] < +\infty$, and $c_0 \not\subset Y$ by assumption, $g_1 \in \mathscr{L}_1(m_{1,(\cdot),(\cdot)})$ by Theorem II.5. Thus according to Definition I.2 the function g_1 is integrable with respect to the measure $m_{1,(\cdot),(\cdot)}$, and $\int_{A_1} g_1 \, dm_{1,(\cdot),(\cdot)} = \lim_{n_1 \to \infty} \int_{A_1} g_{1,n_1} \, dm_{1,(\cdot),(\cdot)} \in Y$ exists for each $A_1 \in \sigma(\mathscr{P}_{g_1})$, hence also for each $A_1 \in \sigma(\mathscr{P}_1)$. Since this is true for every $n_2, \ldots, n_d = 1, 2, \ldots$ and every $A_i \in \sigma(\mathscr{P}_{g_i})$,

$i = 2, \ldots, d$, and since clearly $\int_{A_1} g_{1,n_1} \, \mathrm{d}m_{1,(\cdot),(\cdot)} = \int_{(A_i)} (g_{i,n_i}) \, \mathrm{d}\Gamma$, we immediately obtain that $(g_1, g_{2,n_2}, \ldots, g_{d,n_d}) \in \mathscr{I}_1(\Gamma)$ and

$$\int_{(A_i)} (g_1, g_{2,n_2}, \ldots, g_{d,n_d}) \, \mathrm{d}\Gamma = \lim_{n_1 \to \infty} \int_{(A_i)} (g_{i,n_i}) \, \mathrm{d}\Gamma$$

for each $n_2, \ldots, n_d = 1, 2, \ldots$ and each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Similarly $(g_1, x_2 \cdot \chi_{A_2}, g_{3,n_3}, \ldots, g_{d,n_d}) \in \mathscr{I}_1(\Gamma)$ and

$$\int_{(A_i)} (g_1, x_2 \cdot \chi_{A_2}, g_{3,n_3}, \ldots, g_{d,n_d}) \, \mathrm{d}\Gamma =$$
$$= \lim_{n_1 \to \infty} \int_{(A_i)} (g_{1,n_1}, x_2 \cdot \chi_{A_2}, g_{3,n_3}, \ldots, g_{d,n_d}) \, \mathrm{d}\Gamma$$

for each $x_2 \in X_2$ and each $A_2 \in \sigma(\mathscr{P}_{g_2})$, for any given $n_3, \ldots, n_d$ and $A_1, A_3, \ldots, A_d$. This equality implies that for any given $n_3, \ldots, n_d$ and $A_1, A_3, \ldots, A_d$ the mapping $(A_2, x_2) \to \int_{(A_i)} (g_1, x_2 \cdot \chi_{A_2}, g_{3,n_3}, \ldots, g_{d,n_d}) \, \mathrm{d}\Gamma'$, $A_2 \in \mathscr{P}_{g_2}$, $x_2 \in X_2$, defines a measure $m_{2,(\cdot),(\cdot)}: \mathscr{P}_{g_2} \to L(X_2, Y)$ countably additive in the strong operator topology, whose semivariation on a set $A_2 \in \mathscr{P}_{g_2,k}$ is bounded by k. $\hat{\Gamma}[(g_i), (T_i)] < +\infty$. Continuing as above we obtain that $(g_1, g_2, g_{3,n_3}, \ldots, g_{d,n_d}) \in \mathscr{I}_2(\Gamma)$, and

$$\int_{(A_i)} (g_1, g_2, g_{3,n_3}, \ldots, g_{d,n_d}) \, \mathrm{d}\Gamma = \lim_{n_2 \to \infty} \lim_{n_1 \to \infty} \int_{(A_i)} (g_{i,n_i}) \, \mathrm{d}\Gamma$$

for each $n_3, \ldots, n_d = 1, 2, \ldots$ and each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Continuing in this manner we finally obtain that $(g_i) \in \mathscr{I}_d(\Gamma)$, and

$$\int_{(A_i)} (g_i) \, \mathrm{d}\Gamma = \lim_{n_d \to \infty} \ldots \lim_{n_1 \to \infty} \int_{(A_i)} (g_{i,n_i}) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Let us note that by symmetry in coordinates the analogs are valid for any permutation of $\{1, \ldots, d\}$. Note finally that in Theorem 10 below we show that $\mathscr{L}_1(\Gamma) \subset$ $\subset \mathscr{I}_1(\Gamma)$ in general. The theorem is proved.

Suppose now that each X_i, $i = 1, \ldots, d$. is finite dimensional. Then according to Corollary of Theorem X.5 we have $\mathscr{I}(\Gamma) = \mathscr{I}_1(\Gamma)$. Further $\hat{\Gamma}[(g_i), (T_i)] < +\infty$ for each $(g_i) \in \mathscr{I}(\Gamma)$ by Theorems VIII.2 and VIII.3. Hence we have obtained the following

Corollary. *Let each X_i, $i = 1, 2, \ldots$ be finite dimensional and let $c_0 \not\subset Y$. Then $\mathscr{I}(\Gamma) = \mathscr{I}_1(\Gamma) = \mathscr{L}_1(\Gamma)$.*

One of our most important results is the following

Theorem 6. (The Fubini Theorem in $\mathscr{L}_1(\Gamma)$.) *Let $(g_i) \in \mathscr{L}_1(\Gamma)$, let $d > 1$ and let d_1 be a positive integer such that $1 \leqq d_1 < d$. Then:*

0) $(g_i) \in \mathscr{I}_d(\Gamma)$;

1) $(g_1, \ldots, g_{d_1}, x_{d_1+1} \cdot \chi_{A_{d_1+1}}, \ldots, x_d \cdot \chi_{A_d}) \in \mathscr{I}(\Gamma)$ *for each* $x_i \in X_i$ *and* $A_i \in \mathscr{P}_{g_i}$, $i = d_1 + 1, \ldots, d$.

2) *Let* $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_1) \times \ldots \times \sigma(\mathscr{P}_{d_1})$ *be given. For each* $x_i \in X_i$ *and*

$A_i \in \mathscr{P}_{g_i}$, $i = d_1 + 1, \ldots, d$ *put*

$$\Gamma_{d_1,(A_1,\ldots,A_{d_1})}(A_{d_1+1}, \ldots, A_d)(x_{d_1+1}, \ldots, x_d) =$$
$$= (\int_{(A_1,\ldots,A_{d_1})} (g_1, \ldots, g_{d_1}, \ldots) \, \mathrm{d}\Gamma)(A_{d_1+1}, \ldots, A_d)(x_{d_1+1}, \ldots, x_d) =$$
$$= \int_{(A_i)} (g_1, \ldots, g_{d_1}, x_{d_1+1} \cdot \chi_{A_{d_1+1}}, \ldots, x_d \cdot \chi_{A_d}) \, \mathrm{d}\Gamma \, .$$

Then $\Gamma_{d_1,(\cdot)}: \mathscr{P}_{gd_1+1} \times \ldots \times \mathscr{P}_{gd} \to L^{(d-d_1)}(X_{d_1+1}, \ldots, X_d; Y)$, $\Gamma_{d_1,(\cdot)}$ *is separately countably additive in the strong operator topology, and its semivariation* $\hat{\Gamma}_{d_1,(\cdot)}$ *is finite valued on* $\mathscr{P}_{gd_1+1,k} \times \ldots \times \mathscr{P}_{gd,k}$ *for each* $k = 1, 2, \ldots$. *Hence* $\hat{\Gamma}_{d_1(\cdot)}$ *is locally σ-finite on* $\mathscr{P}_{gd_1+1} \times \ldots \times \mathscr{P}_{gd}$ *and also on* $\sigma(\mathscr{P}_{gd_1+1}) \times \ldots \times \sigma(\mathscr{P}_{gd})$.

3) $(g_{d_1+1}, \ldots, g_d) \in \mathscr{L}_1(\Gamma_{d_1(\cdot)})$ *for each* $(\cdot) = (A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_1) \times \ldots \times \sigma(\mathscr{P}_{d_1})$,

$$\hat{\Gamma}_{d_1,(\cdot)}[(g_{d_1+1}, \ldots, g_d), (A_{d_1+1}, \ldots, A_d)] \leqq \hat{\Gamma}[(g_i), (A_i)], \text{ and}$$
$$\int_{(A_{d_1+1},\ldots,A_d)} (g_{d_1+1}, \ldots, g_d) \, \mathrm{d}\Gamma_{d_1,(\cdot)} = \int_{(A_i)} (g_i) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *and*

4) $$\int_{(A_{d_1+1},\ldots,A_d)} (g_{d_1+1}, \ldots, g_d) \, \mathrm{d}(\int_{(A_1,\ldots,A_{d_1},\ldots)} (g_1, \ldots, g_{d_1}, \ldots) \, \mathrm{d}\Gamma) = \int_{(A_i)} (g_i) \, \mathrm{d}\Gamma =$$
$$= \int_{(A_1,\ldots,A_{d_1})} (g_1, \ldots, g_{d_1}) \, \mathrm{d} \int_{(\ldots,A_{d_1+1},\ldots,A_d)} (\ldots, g_{d_1+1}, \ldots, g_d) \, \mathrm{d}\Gamma)$$

for each $(A_i) \in \chi\, \sigma(\mathscr{P}_i)$, *where the* d_1 *polymeasure*

$$\int_{(\ldots,A_{d_1+1},\ldots,A_d)} (\ldots, g_{d_1+1}, \ldots, g_d) \, \mathrm{d}\Gamma: \mathscr{P}_{g_1} \times \ldots \times \mathscr{P}_{gd_1} \to$$
$$\to L^{(d_1)}(X_1, \ldots, X_{d_1}; Y)$$

is defined similarly as the $(d - d_1)$*-polymeasure in* 2), *and has similar properties as the latter.*

Proof. We now prove the theorem under the additional assumption that $\hat{\Gamma}[(g_i), (T_i)] < +\infty$. In Theorem 8 below we prove that for $d = 2$, $(g_i) \in \mathscr{L}_1(\Gamma) \Rightarrow \Rightarrow \hat{\Gamma}[(g_i), (T_i)] < +\infty$. In the forthcoming Part XIII, Theorem 12 we will prove this implication for an arbitrary dimension d.

Having this additional assumption we first show that the proof of Theorem 5 remains valid in this new situation. To this end we must show that $g_1 \in \mathscr{L}_1(m_{1,(\cdot),(\cdot)})$ in the notation of this proof. According to Theorem 4 we must prove that $f_1: T_1 \to X_1$ is integrable with respect to the measure $m_{1(\cdot),(\cdot)}$ provided f_1 is $\mathscr{P}_{g_1}$ measurable and $|f_1| \leqq |g_1|$. Let f_1 be such a function.

Since $(f_1, g_{2,n_2}, \ldots, g_{d,n_2}) \in \mathscr{I}(\Gamma)$ by the definition of $\mathscr{L}_1(\Gamma)$, the set function v_0, $v_0(A_1) = \int_{(A_1,\ldots,A_d)} (f_1, g_{2,n_2}, \ldots, g_{d,nd}) \, \mathrm{d}\Gamma$, $A_1 \in \sigma(\mathscr{P}_1)$, is a countably additive vector measure by the separate countable additivity of the indefinite integral with respect to Γ, see Theorems IX.3 and IX.4.

Let $f_{1,n}: T_1 \to X_1$, $n = 1, 2, \ldots$ be a sequence of $\mathscr{P}_{g_1}$ measurable functions such that $f_{1,n} \to f_1$ and $|f_{1,n}| \nearrow |f_1|$. Since the semivariation $\hat{m}_{1,(\cdot),(\cdot)}$ is σ-finite on $\mathscr{P}_{g_1}$, each $f_{1,n}$, $n = 1, 2, \ldots$ is integrable with respect to $m_{1,(\cdot),(\cdot)}$.

For $A_1 \in \sigma(\mathscr{P}_{g_1})$ put $v_n(A_1) = \int_{A_1} f_{1,n} \, \mathrm{d}m_{1,(\cdot)(\cdot)} =$

$= \int_{(A_1,\dots,A_d)} (f_1, g_{2,n_2}, \dots, g_{d,n_d})\, \mathrm{d}\Gamma$, $n = 1, 2, \dots$, and let

$$\mu(A_1) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\bar{v}_n(A_1)}{1 + \bar{v}_n(T_1)}\,.$$

Then $\mu: \sigma(\mathscr{P}_{g_1}) \to [0, 2]$ is a subadditive submeasure in the sense of Definition 1 in [21] ($\bar{v}_n(A_1) = \sup\{|v_n(B_1)|,\ B_1 \in \sigma(\mathscr{P}_{g_1}),\ B_1 \subset A_1\}$, i.e. $\bar{v}_n$ is the supremation of v_n, see Definition VIII.2).

Put $F = \{t_1 \in T_1,\ f_1(t_1) \neq 0\} \in \sigma(\mathscr{P}_{g_1})$. According to Egoroff-Lusin theorem, see Section 1.4 in [5], which remains valid for the subadditive submeasure μ, there is a set $N \in \sigma(\mathscr{P}_{g_1})$ with $\mu(N) = 0$ and a sequence of sets $F_k \in \mathscr{P}_{g_1}$, $k = 1, 2, \dots$ such that $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \dots$ the sequence $f_{1,n}$, $n = 1, 2, \dots$ converges uniformly to the function f_1. Since the semivariation $\hat{m}_{1,(\cdot),(\cdot)}(F)$ is σ-finite, without loss of generality we may and will suppose that $\hat{m}_{1,(\cdot),(\cdot)}(F_k) < +\infty$ for each $k = 1, 2, \dots$. But then, clearly, the functions $f_1 \cdot \chi_{F_k \cup N}$, $k = 1, 2, \dots$ are integrable with respect to the measure $m_{1,(\cdot),(\cdot)}$ and

$$\int_{A_1} f_1 \cdot \chi_{F_k \cup N}\, \mathrm{d}m_{1,(\cdot),(\cdot)} = \lim_{n\to\infty} \int_{A_1} f_{1,n} \cdot \chi_{F_k \cup N}\, \mathrm{d}m_{1,(\cdot),(\cdot)} =$$
$$= \lim_{n\to\infty} \int_{(A_1,\dots,A_d)} (f_{1,n} \cdot \chi_{F_k \cup N}, g_{2,n_2}, \dots, g_{d,n_d})\, \mathrm{d}\Gamma =$$
$$= \int_{(A_1,\dots,A_d)} (f_1 \cdot \chi_{F_k \cup N}, g_{2,n_2}, \dots, g_{d,n_d})\, \mathrm{d}\Gamma = v_0(A_1 \cap (F_k \cup N))$$

for each $A_1 \in \sigma(\mathscr{P}_{g_1})$ and each $k = 1, 2, \dots$. Since $f_1 \cdot \chi_{F_k \cup N} \to f_1$, and since the indefinite integrals $\int f_1 \cdot \chi_{F_k \cup N}\, \mathrm{d}m_{1(\cdot),(\cdot)} = v_0(\cdot \cap (F_k \cup N))$, $k = 1, 2, \dots$ are uniformly countably additive by the countable additivity of the vector measure v_0: $\sigma(\mathscr{P}_{g_1}) \to Y$, f_1 is integrable with respect to the measure $m_{1,(\cdot),(\cdot)}$ and $\int_{A_1} f_1\, \mathrm{d}m_{1,(\cdot),(\cdot)} = v_0(A_1)$ for each $A_1 \in \sigma(\mathscr{P}_{g_1})$ by Theorem I.16. Hence $g_1 \in \mathscr{L}_1(m_{1,(\cdot),(\cdot)})$, which we wanted to show.

Thus the rest of the proof of Theorem 5 remains valid under the given assumptions of this proof. Hence $(g_i) \in \mathscr{I}_d(\Gamma)$ and

$$(*) \qquad \int_{(A_i)} (g_i)\, \mathrm{d}\Gamma = \lim_{n_{p(d)}\to\infty} \dots \lim_{n_{p(1)}\to\infty} \int_{(A_i)} (g_{i,n_i})\, \mathrm{d}\Gamma$$

(for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each permutation p of $\{1, \dots, d\}$.

Now 0) immediately follows from (*).

1) is clear from the definitions of $\mathscr{P}_{g_i}$ and of $\mathscr{L}_1(\Gamma)$.

2) follows from (*) by the uniform boundedness principle and by the Vitali-Hahn-Saks-Nikodým-(VHSN)-Theorem for polymeasures, see the beginning of Part VIII.

3) and 4) are direct consequences of (*) and of the corresponding definitions. The theorem is proved.

We immediately have the following

Corollary 1. *Let $(g_i) \in \mathscr{L}_1(\Gamma)$. Then the integrals on the right hand side below exist, the polymeasures obtained are separately countably additive in the strong* *operator topologies and have σ-finite semivariations, and*

$$\int_{(A_i)} (g_i)\, \mathrm{d}\Gamma = \int_{A_d} g_d\, \mathrm{d}\left(\int_{(A_{d-1},\cdot)} (g_{d-1}, \cdot)\, \mathrm{d}(\ldots \mathrm{d}\left(\int_{(A_2,\ldots)} (g_2, \ldots) \,.\right.\right.$$
$$\left.\left. . \,\mathrm{d}\left(\int_{(A_1,\ldots)} (g_1, \ldots)\, \mathrm{d}\Gamma\right)\right) \ldots\right)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{g_i})$. The analogs hold for all permutations of $\{1, \ldots, d\}$ and all decompositions of d as a sum of positive integers.

The next corollary requires the following

Remark. Let $m: \mathscr{P}_0 \to L(X, Y)$ be countably additive in the strong operator topology, let $g: T \to X$ be $\mathscr{P}_0$-measurable, and let its L_1-pseudonorm $\hat{m}(g, \cdot)$: $\sigma(\mathscr{P}_0) \to [0, +\infty]$ be continuous. Then $\hat{m}(g, T) < +\infty$ by Corollary of Theorem II.5 (now the simple proof at the beginning of this section does not work since the semivariation $\hat{m}$ may take the value $+\infty$ on some sets of $\mathscr{P}_0$, see Section 1.1 in Part I). Hence $\mathscr{P}_{g,k} \subset \tilde{\mathscr{P}} \subset \mathscr{P}$ for each $k = 1, 2, \ldots$ by the Tschebyscheff inequality, see Corollary of Theorem II.1. Thus $\mathscr{P}_g \subset \sigma(\mathscr{P})$, hence g is $\mathscr{P}$-measurable. This implies that $g \in \mathscr{L}_1(m)$. In this way the requirement of σ-finiteness of the semivariation $\hat{m}$ is not needed for the definition of $\mathscr{L}_1(m)$. We use this fact in the following

Corollary 2. *Let g_i: $T_i \to X_i$ be $\mathscr{P}_i$-measurable, $i = 1, \ldots, d$. Then the following two conditions are equivalent:*

a) *$(g_i) \in \mathscr{L}_1(\Gamma)$, and*

b) *the following d conditions hold:*

1) *$g_1 \in \mathscr{L}_1(_{(\cdot,A_2,\ldots,A_d)} (\cdot, x_2 \,.\, \chi_{A_2}, \ldots, x_d \,.\, \chi_{A_d})\, \mathrm{d}\Gamma)$ for each $(A_2, \ldots, A_d) \in \mathscr{P}_{g_2} \times \ldots$ $\ldots \times \mathscr{P}_{g_d}$ and each $(x_2, \ldots, x_d) \in X_2 \times \ldots \times X_d$ in the sense of the preceding Remark. This implies:*

A) *For each $(A_2, \ldots, A_d) \in \mathscr{P}_{g_2} \times \ldots \times \mathrm{P}_{g_d}$ and each $(x_2, \ldots, x_d) \in X_2 \times \ldots \times X_d$ the measure $\int_{(\cdot,A_2,\ldots,A_d)} (\cdot, x_2 \,.\, \chi_{A_2}, \ldots, x_d \,.\, \chi_{A_d})\, \mathrm{d}\Gamma$: $\mathscr{P}_{g_1} \to L(X_1, Y)$, countably additive in the strong operator topology, has σ-finite semivariation on $\mathscr{P}_{g_1}$, g_1 is integrable with respect to this measure, $(g_1, x_2 \,.\, \chi_{A_2}, \ldots, x_d \,.\, \chi_{A_d}) \in \mathscr{I}(\Gamma)$, and*

$$\int_{A_1} g_1\, \mathrm{d}\left(\int_{(\cdot,A_2,\ldots,A_d)} (\cdot, x_2 \,.\, \chi_{A_2}, \ldots, x_d \,.\, \chi_{A_d})\, \mathrm{d}\Gamma\right) =$$
$$= \int_{(A_i)} (g_1, x_2 \,.\, \chi_{A_2}, \ldots, x_d \,.\, \chi_{A_d})\, \mathrm{d}\Gamma$$

for each $A_1 \in \sigma(\mathscr{P}_{g_i})$;

B) *for each $A_1 \in \sigma(\mathscr{P}_{g_1})$ we have $\int_{(A_1,\ldots)} (g_1, \ldots)\, \mathrm{d}\Gamma$: $\mathscr{P}_{g_2} \times \ldots \times \mathscr{P}_{g_d} \to$ $\to L^{(d-1)}(X_2, \ldots, X_d; Y)$, and it is separately countably additive in the strong operator topology.*

2) *$g_2 \in \mathscr{L}_1(\int_{(A_1,\cdot,A_3,\ldots,A_d)} (g_1, \cdot, x_3 \,.\, \chi_{A_3}, \ldots, x_d \,.\, \chi_{A_d})\, \mathrm{d}\Gamma$ for each $(A_1, A_3, \ldots, A_d) \in \sigma(\mathscr{P}_{g_1}) \times \mathscr{P}_{g_3} \times \ldots \times \mathscr{P}_{g_d}$ and each $(x_3, \ldots, x_d) \in X_3 \times \ldots \times X_d$ in the sense of the preceding Remark. This implies:*

. . . .

… d) *$g_d \in \mathscr{L}_1(\int_{(A_1,\ldots,A_{d-1},\cdot)} (g_1, \ldots, g_{d-1}, \cdot)\, \mathrm{d}\Gamma)$ for each $(A_1, \ldots, A_{d-1}) \in$ $\in \sigma(\mathscr{P}_{g_1}) \times \ldots \times \sigma(\mathscr{P}_{g_{d-1}})$ in the sense of the preceding Remark. This implies*

A) *For each* $(A_1, \dots, A_{d-1}) \in \sigma(\mathscr{P}_{g_1}) \times \dots \times \sigma(\mathscr{P}_{g_{d-1}})$ *the measure* $\int_{(A_1,\dots,A_{d-1},\cdot)} (g_1, \dots, g_{d-1}, \cdot) \, \mathrm{d}\Gamma : \mathscr{P}_{g_d} \to L(X_d, Y)$, *countably additive in the strong operator topology, has* σ*-finite semivariation on* $\mathscr{P}_{g_d}$, g_d *is integrable with respect to this measure,* $(g_i) \in \mathscr{I}(\Gamma)$, *and*

$$\int_{A_d} g_d \, \mathrm{d}\left(\int_{(A_1,\dots,A_{d-1},\cdot)} (g_1, \dots, g_{d-1}, \cdot) \, \mathrm{d}\Gamma\right) = \int_{(A_i)} (g_i) \, \mathrm{d}\Gamma$$

for each $A_d \in \sigma(\mathscr{P}_{g_d})$.

If a) holds, then the analogs of b) hold for all permutations p of $\{1, \dots, d\}$.

If $X = K$ is the space of scalars, and $m: \mathscr{P} \to L(K, Y) = Y$ is a countably additive vector measure, then $\mathscr{I}(m) = \mathscr{L}_1(m)$, see Part II. Hence we easily obtain

Corollary 3. *Let each* X_i, $i = 1, \dots, d$ *be a finite dimensional Banach space. Then in* b) *of the preceding Corollary 2 we may replace the requirements* $g_i \in \mathscr{L}_1(\dots)$, $i = 1, \dots, d$ *by the requirements* $g_i \in \mathscr{I}(\dots)$, $i = 1, \dots, d$.

For the particular case $d = 2, X_1 = X_2 = Y = C$ – the field of complex numbers, and a separately countably additive bimeasure $\beta: \mathscr{P}_1 \times \mathscr{P}_2 \to C$ in Definition 2.6 in [1], see also [2], the concept of the strict β-integrability of a pair (g_1, g_2) of $\mathscr{P}_i$-measurable functions $g_i: T_i \to C$, $i = 1, 2$ was introduced by the following three requirements. in our notation:

(i) $g_1 \in \mathscr{I}(\beta(\cdot, A_2))$ for each $A_2 \in \mathscr{P}_{g_2}$, and $g_2 \in \mathscr{I}(\int_{(A_1,\cdot)} (g_1, \cdot) \, \mathrm{d}\beta)$ for each $A_1 \in \sigma(\mathscr{P}_{g_1})$,

(ii) $g_2 \in \mathscr{I}(\beta(A_1, \cdot))$ for each $A_1 \in \mathscr{P}_{g_1}$, and $g_1 \in \mathscr{I}(\int_{(\cdot,A_2)} (\cdot, g_2) \, \mathrm{d}\beta$ for each $A_2 \in \sigma(\mathscr{P}_{g_2})$, and

(iii) $\int_{A_2} g_2 \, \mathrm{d}(\int_{(A_1,\cdot)} (g_1, \cdot) \, \mathrm{d}\beta) = \int_{A_1} g_1 \, \mathrm{d}(\int_{(\cdot,A_2)} (\cdot, g_2) \, \mathrm{d}\beta)$ for each $(A_1, A_2) \in \sigma(\mathscr{P}_{g_1}) \times \sigma(\mathscr{P}_{g_2})$.

By Corollary 3 above (i) $\Leftrightarrow (g_1, g_2) \in \mathscr{L}_1(\beta) \Leftrightarrow$ (ii), and they imply (iii). Note further that according to Theorem 5 and Theorem 8 below, $(g_1, g_2) \in \mathscr{L}_1(\beta)$ if and only if $\hat{\beta}[(g_1, g_2), (T_1, T_2)] < +\infty$.

We shall need the following useful theorem, which by virtue of Theorem 5 is a generalization of Theorem VIII.5.

Theorem 7. *Let* $(g_i) \in \mathscr{L}_1(\Gamma)$, *let* $A_{i,n} \in \sigma(\mathscr{P}_i)$, $n = 1, 2, \dots$, $i = 1, \dots, d$, *and let* $A_{i,n} \to \emptyset$ *for each* $i = 1, \dots, d$. *Then* $\hat{\Gamma}[(g_i), (A_{i,n})] \to 0$ *as* $n \to \infty$.

Proof. Since $A_{i,n} \to \emptyset$ if and only if $\limsup_n A_{i,n} \searrow \emptyset$, we may and will suppose that $A_{i,n} \searrow \emptyset$ for each $i = 1, \dots, d$. Suppose $\hat{\Gamma}[(g_i), (A_{i,n})] > a > 0$ for each $n = 1, 2, \dots$. Put $n_0 = 1$. By the Fatou property of the multiple L_1-gauge $\hat{\Gamma}[(\cdot), (\cdot)]$, see Theorem VIII.4, there is an $n_1 > n_0$ such that

$$\hat{\Gamma}[(g_i), (A_{i,n_0} - A_{i,n_1})] > a .$$

By the definition of $\hat{\Gamma}[(\cdot), (\cdot)]$, see Definition VIII.3, there are $u_{i,1} \in S(\mathscr{P}_i, X_i)$ with $|u_{i,1}| \leqq |g_i| \cdot \chi_{A_{i,n_0} - A_{i,n_1}}$, $i = 1, \dots, d$, such that

$$\left|\int_{(T_i)} (u_{i,1}) \, \mathrm{d}\Gamma\right| > a .$$

Repeating the above consideration we obtain a subsequence $\{n_k\} \subset \{n\}$, and for each $k = 2, 3, \ldots$ we obtain functions $u_{i,k} \in S(\mathscr{P}_i, X_i)$, $i = 1, \ldots, d$, such that $|u_{i,k}| \leqq |g_i| \cdot \chi_{A_{i,n_{k-1}} - A_{i,n_k}}$ for each $i = 1, \ldots, d$, and

$$\left|\int_{(T_i)} (u_{i,k}) \, \mathrm{d}\Gamma\right| > a \, .$$

Put $u_i = \sum_{k=1}^{\infty} u_{i,k}$, $i = 1, \ldots, d$. Then $|u_i| \leqq |g_i|$ for each $i = 1, \ldots, d$, hence $(u_i) \in \mathscr{I}(\Gamma)$. Since $u_{i,k} = u_i \cdot \chi_{A_{i,n_{k-1}} - A_{i,n_k}}$ for each $i = 1, \ldots, d$ and each $k = 1, 2, \ldots$, and since $A_{i,n_{k-1}} - A_{i,n_k} \to \emptyset$ ad $k \to \infty$ for each $i = 1, \ldots, d$, according to Theorems VIII.1, IX.3 and IX.4 we obtain that

$$a < \left|\int_{(T_i)} (u_{i,k}) \, \mathrm{d}\Gamma\right| = \left|\int_{(A_{i,n_{k-1}} - A_{i,n_k})} (u_i) \, \mathrm{d}\Gamma\right| \to 0$$

as $k \to \infty$, a contradiction. The theorem is proved.

In the forthcoming Part XIII we will prove the analog of the next result for arbitrary d.

Theorem 8. *Let $d = 2$ and let $(f_1, f_2) \in \mathscr{L}_1(\Gamma)$. Then $\hat{\Gamma}[(f_1, f_2), (T_1, T_2)] < +\infty$.*

Proof. For $i = 1, 2$ put $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$. By the assumed local σ-finiteness of the semivariation $\hat{\Gamma}$ on $F_1 \cap \sigma(\mathscr{P}_1) \times F_2 \cap \sigma(\mathscr{P}_2)$ there are $F'_{i,r} \in \mathscr{P}_i$, $r = 1, 2, \ldots$, $i = 1, 2$ such that $F'_{i,r} \nearrow F_i$ as $r \to \infty$ for both $i = 1, 2$, and $\hat{\Gamma}(F'_{1,r}, F'_{2,r}) < +\infty$ for each $r = 1, 2, \ldots$. Put

$$F_{i,r} = \{t_i \in F_i, |f_i(t_i)| \leqq r\} \cap F'_{i,r}$$

for $r = 1, 2, \ldots$ and $i = 1, 2$. Obviously

$$\begin{aligned}\hat{\Gamma}[(f_1, f_2), (T_1, T_2)] &= \hat{\Gamma}[(f_1, f_2), (F_1, F_2)] \leqq \\ &\leqq \hat{\Gamma}[(f_1, f_2), (F_{1,r}, F_{2,r})] + \hat{\Gamma}[(f_1, f_2), (F_1 - F_{1,r}, F_{2,r})] + \\ &+ \hat{\Gamma}(f_1, f_2), (F_{1,r}, F_2 - F_{2,r})] + \hat{\Gamma}[(f_1, f_2), (F_1 - F_{1,r}, F_2 - F_{2,r})]\end{aligned}$$

for each $r = 1, 2, \ldots$. Clearly $\hat{\Gamma}[(f_1, f_2), (F_{1,r}, F_{2,r})] \leqq r^2 \, . \, \hat{\Gamma}(F_{1,r}, F_{2,r}) < +\infty$ for each $r = 1, 2, \ldots$. Since $F_i - F_{i,r} \searrow \emptyset$ as $r \to \infty$ for both $i = 1, 2$, according to Theorem 7 there is an r'_0 such that $\hat{\Gamma}[(f_1, f_2), (F_1 - F_{1,r}, F_2 - F_{2,r})] \leqq \hat{\Gamma}[(f_1, f_2), (F_1 - F_{1,r_0'}, F_2 - F_{2,r_0'})] < 1$ for each $r \geqq r'_0$. Hence to prove the theorem it suffices to show that there is an $r_0 \geqq r'_0$ such that $\hat{\Gamma}[(f_1, f_2), (F_1 - F_{1,r_0}, F_{2,r_0})] + \hat{\Gamma}[(f_1, f_2), (F_{1,r_0}, F_2 - F_{2,r_0})] < +\infty$. Suppose the contrary. Then either $\hat{\Gamma}[(f_1, f_2), (F_1 - F_{1,r_k}, F_{2,r_k})] = +\infty$ for an infinite subsequence r_k, $k = 1, 2, \ldots$ with $r_1 \geqq r'_0$, or $\hat{\Gamma}[(f_1 \, . \, f_2), (F_{1,r_k}, F_2 - F_{2,r_k})] = +\infty$ for an infinite subsequence r_k, $k = 1, 2, \ldots$ with $r_1 \geqq r'_0$.

By symmetry in coordinates it is enough to suppose that $\hat{\Gamma}[(f_1, f_2), (E_{1,k}, F_2 - E_{2,k})] = +\infty$ for each $k = 1, 2, \ldots$, where $E_{i,k} = F_{i,r_k}$ and $r_1 \geqq r'_0$, $i = 1, 2$, $k = 1, 2, \ldots$. Put $k_0 = 1$. By the definition of the multiple L_1-gauge, see Definition VIII.3, there is a pair $(u'_{1,1}, u'_{2,1}) \in S(\mathscr{P}_1, X_1) \times S(\mathscr{P}_2, X_2)$ such that $|u'_{1,1}| \leqq |f_1| \cdot \chi_{E_{1,1}} \leqq r_1$, $|u'_{2,1}| \leqq |f_2| \cdot \chi_{F_2 - E_{2,1}}$ and

$$\left|\int_{(F_1, F_2 - E_{2,k_0})} (u'_{1,1}, u'_{2,1}) \, \mathrm{d}\Gamma\right| > 3.4.r_1 \, .$$

Put

$$u_{1,1} = u'_{1,1} \frac{1}{2r_1} .$$

Let $\mathscr{P}'_1 = \bigcup_{k=1}^{\infty} E_{1,k} \cap \mathscr{P}_1$ and $\mathscr{P}'_2 = \bigcup_{k=1}^{\infty} E_{2,k} \cap \mathscr{P}_2$. For $E_2 \in \mathscr{P}'_2$ and $x_2 \in X_2$ put

$$m_{u_{1,1}}(E_2)\, x_2 = \int_{(E_{1,1},E_2)} (u_{1,1}, x_2 \,.\, \chi_{E_2})\, \mathrm{d}\Gamma .$$

Clearly $m_{u_{1,1}}\colon \mathscr{P}'_2 \to L(X_2, Y)$, and it is countably additive in the strong operator topology. Further, since $u_{1,1}$ is a $\mathscr{P}'_1$ – simple function, and since the semivariation $\hat{\Gamma}$ is finite on $\mathscr{P}'_1 \times \mathscr{P}'_2$, the semivariation $\hat{m}_{u_{1,1}}$ is finite on $\mathscr{P}'_2$. Similarly as we showed that $g_1 \in \mathscr{L}_1(m_{1,(\cdot),(\cdot)})$ in the proof of Theorem 6 we conclude that $f_2 \in \mathscr{L}_1(m_{u_{1,1}})$. Put

$$a_1 = \hat{m}_{u_{1,1}}(f_2, T_2) < +\infty .$$

Since $\left|\int_{(F_1,F_2-E_{2,k_0})} (u_{1,1}, u'_{2,1})\, \mathrm{d}\Gamma\right| > 2.3$ and $E_{2,k} \nearrow F_2$ as $k \to \infty$, by the separate countable additivity of Γ in the strong operator topology there is a $k_1 > k_0 = 1$ such that

$$\left|\int_{(F_1,E_{2,k_1}-E_{2,k_0})} (u_{1,1}, u'_{2,1})\, \mathrm{d}\Gamma\right| > 2.3 .$$

Put

$$u_{2,1} = u'_{2,1} \cdot \chi_{E_{2,k_1}-E_{2,k_0}} .$$

Let

$$l_{u_{2,1}}(E_1)\, x_1 = \int_{(E_1,F_2)} (x_1 \,.\, \chi_{E_1}, u_{2,1})\, \mathrm{d}\Gamma$$

for $E_1 \in \mathscr{P}'_1$ and $x_1 \in X_1$. Then $l_{u_{2,1}}\colon \mathscr{P}'_1 \to L(X_1, Y)$, it is countably additive in the strong operator topology, and has finite semivariation $\hat{l}_{u_{2,1}}$ on $\mathscr{P}'_1$. Now similarly as above we obtain that $f_1 \in \mathscr{L}_1(l_{u_{2,1}})$. Put

$$b_1 = \hat{l}_{u_{2,1}}(f_1, T_1) < +\infty .$$

By assumption $\hat{\Gamma}[(f_1, f_2), (E_{1,k_1}, F_2 - E_{2,k_1})] = +\infty$. For $n = 2, 3, \ldots$ we proceed successively in the following way: given $(u_{1,n-1}, u_{2,n-1}) \in S(\mathscr{P}'_1, X_1) \times S(\mathscr{P}'_2, X_2)$, a_{n-1} and b_{n-1}, we have $\hat{\Gamma}[(f_1, f_2), (E_{1,k_{n-1}}, F_2 - E_{2,k_{n-1}})] = +\infty$ by assumption. Hence there are $(u'_{1,n}, u'_{2,n}) \in S(\mathscr{P}'_1, X_1) \times S(\mathscr{P}'_2, X_2)$ and $k_n > k_{n-1}$ such that $|u'_{1,n}| \leqq |f_1| \cdot \chi_{E_{1,k_{n-1}}}$, $|u'_{2,n}| \leqq |f_2| \cdot \chi_{E_{2,k_n}-E_{2,k_{n-1}}}$ and

$$\left|\int_{(F_1,F_2)} (u'_{1,n}, u'_{2,n})\, \mathrm{d}\Gamma\right| > 2^n \,.\, r_{k_{n-1}} \,.\, 3 \,.\, (1 + a_{n-1})(1 + b_1) \ldots$$
$$\ldots (1 + b_{n-1}) .$$

Put

$$u_{1,n} = 2^{-n} \,.\, r_{k_{n-1}}^{-1} \cdot (1 + b_1)^{-1} \ldots . (1 + b_{n-1})^{-1} \,.\, u'_{1,n}$$

and

$$u_{2,n} = u'_{2,n} \cdot \chi_{E_{2,k_n}-E_{2,k_{n-1}}} .$$

Clearly

$$|u_{1,n}| \leqq 2^{-n} \,.\, (1 + b_1)^{-1} \ldots . (1 + b_{n-1})^{-1} .$$

Similarly as above,

$$a_n = \hat{m}_{(\sum_{j=1}^{n} u_{1,j})}(f_2, T_2) < +\infty \quad \text{and} \quad b_n = \hat{l}_{u_{2,n}}(f_1, T_1) < +\infty .$$

Obviously $\sum_{n=1}^{\infty} |u^i{}_{,n}(t_i)| \leqq |f_i(t_i)| < +\infty$ for each $t_i \in T_i$, for both $i = 1, 2$. Put $u_i = \sum_{n=1}^{\infty} u_{i,n}$, $i = 1, 2$. Then obviously u_i: $T_i \to X_i$ is $\mathscr{P}'_i$-measurable and $|u_i| \leqq |f_i|$ for both $i = 1, 2$. Hence $(u_1, u_2) \in \mathscr{I}(\Gamma)$ by the definition of $\mathscr{L}_1(\Gamma)$. Let $\gamma(A_1, A_2) = \int_{(A_1, A_2)} (u_1, u_2)\, d\Gamma$, $(A_1, A_2) \in \sigma(\mathscr{P}'_1) \times \sigma(\mathscr{P}'_2)$. Then γ: $\sigma(\mathscr{P}'_1) \times \sigma(\mathscr{P}'_2) \to Y$ is a separately countably additive vector bimeasure, see Theorem IX.4. Put $A_{2,n} = E_{2,k_n} - E_{2,k_{n-1}}$, $n = 1, 2, \ldots$. Then $A_{2,n}$, $n = 1, 2, \ldots$ are pairwise disjoint sets from $\sigma(\mathscr{P}'_2)$, hence $\gamma(F_1, A_{2,n}) \to 0$ as $n \to \infty$. Let n_0 be such that $|\gamma(F_1, A_{2,n_0})| < 1$. Then

$$\begin{aligned}
&3(1 + a_{n_0-1}) < \left|\int_{(F_1, A_{2,n_0})} (u_{1,n_0}, u_{2,n_0})\, d\Gamma\right| = \\
&= \left|\int_{(F_1, A_{2,n_0})} (u_1 - \sum_{j=n_0+1}^{\infty} u_{1,j} - \sum_{j=1}^{n_0-1} u_{1,j}, u_{2,n_0})\, d\Gamma\right| < \\
&< 1 + \left|\int_{(F_1, A_{2,n_0})} (\sum_{j=n_0+1}^{\infty} u_{1,j}, u_{2,n_0})\, d\Gamma\right| + \\
&+ \left|\int_{(F_1, A_{2,n_0})} (\sum_{j=1}^{n_0-1} u_{1,j}, u_{2,n_0})\, d\Gamma\right| < \\
&< 1 + 2^{-n_0} b_{n_0}(1 + b_{n_0})^{-1} + a_{n_0-1} ,
\end{aligned}$$

a contradiction. The theorem is proved.

The analog of the next theorem for $\hat{\mathscr{L}}_1(\Gamma)$ is evidently valid.

Theorem 9. *Let $(f'_i) \in \mathscr{L}_1(\Gamma)$, for each $i = 1, \ldots, d$, let $\mathscr{P}'_i \subset \mathscr{P}_i$ be a δ-subring and suppose f'_i is $\mathscr{P}'_i$-measurable. Denote by Γ' the restriction $\Gamma' = \Gamma$: $\mathsf{X}\mathscr{P}'_i \to L^{(d)}(X_i; Y)$, and suppose that the semivariation $\hat{\Gamma}$ is locally σ-finite on $\mathsf{X}\sigma(\mathscr{P}'_i)$. Then $(f'_i) \in \mathscr{L}_1(\Gamma')$, and*

(1) $$\int_{(A_i')} (f'_i)\, d\Gamma = \int_{(A_i')} (f'_i)\, d\Gamma$$

for each $(A'_i) \in \mathsf{X}\sigma(\mathscr{P}'_i)$. ($\hat{\Gamma}'$ is also locally σ-finite on $\mathsf{X}\sigma(\mathscr{P}'_i)$.)

Proof. Put $F'_i = \{t_i \in T_i, f'_i(t_i) \neq 0\} \in \sigma(\mathscr{P}'_i)$, $i = 1, \ldots, d$. By the assumed local σ-finiteness of the semivariation $\hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}'_i)$ there are $F''_{i,r} \in \mathscr{P}'_i$, $r = 1, 2, \ldots$, $i = 1, \ldots, d$ such that $F''_{i,r} \nearrow F'_i$ as $r \to \infty$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(F''_{i,r}) < +\infty$ for each $r = 1, 2, \ldots$. Define $F'_{i,r} = \{t_i \in T_i, |f'_i(t_i)| \leqq r\} \cap F''_{i,r}$, $i = 1, \ldots, d$ and $r = 1, 2, \ldots$. Then $(F'_{i,r}) \in \mathsf{X}\sigma(\mathscr{P}'_i)$, $\hat{\Gamma}(F'_{i,r}) < +\infty$ and $\hat{\Gamma}[(f'_i), (F'_{i,r})] \leqq r^d\, \hat{\Gamma}(F'_{i,r}) < +\infty$ for each $r = 1, 2, \ldots$, and $F'_{i,r} \nearrow F'_i$ for each $i = 1, \ldots, d$.

Let u'_i: $T_i \to X_i$ be $\mathscr{P}'_i$-measurable, $i = 1, \ldots, d$, and let $|u'_i| \leqq |f'_i|$ for each i. To prove the theorem we have to show that $(u'_i) \in \mathscr{I}(\Gamma')$ and that (1) holds. Since $(f'_i) \in \mathscr{L}_1(\Gamma)$, we have $(u'_i) \in \mathscr{I}(\Gamma)$. Hence γ: $\mathsf{X}(F'_i \cap \sigma(\mathscr{P}_i)) \to Y$, $\gamma(A_i) = \int_{(A_i)} (u'_i)\, d\Gamma$, is a separately countably additive vector d-polymeasure. Now, to show that $(u'_i) \in \mathscr{I}(\Gamma')$, according to Corollary 2 of Theorem IX.4 it suffices to prove that $(u'_i \,.\, \chi_{F_{i,r'}}) \in$

$\in \mathscr{I}(\Gamma')$ for each $r = 1, 2, \ldots$, and that for a given r, $\gamma(A_i' \cap F_{i,r}') = $
$= \int_{(A_i')} (u_i' \cdot \chi_{F'_{i,r}}) \, d\Gamma'$ for each $(A_i') \in (F_i' \cap \sigma(\mathscr{P}_i'))$, hence for each $(A_i') \in \mathsf{X}(F_{i,r}' \cap \mathscr{P}_i')$.

For each $i = 1, \ldots, d$ take a sequence $u_{i,n}' \in S(\mathscr{P}_i', X_i)$, $n = 1, 2, \ldots$ such that $u_{i,n}' \to u_i'$ and $|u_{i,n}'| \nearrow |u_i'|$. Let $r \in \{1, 2, \ldots\}$ be fixed. Since $\hat{\Gamma}[(u_i'), (F_{i,r}')] \leqq$ $\leqq r^d \hat{\Gamma}(F_{i,r}') < +\infty$ and $(u_i' \cdot \chi_{F'_{i,r}}) \in \mathscr{L}_1(\Gamma)$, by the proof of Theorem 6 we obtain that

$$\lim_{n_d \to \infty} \ldots \lim_{n_1 \to \infty} \int_{(A_i)} (u_{i,n_i}' \cdot \chi_{F'_{i,r}}) \, d\Gamma =$$
$$= \lim_{n_d \to \infty} \ldots \lim_{n_2 \to \infty} \int_{(A_i)} (u_1' \cdot \chi_{F'_{1,r}}, u_{2,n_2}' \cdot \chi_{F'_{2,r}}, \ldots, u_{d,n_d}' \cdot \chi_{F'_{d,r}}) \, d\Gamma = \ldots$$
$$\ldots = \int_{(A_i)} (u_i' \cdot \chi_{F'_{i,r}}) \, d\Gamma = \gamma(A_i \cap F_{i,r}')$$

for each $(A_i) \in \mathsf{X}(F_{i,r}' \cap \mathscr{P}_i)$, particularly for each $(A_i) \in \mathsf{X}(F_{i,r}' \cap \mathscr{P}_i')$. But in the last case we may replace $d\Gamma$ by $d\Gamma'$, hence $(u_i' \cdot \chi_{F'_{i,r}}) \in \mathscr{I}_d(\Gamma')$, and the analog of (1) holds, which we wanted to show. Hence $(u_i') \in \mathscr{I}(\Gamma')$ and the analog of (1) holds for (u_i'). Taking $(u_i') = (f_i')$ we obtain (1). The theorem is proved.

Let us note that if $\Gamma(\ldots)(x_i): \mathsf{X}\mathscr{P}_i \to Y$ has a locally control d-polymeasure for each $(x_i) \in \mathsf{X}X_i$, then the assertion of the preceding theorem is a consequence of Theorem X.13.

We are now ready to prove

Theorem 10. (Lebesgue dominated convergence theorem in $\mathscr{L}_1(\Gamma)$.) *Let $f_i, f_{i,n}$: $T_i \to X_i$, $n = 1, 2, \ldots$ be $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$, let the sequence of d-tuples $(f_{i,n})$, $n = 1, 2, \ldots$ converge Γ-almost everywhere to the d-tuple (f_i), and let there exist a d-tuple $(g_i) \in \mathscr{L}_1(\Gamma)$ such that $|f_{i,n}| \leqq |g_i|$, $i = 1, \ldots, d$, Γ-almost everywhere for each $n = 1, 2, \ldots$. Then (f_i), $(f_{i,n}) \in \mathscr{L}_1(\Gamma)$, $n = 1, 2, \ldots$. and*

$$(1) \qquad \lim_{n_1, \ldots, n_d \to \infty} \int_{(A_i)} (f_{i,n_i}) \, d\Gamma = \int_{(A_i)} (f_i) \, d\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

If in each of the d coordinates either the convergence $f_{i,n}(t_i) \to f_i(t_i)$ is uniform with respect to $t_i \in T_i$, or the multiple L_1-gauge $\hat{\Gamma}[(g_i), (\ldots, T_{i-1}, \cdot, T_{i+1}, \ldots)]$: $\sigma(\mathscr{P}_i) \to [0, +\infty)$ is continuous in that coordinate, then the limit in (1) is uniform with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Without loss of generality we may suppose that the second and third assumptions of the theorem hold everywhere. But then $(f_i), (f_{i,n}) \in \mathscr{L}_1(\Gamma)$, $n =$ $= 1, 2, \ldots$, by the definition of $\mathscr{L}_1(\Gamma)$.

Next, the last assertion of the theorem follows easily from the proof of Theorem IX.7 in [13].

Put $G_i = \{t_i \in T_i,\ g_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$. By the assumed local σ-finiteness of the semivariation $\hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}_i)$, see the beginning of Part IX, there are $(G_{i,n}) \in \mathsf{X}\mathscr{P}_i$, $n = 1, 2, \ldots$ such that $G_{i,n} \nearrow G_i$ as $n \to \infty$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(A_{i,n}) < +\infty$ for each $n = 1, 2, \ldots$.

Let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. From the definition of $\mathscr{P}$-measurable functions, see Section 1.2

in Part I, we immediately see that for each $i = 1, \ldots, d$ there is a countably generated δ-ring $\mathscr{P}_{i,(A_i)}$ such that $\{G_{i,n}\}_{n=1}^{\infty} \subset \mathscr{P}_{i,(A_i)} \subset \mathscr{P}_i$, and the functions $g_i, \chi_{A_i}, f_{i,n}$, $n = 1, 2, \ldots$ are $\mathscr{P}_{i,(A_i)}$-measurable (hence also the functions $f_{i,n} \cdot \chi_{A_i}$ and $f_i \cdot \chi_{A_i}$, $n = 1, 2, \ldots$ are $\mathscr{P}_{i,(A_i)}$-measurable). Further, take separable closed subspaces $X_i' \subset X_i$, $i = 1, \ldots, d$, such that $g_i(T_i) \cup \bigcup_{n=1}^{\infty} f_{i,n}(T_i) \subset X_i'$ for each i. Obviously, in our consideration we may replace X_i by X_i' for each $i = 1, \ldots, d$. Denote by $\Gamma_{(A_i)}$ the restriction $\Gamma_{(A_i)} = \Gamma\colon \mathsf{X}(G_i \cap \mathscr{P}_{i,(A_i)}) \to L^{(d)}(X_i'; Y)$. Evidently the semivariation $\hat{\Gamma}_{(A_i)}(G_i) \leqq \hat{\Gamma}(G_i)$ is σ-finite. Hence, according to Theorem 9, $(g_i), (f_i), (f_{i,n}) \in \mathscr{L}_1(\Gamma_{(A_i)})$, $n = 1, 2, \ldots$, and on both sides of (1) we may replace $\mathrm{d}\Gamma$ by $\mathrm{d}\Gamma_{(A_i)}$.

Owing to Corollary of Theorem VIII.11 and Theorems VIII.17 and VIII.19 the semivariation $\hat{\Gamma}_{(A_i)}\colon \mathsf{X}(G_i \cap \mathscr{P}_{i,(A_i)}) \to [0, +\infty]$ has a control d-polymeasure, say $\lambda_1 \times \ldots \times \lambda_d\colon \mathsf{X}(G_i \cap \mathscr{P}_{i,(A_i)}) \to [0, 1]$.

Now it is easy to check that an analog of the proof of Theorem 3 in [18] yields (1). Namely, instead of (2) in that proof, by Lemma 2 and Theorem 7 there is an integer k_0 such that

$$\left|\int_{(A_i - N_i - G'_{i,k_0})} (f_{i,n_i} - f_i)\, \mathrm{d}\Gamma_{(A_i)}\right| \leqq$$
$$\leqq 2^d \cdot \hat{\Gamma}_{(A_i)}\left[(g_i), (A_i - N_i - G'_{i,k_0})\right] < \varepsilon/4\,,$$

where $G'_{i,k_0} = G_{i,k} \cap \{t_i \in T_i, |g_i(t_i)| \leqq k_0\}$, and G_{i,k_0} is as in that proof. By Fubini's theorem (Theorem 6) and the inductive assumption we obtain the analog of (3) from the original proof. The rest follows similarly as in [18]. Thus the theorem is proved.

Since for each $\mathscr{P}_i$-measurable g_i, $i = 1, \ldots, d$, there is a sequence of $\mathscr{P}_i$-simple functions $g_{i,n}$, $n = 1, 2, \ldots$ such that $g_{i,n}(t_i) \to g_i(t_i)$ and $|g_{i,n}(t_i)| \nearrow |g_i(t_i)|$ for each $t_i \in T_i$, we immediately obtain

Corollary 1. $\mathscr{L}_1(\Gamma) \subset \mathscr{I}_1(\Gamma)$.

The next corollaries are also immediate.

Corollary 2. (Lebesgue bounded convergence theorem in $\mathscr{L}_1(\Gamma)$.) *Let* $(x_i \cdot \chi_{A_i}) \in \mathscr{L}_i(\Gamma)$ *for each* $(x_i) \in \mathsf{X}X_i$ *and each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. *Then the assertions of the theorem hold if* $\sup_{i,n} \|f_{i,n}\|_{T_i} < +\infty$.

Corollary 3. (Special case of LBCT in $\mathscr{L}_1(\Gamma)$, se Theorem 3 in [18].) *Let each of* X_i, $i = 1, \ldots, d$, *be a finite dimensional Banach space, and let each* $\mathscr{P}_i$, $i = 1, \ldots, d$, *be a* σ-*ring. Then the assertions of the theorem hold if* $\sup_{i,n} \|f_{i,n}\|_{Ti} < +\infty$.

Evidently, our assertion (1) of the Lebesgue dominated convergence theorem in $\mathscr{L}_1(\Gamma)$ is stronger than the result of Corollary 2.9 in [1], obtained for scalar bimeasures, see the paragraph after Corollary 3 of Theorem 6. It is an interesting novelty that the proof of LDCT in $\mathscr{L}_1(\Gamma)$ requires the Fubini theorem in $\mathscr{L}_1(\Gamma)$, whose proof requires the weaker (iterated limit) version of LDCT.

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