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SOME REMARKS ON THE SURJECTIVITY SPECTRUM OF LINEAR OPERATORS

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INTRODUCTION

The spectrum of a linear operator $\boldsymbol{T}$ can be divided into subsets in many different ways, depending on the purpose of the inquiry. For the study of spectral subspaces (the precise definition of this and other terms used rather freely in this introduction may be found in the main body of the paper), i.e. the set of points λ for which the operator $\boldsymbol{T} - \lambda$ is not a surjection has proved quite useful (cf. e.g. ref. [8], [9]).

This paper studies the basics of this concept and relates it to local spectra and to algebraic subspaces (surjectivity spaces) $E_T(\bar{F})$; moreover, the relationship between this class of subspaces and the class of analytic spectral subspaces $X_T(F)$ is investigated.

The urge to study connections between the E_T and the X_T spaces derives from automatic continuity: the E_T spaces are defined in algebraic terms and are thus particularly well suited to situations where continuity of certain linear maps is not assumed. On the other hand, as the X_T spaces are analytically defined, their structure is much more readily accessible to analysis. If the two classes can be shown to coincide for a given $\boldsymbol{T}$, then stronger conclusions are at hand. This is illustrated quite well in for instance [5, 6, 7].

Here we study the concepts mentioned in their own right. To begin with, the surjectivity spectrum $\sigma_s(\boldsymbol{T})$ of $\boldsymbol{T}$ is introduced. If $\boldsymbol{T}$ is bounded on a normed linear space X then $\sigma_s(\boldsymbol{T})$ is a non-empty subset of the spectrum $\sigma(\boldsymbol{T})$. If X is complete then $\sigma_s(\boldsymbol{T})$ is the union of all the local spectra $\sigma_T(x)$ (as x ranges over X).

A decomposability of $\boldsymbol{T}$ expressed in terms of surjectivity spectra is formally weaker than the usual decomposability, but we show immediately that in fact the two concepts coincide.

We then introduce the surjectivity spectra, here most conveniently defined in

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terms of surjectivity spectra. These spaces are identified for a class of operators which includes the generalized spectral operators. The formula is

$$E_T(A) = \bigcap_{n \in N, \lambda \notin A} (T - \lambda)^n X\,, \quad \text{for any} \quad A \subseteq \mathbb{C}\,.$$

An analytic spectral subspace $X_T(A)$ is always a subspace of the corresponding $E_T(A)$, but for well-decomposable operators the two classes are intimately related, as shown in Proposition 4. This result is best possible, in a manner specified.

The last part of the paper is concerned with the consequences of closedness of $E_T(F)$, for a closed set F. For instance, if F contains the set where the single valued extension property fails, then $E_T(F)$ is spectral maximal. Closedness of $E_T(F)$ alone is sufficient for coincidence with $X_T(F)$.

For a linear operator T on a complex vector space X we define the *surjectivity spectrum*

$$\sigma_s(T; X) := \{\lambda \in \mathbb{C} \mid (T - \lambda) X \neq X\}\,.$$

If the context is clear we usually write just $\sigma_s(T)$. The surjectivity spectrum has been called other things by other authors, e.g. approximate defect spectrum in [3], but terminology does not appear to be standard, and as this is clearly a purely algebraic notion, we shall stay with the term surjectivity spectrum. Nevertheless, in this paper we shall concentrate on the bounded linear operators. In this setting it is easy to relate the surjectivity spectrum to the spectrum $\sigma(T)$.

Lemma 1. *If T is a bounded linear operator on a normed linear space X then $\sigma_s(T) \subseteq \sigma(T)$. If X is a Banach space then $\sigma_s(T)$ is a compact set containing the topological boundary of $\sigma(T)$; in particular, $\sigma_s(T)$ is non-empty and $\sigma(T) \subseteq \sigma_s(T)^\wedge$ (where $^\wedge$ denotes polynomially convex hull).*

Proof. The first claim is trivial. The second is a standard application of the open mapping theorem, cf. e.g. [4]. The third claim is standard topology, once we know that $\sigma_s(T)$ contains the boundary of $\sigma(T)$.

Thus, if $\sigma_s(T)$ and $\sigma(T)$ are different then $\sigma(T)$ is obtained from $\sigma_s(T)$ by filling in one or more of the bounded components of the complement of $\sigma_s(T)$. But more can be said: recall that if T is a bounded linear operator on a Banach space X and if $x \in X$ then the local resolvent set $\varrho_T(x)$ is

$\varrho_T(x) := \{\lambda \in \mathbb{C}$: there exists a neighbourhood N_λ and a holomorphic function $f: N_\lambda \to X$ so that $(T - \mu) f(\mu) = x$ for all $\mu \in N_\lambda\}$

and the local spectrum of x is $\sigma_T(x) := \mathbb{C} \setminus \varrho_T(x)$. We have the following

Lemma 2. *If T is a bounded linear operator on the Banach space then*

$$\sigma_s(T) = \bigcup_{x \in X} \sigma_T(x)\,,$$

and $\{x \mid \sigma_T(x) = \sigma_s(T)\}$ is of the second category in X.

Proof. If $\lambda \notin \bigcup_{x \in X} \sigma_T(x)$, then $\lambda \in \varrho_T(x)$ for every $x \in X$, hence $T - \lambda$ is surjective. Thus $\lambda \notin \sigma_s(T)$. The converse is, like parts of Lemma 1, a consequence of the open mapping theorem: if $\lambda \notin \sigma_s(T)$ then $T - \lambda$ is an open mapping of X onto X, so there is a constant $C \in \mathbb{R}_+$ so that for every $x \in X$ there exists $y \in X$ with $(T - \lambda)\, y = x$ and $\|y\| < C\|x\|$. Given $x_0 \in X$, we may choose (x_n) so that $(T - \lambda)x_{n+1} = x_n$ and $\|x_{n+1}\| < C\|x_n\|$, $n = 0, 1, 2, \ldots$. On the open disc $\{|\lambda - \mu| < C^{-1}\}$ the formula

$$f(\mu) := \sum_{n=0}^{\infty} x_{n+1}(\mu - \lambda)^n$$

defines a holomorphic function for which

$$(T - \mu)f(\mu) = (T - \mu)f(\mu) + (\lambda - \mu)f(\mu) =$$
$$= \sum_{n=0}^{\infty} x_n(\mu - \lambda)^n - \sum_{n=1}^{\infty} x_n(\mu - \lambda)^n = x_0\,,$$

for any μ with $|\mu - \lambda| < C^{-1}$. This shows that $\lambda \in \varrho_T(x_0)$ and since x_0 is arbitrary this part of the proof is done.

To show that the set $\{x \in X |\ \sigma_T(x) = \sigma_s(T)\}$ is of the second category in X we shall follow the same argument as in [9]. Let $\mathcal{M}$ be a countable dense set in $\sigma_s(T)$. Since $(T - \lambda)\, X \neq X$ for $\lambda \in \mathcal{M}$ it follows that $(T - \lambda)\, X$ is of the first category in X and so is the set $\bigcup_{\lambda \in \mathcal{M}}(T - \lambda)\, X$. If $x \notin \bigcup_{\lambda \in \mathcal{M}} (T - \lambda)\, X$ then $\mathcal{M} \subseteq \sigma_T(x)$ and, consequently, $\sigma_s(T) = \mathcal{M}^- \subseteq \sigma_T(x)$.

Example. Consider the left shift L on $\ell^1 := \ell^1(\mathbb{N})$. This operator has spectrum equal to the unit disc, but surjectivity spectrum equal to the unit circle: if λ is a complex number with $|\lambda| < 1$ and we consider the equation $(L - \lambda)\,(y_n) = (x_n)$, where $(x_n) \in \ell^1$ is given, then $y_{n+1} = \lambda y_n + x_n$ means that

$$y_{n+1} = \lambda^n y_1 + \sum_{p=0}^{n-1} \lambda^p x_{n-p} \quad \text{for} \quad n = 1, 2, \ldots.$$

Since $(\lambda^p) \in \ell^1$ a standard argument involving convolution shows that the second term on the right hand side of the last equation is the general term of an element in ℓ^1. Hence $L - \lambda$ is onto ℓ^1 for each λ, $|\lambda| < 1$.

What happens here is actually typical of the way in which the holes are filled in when we pass from $\sigma_s(T)$ to $\sigma(T)$. To see this, we need to introduce the *analytic residuum* $S(T)$:

$S(T) := \{\lambda \in \mathbb{C} |$ for every neighborhood N_λ there is a neighborhood $N'_\lambda \subseteq N_\lambda$ and a non-zero holomorphic function $f: N'_\lambda \to X$ satisfying $(T - \mu)f(\mu) = 0\}$

Note that in contrast with [9] we define $S(T)$ to be open. Clearly, $S(T) \subseteq S_T$ (for the definition of this latter set, see [9]) and $S(T)^- = S_T$.

Essentially the same open mapping argument as before then yields the following

Lemma 3. *If T is a bounded linear operator on a Banach space X then*

$$\sigma(T) = S(T) \cup \sigma_s(T)\,.$$

In particular, $\sigma_s(T)$ contains the topological boundary of $S(T)$.

Proof. If $\lambda \in S(T)$ then every neighborhood of λ contains eigenvalues of T; this yields the inclusion $\supseteq$. For the inclusion $\subseteq$, suppose $\lambda \notin S(T)$. Refer to the proof of Lemma 2 and the notation used there. Take $x_0 = 0$. Since $\lambda \notin S(T)$, $f = 0$ in a neighbourhood of λ, hence $x_1 = 0$. Thus $\boldsymbol{T} - \lambda$ is 1-1 and, since $\lambda \notin \sigma_s(\boldsymbol{T})$, also onto, so is invertible; thus $\lambda \in \varrho(\boldsymbol{T})$. The last claim is obvious, since $S(\boldsymbol{T})^- \subseteq \sigma(\boldsymbol{T})$

If $S(\boldsymbol{T}) = \emptyset$ then $\boldsymbol{T}$ is said to have the *single valued extension property* (SVEP) Thus we this obvious corollary

Corollary. *If $\boldsymbol{T}$ has* SVEP *then $\sigma(\boldsymbol{T}) = \sigma_s(\boldsymbol{T})$.*

This corollary allows us to give a formally weaker condition for decomposability of the operator $\boldsymbol{T}$ by replacing the spectrum by the surjectivity spectrum. One part of the argument is conveniently singled out as a general lemma.

Lemma 4. *Suppose $\boldsymbol{Z} \subseteq \boldsymbol{X}$ is a closed $\boldsymbol{T}$-invariant subspace of the Banach space $\boldsymbol{X}$. Then $\sigma_s(\boldsymbol{T} \mid \boldsymbol{Z})^\wedge \subseteq \sigma_s(\boldsymbol{T})^\wedge$.*

Proof. If δ denotes topological boundary then by Lemma 1 $\sigma_s(\boldsymbol{T})^\wedge = \sigma(\boldsymbol{T})^\wedge = (\delta\sigma(\boldsymbol{T}))^\wedge$. Moreover, since $\delta\sigma(\boldsymbol{T} \mid \boldsymbol{Z})$ (for any $\boldsymbol{T}$-invariant $\boldsymbol{Z}$) is a subset of the approximate point spectrum we get

$$\sigma_s(\boldsymbol{T} \mid \boldsymbol{Z})^\wedge = (\delta\sigma(\boldsymbol{T} \mid \boldsymbol{Z}))^\wedge \subseteq \sigma(\boldsymbol{T})^\wedge = \sigma_s(\boldsymbol{T})^\wedge .$$

Proposition. *If a bounded linear operator $\boldsymbol{T}$ on a Banach space X satisfies the following: for every open cover G_1, G_2 of $\mathbb{C}$ there are closed $\boldsymbol{T}$-invariant subspaces $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ such that*

$$\boldsymbol{X} = \boldsymbol{Y}_1 + \boldsymbol{Y}_2$$

and

$$\sigma_s(\boldsymbol{T} \mid \boldsymbol{Y}_i) \subseteq G_i , \quad i = 1, 2 ,$$

then T is decomposable.

Proof. By the Corollary just stated it suffices to show that $\boldsymbol{T}$ has SVEP. Take an open cover G_i, $i = 1, 2$, of $\mathbb{C}$ with G_2 polynomially convex (this means that for every every compact set $K \subseteq G_2$ with polynomially convex hull $K^\wedge$ we have $K^\wedge \subseteq G_2$). Suppose $\alpha \in G_1 \setminus G_2$ and $x \in \boldsymbol{X}$ are given such that $(\boldsymbol{T} - \alpha)\, x = 0$; we claim that $x \in \boldsymbol{Y}_1$. Write $x = y_1 + y_2$ with $y_i \in \boldsymbol{Y}_i$, $i = 1, 2$. By Lemma 1, $\sigma(\boldsymbol{T} \mid \boldsymbol{Y}_2) \subseteq G_2$, hence $(\boldsymbol{T} - \alpha) \mid \boldsymbol{Y}_2$ is invertible. Consider the $\boldsymbol{T}$-invariant subspace $\boldsymbol{Y}_1 \cap \boldsymbol{Y}_2$. By Lemma 4, $\sigma(\boldsymbol{T} \mid \boldsymbol{Y}_1 \cap \boldsymbol{Y}) \subseteq G_2$, hence $(\boldsymbol{T} - \alpha) \mid \boldsymbol{Y}_1 \cap \boldsymbol{Y}_2$ is invertible as well. But $(\boldsymbol{T} - \alpha)\, x = 0$ implies that $(\boldsymbol{T} - \alpha)\, y_1 = -(\boldsymbol{T} - \alpha)\, y_2$, hence $(\boldsymbol{T} - \alpha)\, y_2 \in \boldsymbol{Y}_1 \cap \boldsymbol{Y}_2$ and $y_2 = ((\boldsymbol{T} - \alpha) \mid \boldsymbol{Y}_1 \cap \boldsymbol{Y}_2)^{-1} (\boldsymbol{T} - \alpha)\, y_2 \in \boldsymbol{Y}_1$. This shows that $x \in \boldsymbol{Y}_1$.

Suppose now that $x(\ )$ is a non-zero holomorphic function defined on an open connected set $G \subseteq \mathbb{C}$ with values in $\boldsymbol{X}$ and suppose $(\boldsymbol{T} - \alpha)\, x(\alpha) = 0$ for all $\alpha \in G$. A standard analytic continuation argument shows that if $x(\alpha)$, for infinitely many different $\alpha \in G$, belong to a given closed linear $\boldsymbol{T}$-invariant subspace $\boldsymbol{Z}$ then $x(G) \subseteq \boldsymbol{Z}$. Let $\mathbb{C}$ be covered by two open halfplanes H_1 and H_2 chosen so that $G \cap H_1$ and $G \cap H_2$ are both non-empty. Two applications of the above shows that $\boldsymbol{G} \subseteq H_1 \cap H_2$ Since the strip $H_1 \cap H_2$ is arbitrary, this shows that $x(\ ) = 0$.

We now come to the concept that will dominate the rest of this paper, namely that of the surjectivity spaces $E_{\boldsymbol{T}}(F)$. These spaces go back to [4] where their significance in automatic continuity was first exploited. This work has been pursued in [6, 5, 1, 8]. In terms of surjectivity spectra the following definition is natural.

Definition. Let T be a linear operator on a complex vector space X. Let $\operatorname{lat}(T)$ denote the collection of T-invariant subspaces of X. Let $A \subseteq \mathbb{C}$. Then

$$E_T(A) := \operatorname{span}\{Y \in \operatorname{lat}(T) \mid \sigma_s(T/Y) \subseteq A\}.$$

It is straightforward to check that $E_T(A)$ is the largest T-invariant subspace of X for which the surjectivity spectrum of T is a subset of A. Equivalently, $E_T(A)$ is the largest T-invariant subspace of X on which all restrictions $T - \lambda$, $\lambda \notin A$, are surjective. This latter characterization is the usual definition.

Evidently, each $x_0 \in E_T(A)$ and each $\lambda \notin A$ allow the choice of a chain (x_n) of preimages in $E_T(A)$: for each $n \geqq 0$ there is $x_{n+1} \in E_T(A)$ such that $x_n = (T - \lambda)\, x_{n+1}$. It is easy to see that the converse also holds.

Proposition. *Let $x_0 \in X$ and $A \subseteq \mathbb{C}$ be given. Suppose that for each $\alpha \notin A$ there exists a chain $(x_n(\alpha))$ such that $x_n(\alpha) = (T - \alpha)\, x_{n+1}(\alpha)$, $n = 0, 1, 2, \ldots$* (here of course $x_0(\alpha) = x_0$ for each $\alpha \notin A$). *Then $x_0 \in E_T(A)$.*

Proof. Fix $\alpha \notin A$ and let $x_n := x_n(\alpha)$ be a chain of preimages, as above. Let $M := \operatorname{span}_{k,n \geqq 0}(T^k x_n)$. Clearly M is T-invariant and $M \subseteq (T - \alpha)\, M$. Thus, by [8], $x_0 \in M \subseteq E_T(\mathbb{C} \setminus \{\alpha\})$. But then $x_0 \in \bigcap_{\alpha \notin A} E_T(\mathbb{C} \setminus \{\alpha\}) = E_T(A)$.

This observation allows the introduction of a purely algebraic notion of local spectrum: Say that $\lambda \in \mathbb{C}$ is in the algebraic resolvent of x_0 (with respect to T) if there is a chain (x_n) such that $x_n = (T - \lambda)\, x_{n+1}$, $n = 0, 1, 2, \ldots$. Note that $\lambda \in \varrho_T(x)$ if and only if there is such a chain for which $\|x_n\|^{1/n}$ is bounded. Take the algebraic local spectrum of x_0 as the complement of the set just introduced. It is obvious that the algebraic local spectrum of x_0 may be characterized as the smallest set $A \subseteq \mathbb{C}$ for which $x_0 \in E_T(A)$.

In some instances simple characterizations of E_T-spaces are available. Here is a sufficient condition that E_T-spaces are "large", i.e. countable intersections of ranges of the operator T.

Lemma. *Let T be a bounded linear operator on a Banach space X for which $\ker T^n = \ker T^{n+1}$ for some $n \in \mathbb{N}$. Then*

$$E_T(\mathbb{C} \setminus \{0\}) = \bigcap_{k=1}^{\infty} T^k X.$$

Proof. Letting $Y := \bigcap_{k=1}^{\infty} T^k X$ we shall show that

$$E_T(\mathbb{C} \setminus \{0\}) \subseteq Y \subseteq E_{T^n}(\mathbb{C} \setminus \{0\}) \subseteq E_T(\mathbb{C} \setminus \{0\}).$$

The inclusion on the left is obvious from the definitions of $E_T(\mathbb{C} \setminus \{0\})$ and of Y. The inclusion on the right is a consequence of this observation: with $Z := ET_n(\mathbb{C} \setminus \{0\}$

$$Z \supseteq TZ \supseteq T^2 Z \supseteq \ldots \supseteq T^n Z = Z$$

so that $Z \subseteq E_T(\mathbb{C} \setminus \{0\})$, by maximality.

To show that $Y \subseteq E_{T^n}(\mathbb{C} \setminus \{0\})$, observe first that $Y = \bigcap_{k=1}^{\infty} (T^n)^k X$ and second that

$\ker(T^n)^2 = \ker T^n$. Thus there is no loss of generality in assuming that $n = 1$, so that $\ker T = \ker T^2$. Since $TY \subseteq Y$, we have to show that $Y \subseteq TY$. Let $x = T^n y_n \in Y$, $n = 1, 2, \ldots$ and note that $0 = T^2(T^{n-2}y_n - T^{n-1}y_{n+1})$ for $n = 2, 3, \ldots$. Hence $T^{n-2}y_n - T^{n-1}y_{n+1} \in \ker T^2 = \ker T$ so that $T^{n-1}y_n = T^n y_{n+1}$, $n = 2, 3, \ldots$. Let the constant value of this latter sequence be y. Evidently $y \in Y$ and $Ty = T(T^{n-1}y_n) = T^n y_n = x$. This completes the proof.

Corollary. *Let T be a generalized scalar operator. Then*

$$E_T(A) = \bigcap_{n \in N, \lambda \notin A} (T - \lambda)^n X \quad \text{for any} \quad A \subseteq C$$

In particular this holds if T is a normal operator on a Hilbert space.

Proof. It is sufficient to show that $\ker T^n = \ker T^{n+1}$ for some $n \in N$. If T is generalized scalar then there exists an n such that $E_T(\{\lambda\}) \subseteq \ker(T - \lambda)^n$ (cf. e.g. [10] and [6]). On the other hand, $\ker(T - \lambda)^k \subseteq E_T(\{\lambda\})$ for all $k \geqq 0$. Thus $\ker(T - \lambda)^k = \ker(T - \lambda)^n$ for $k \geqq n$.

In describing the surjectivity spaces, and in light of Lemma 2, it is also relevant to introduce here the $X_T(A)$-spaces:

Definition. For a bounded linear operator T on a Banach space X and for $A \subseteq C$

$$X_T(A) := \{x \in X \mid \sigma_T(x) \subseteq A\}$$

It is immediate from the definition that $X_T(A)$ is a linear subspace of X; also, $X_T(A) \in \operatorname{lat}(T)$ for any subset $A \subseteq C$. The proofs are identical to the classical ones [2, Proposition 1.1.2]. Moreover, as pointed out in [5, 8],

$$X_T(A) \subseteq E_T(A),$$

because $\sigma_s(T; X_T(A)) \subseteq A$, for any $A \subseteq C$. Additionally, if $A_1 \subseteq A_2$, then $X_T(A_1) \subseteq X_T(A_2)$ and $E_T(A_1) \subseteq E_T(A_2)$. Note also that it is an immediate consequence of Lemma 2 that $X_T(A) = X_T(A \cap \sigma_s(T))$ for any subset $A \subseteq C$.

The precise relationship between $X_T(A)$ and $E_T(A)$ for a given $A \subseteq C$ remains unknown, but we can offer the following partial answer. A bounded linear operator T on a Banach space X is said to be *well-decomposable* [1] if for every open cover U, V of C there is a linear operator R on X, closed T-invariant subspaces Y, Z of X such that $\sigma(T \mid Y) \subseteq U$, $\sigma(T \mid Z) \subseteq V$, $RX \subseteq Y$, $(I - R) X \subseteq Z$, and an integer n for which $C(T)^n R = 0$, where $C(T) R := TR - RT$. We have the following

Proposition 4. Let T be well-decomposable and let $F \subseteq C$ be closed. Then

$$E_T(F) = \bigcap_{U \supseteq F} (X_T(U) + E_T(\emptyset))$$

where U ranges over all neighbourhoods of F.

Proof. Since $X_T(U) + E_T(\emptyset) \subseteq E_T(U)$ and since $\bigcap_{U \supseteq F} E_T(U) = E_T(F)$ [5], the inclusion $\supseteq$ is immediate. For the converse, let U be an open neighborhood of F and choose R as in the definition of well-decomposability, corresponding to the open cover U, $C \setminus F$. Then $RE_T(F) \subseteq RX \subseteq Y$ and since Y is closed and $\sigma(T \mid Y) \subseteq U$ we get $RE_T(F) \subseteq X_T(U)$. On the other hand, by [7, Corollary 1.2] $RE_T(F) \subseteq E_T(F)$, hence $(I - R)\,E_T(F) \subseteq E_T(F)$ and since $(I - R)\,E_T(F) \subseteq Z \subseteq E_T(\mathbb{C} \setminus F)$ we get that $(I - R)\,E_T(F) \subseteq E_T(F) \cap E_T(\mathbb{C} \setminus F) = E_T(\emptyset)$, hence $E_T(F) \subseteq RE_T(F) + (I - R)\,E_T(F) \subseteq X_T(U) + E_T(\emptyset)$. This proves the other inclusion.

This cannot be improved to $E_T(F) = X_T(F) + E_T(\emptyset)$, even for the smaller class of super-decomposable operators. These are defined as the subclass of the well-decomposable operators for which n in the definition can be taken to be 1. We have the following concrete example.

Proposition 5. *There is a super-decomposable operator T on a Hilbert space H for which $E_T(\{0\}) \neq X_T(\{0\}) + E_T(\emptyset)$.*

Proof. On a Hilbert space H_0 let T_0 be a quasi-nilpotent operator with dense $E_{T_0}(\emptyset)$, e.g. T_0 can be a weighted left shift on $\ell^2(\mathbb{N})$, weighted so that $\sigma(T_0) = \{0\}$; in this case it easy to see that T_0 maps the subspace of finitely supported elements of ℓ^2 onto itself, hence $E_{T_0}(\emptyset)$ is indeed dense. Take a sequence $\{\alpha_n\}$ of strictly positive numbers converging to zero and let

$$T := \oplus_{i=1}^{\infty} (T + \alpha_i)$$

on the Hilbert space $H := \oplus_{i=1}^{\infty} H_0$. Then the spectrum $\sigma(T)$ is a convergent sequence and hence [6, p. 40] T is super-decomposable.

Let $Y := \oplus_{i=1}^{\infty} E_{T_0}(\emptyset)$. We claim that $Y \subseteq E_T(\{0\})$. Let $\alpha \in \mathbb{C}$ be non-zero and let $x = \oplus_{i=1}^{\infty} x_i \in Y$; choose $y_i \in E_{T_0}(\emptyset)$ so that $(T_0 - \alpha_i - \alpha)\, y_i = x_i$ for $i = 1, 2, \dots$. Since $\alpha_i \to 0$ we can find $n_0 \in \mathbb{N}$ so that $(T_0 + \alpha_i - \alpha)^{-1}$ is defined for $i \geqq n_0$. Note also that there is a constant $C \in \mathbb{R}_+$ so that $\|(T_0 + \alpha_i - \alpha)^{-1}\| < C$ for all $i \geqq n_0$. Since $y_i = (T_0 + \alpha_i - \alpha)^{-1}\, x_i$ for $i \geqq n_0$ we get $\sum_{i > n_0} \|y_i\|^2 < C^2 \sum_{i > n_0} \|x_i\|^2$ and hence $y := \oplus_{i=1}^{\infty} y_i \in Y$ and $(T - \alpha)\, y = x$. This shows that $Y \subseteq (T - \alpha)\,Y$ for every $\alpha \neq 0$. Since the inclusion $(T - \alpha)\,Y \subseteq Y$ is trivial we have shown that $Y \subseteq E_T(\{0\})$.

On the other hand we next show that Y is not contained in $E_T(\emptyset)$. For $\alpha > 0$ it is obvious that $\|(T_0 + \alpha)^{-1}\| \geqq \alpha^{-1}$ and since $E_{T_0}(\emptyset)$ is dense in H_0 we can find $x_\alpha \in E_T(\emptyset)$ for which $\|x_\alpha\| \leqq 1$ and $\|(T + \alpha)^{-1}\, x_\alpha\| \geqq (2\alpha)^{-1}$. Pick a square summable subsequence $\{\alpha_{n_k}\}$ and define $y_i := x_i / \|(T_0 + \alpha_i)^{-1}\, x_i\|$ for $i \in \{n_k\}$ and $y_i = 0$ for all other values of i. Then

$$\sum \|y_i\|^2 = \sum \|x_i\|^2 / \|(T_0 + \alpha_i)^{-1}\, x_i\|^2 \leqq \sum 4\alpha_{n_k}^2 < \infty\,.$$

This shows that $y := \oplus_{i=1}^{\infty} y_i \in Y$. However, y is not in the range of T, because if $y = Tz$ for some $z = \oplus_{i=1}^{\infty} z_i$ then $y_i = (T_0 + \alpha_i)\, z_i$ so that for $i \in \{n_k\}$ $z_i = (T_0 + \alpha_i)^{-1}\, y_i$. By definition of y_i, $\|z_i\| = 1$ for $i \in \{n_k\}$, so $\sum \|z_i\|^2 = \infty$.

If $\varepsilon \in \mathbb{R}_+$ and $P_\varepsilon(\oplus x_i) := \oplus_{|\alpha_i| \leqq \varepsilon} x_i$, then P_ε is a spectral projection and by [2, Proposition 1.3.10 and Corollary 1.3.4] $X_T(\{0\}) \subseteq P_\varepsilon H$ for each $\varepsilon \in \mathbb{R}_+$. Hence

$$X_T(\{0\}) \subseteq \bigcap_{\varepsilon > 0} P_\varepsilon H = \{0\}\,.$$

Since $E_T(\emptyset) = X_T(\{0\}) + E_T(\emptyset)$ and since $Y \subseteq E_T(\{0\})$, but Y is not a subset of

$E_T(\emptyset)$ it follows that

$$E_T(\{0\}) \neq X_T(\{0\}) + E_T(\emptyset) .$$

Remark. Suppose T possesses the following property which might be termed *strong algebraic decomposability*: for every closed set $F \subseteq \mathbb{C}$ and every open cover U, V of $\mathbb{C}$

$$E_T(F) = [E_T(F) \cap X_T(U)] + [E_T(F) \cap X_T(V)] .$$

(Note that the proof of Proposition 4 shows that every well-decomposable operator has this property). Then

$$E_T(F) = \bigcap_{U \supseteq F} (X_T(U) + E_T(\emptyset))$$

where U ranges over all neighborhoods of F. The proof of this is simple: by assumption, if $U \supseteq F$ is an open neighborhood then

$$E_T(F) = (E_T(F) \cap X_T(U)) + (E_T(F) \cap X_T(\mathbb{C} \setminus F)) \subseteq$$
$$\subseteq X_T(U) + (E_T(F) \cap E_T(\mathbb{C} \setminus F)) = X_T(U) + E_T(\emptyset) .$$

As a corollary we obtain that if T is strongly algebraically decomposable and if $E_T(\emptyset) = \{0\}$ then $E_T(F) = X_T(F)$ for all closed sets $F \subseteq \mathbb{C}$.

Without some kind of decomposability it is not sufficient to assume that $E_T(\emptyset) = \{0\}$ in order to conclude that $E_T(F) = X_T(F)$. But it remains an open question whether for a (strongly) decomposable operator T with $E_T(\emptyset) = \{0\}$ we have $E_T(F) = X_T(F)$.

If, however, $E_T(F)$ is known to be closed then we shall show equality in general (proposition 10 below). If $E_T(\emptyset) = \{0\}$, this also means that $E_T(F)$ is spectral maximal [5, Proposition 3.3] Recall that a closed linear subspace $Y \subseteq X$ is called spectral maximal if Y is T-invariant and if for any other T-invariant subspace $Z \subseteq X$, if $\sigma(T \mid Z) \subseteq \sigma(T \mid Y)$ then $Z \subseteq Y$.

It is easy to see that if T has SVEP and if $F \subseteq \mathbb{C}$ is closed then $E_T(F)$ is spectral maximal if and only if $E_T(F)$ is closed: obviously, if T has SVEP then $S(T \mid Z) = \emptyset$ for any closed T-invariant subspace Z, so if $E_T(F)$ is closed and if Z is a closed T-invariant subspace with $\sigma(T \mid Z) \subseteq \sigma(T \mid E_T(F))$ then $\sigma_s(T; Z) \subseteq \sigma_s(T; E_T(F)) \subseteq F$, hence $Z \subseteq E_T(F)$. This observation will be generalized below, in Corollary 8.

Example. Without SVEP closedness of $E_T(F)$ is no longer sufficient for spectral maximality: consider the left shift L on $X := \ell^1$, as in the previous Example. Let C denote the unit circle; then $E_L(C) = X$. Now let Z be any non-zero Banach space and let $Y := X \oplus Z$. Define $T: Y \to Y$ by $T(x, z) := (Tx, 0)$ for all $(x, z) \in Y$. Evidently $\sigma(T) = D$, the unit disc. Moreover, $E_T(C) = X \oplus \{0\}$. But since $\sigma(T \mid E_T(C)) = D$, $E_T(C)$ cannot be spectral maximal.

We can pursue the connection between closedness of $E_T(F)$ and spectral maximality a bit further. We have the following improvement of [5, Proposition 3.5). First a lemma.

Lemma 6. $S(T) = \{\lambda \in \mathbb{C}$: *there is a neighborhood* N_λ *and a holomorphic function* $f: N_\lambda \to X$ *without zeros for which* $(T - \mu) f(\mu) = 0$ *on* $N_\lambda\}$.

Proof. Denote by $\mathcal{M}$ the set on the right. Clearly $\mathcal{M} \subseteq S(\boldsymbol{T})$. To show the other inclusion, take $\lambda \in S(\boldsymbol{T})$ so that there is a connected neighborhood G of λ and a non-zero holomorphic function $f\colon G \to \boldsymbol{X}$ such that $(\boldsymbol{T} - \mu) f(\mu) = 0$ on G. Since f is not identically zero and G is connected we can find a minimal m for which $f^{(m)}(\lambda) \neq 0$. Differentiation (n times) of the equation $(\boldsymbol{T} - \mu) f(\mu) = 0$ yields

$$f^{(n)}(\mu) = 1/(n+1)\ (\boldsymbol{T} - \mu) f^{(n+1)}(\mu) \quad \text{for} \quad n \geqq 0 \quad \text{and} \quad \mu \in G\,.$$

The sequence

$$x_n := m!/(m+n-1)!\ f^{(m+n-1)}(\lambda)$$

satisfies $x_n = (\boldsymbol{T} - \lambda)\, x_{n+1}$ $(n > 0)$ and $(\boldsymbol{T} - \lambda)\, x_1 = 0$. Since $\limsup \|x_n\|^{1/n} = \limsup \|m!/(m+n-1)!\, f^{(m+n-1)}(\lambda)\|^{1/n} < \infty$, the series

$$g(\mu) := \sum_{n=1}^{\infty} x_n (\mu - \lambda)^{n-1}$$

converges for $|\mu - \lambda| < (\limsup \|x_n\|^{1/n})^{-1}$ and

$$\begin{aligned}(\boldsymbol{T} - \mu)\, g(\mu) &= (T - \lambda) \sum_{n=1}^{\infty} x_n (\mu - \lambda)^{n-1} - \sum_{n=1}^{\infty} x_n (\mu - \lambda)^n = \\ &= \sum_{n=2}^{\infty} x_{n-1} (\mu - \lambda)^{n-1} - \sum_{n=1}^{\infty} x_n (\mu - \lambda)^n = 0\,.\end{aligned}$$

Moreover, $g(\lambda) = x_1 = f^{(m)}(\lambda) \neq 0$. Thus N_λ may be chosen so that g has no zeros in N_λ.

Proposition. *Let $A \subseteq \mathbb{C}$ be any subset and let $\sigma(\boldsymbol{T} \mid E_{\boldsymbol{T}}(A)) := \{\lambda \in \mathbb{C} \mid \boldsymbol{T} - \lambda$ is not a bijection on $E_{\boldsymbol{T}}(A)\}$. Then $S(\boldsymbol{T}) \subseteq \sigma(\boldsymbol{T} \mid E_{\boldsymbol{T}}(A))$.*

Proof. We may suppose $S(\boldsymbol{T}) \neq \emptyset$. If $\lambda \in S(\boldsymbol{T})$ then, by Lemma 6, there is an open set G containing λ and a holomorphic function $f\colon G \to \boldsymbol{X} \setminus \{0\}$ which satisfies $(\boldsymbol{T} - \mu) f(\mu) = 0$ on G. Now define

$$g(\mu) := \begin{cases} (f(\mu) - f(\lambda))/(\mu - \lambda) & \mu \in G \setminus \{\lambda\} \\ f'(\lambda) & \mu = \lambda \end{cases}$$

Then $f(\mu) = (\boldsymbol{T} - \mu)\, g(\mu)$ for every $\mu \in G$; this shows that $G \subseteq \varrho_{\boldsymbol{T}}(f(\lambda))$, hence $\sigma_{\boldsymbol{T}}(f(\lambda)) \cap G = \emptyset$. Since $(\boldsymbol{T} - \lambda) f(\lambda) = 0$, $f(\lambda) \in E_{\boldsymbol{T}}(G)$ by absorbency [5, Proposition 2.2] and so $f(\lambda) \in \boldsymbol{X}_{\boldsymbol{T}}(\sigma_{\boldsymbol{T}}(f(\lambda))) \cap E_{\boldsymbol{T}}(G) = E_{\boldsymbol{T}}(\emptyset) \subseteq E_{\boldsymbol{T}}(A)$. Thus $\boldsymbol{T} - \lambda$ is not 1-1 on $E_{\boldsymbol{T}}(A)$, so that $\lambda \in \sigma(\boldsymbol{T} \mid E_{\boldsymbol{T}}(A))$.

Lemma 7. *Suppose $E_{\boldsymbol{T}}(F)$ is closed. Then*

$$S(\boldsymbol{T}) \subseteq \sigma(\boldsymbol{T} \mid E_{\boldsymbol{T}}(F)) \subseteq F \cup S(\boldsymbol{T})\,.$$

Proof. This follows immediately from Lemma 3, since $\sigma_s(\boldsymbol{T} \mid E_{\boldsymbol{T}}(F)) \subseteq F$ and since, by (the proof of) the above Proposition, $S(\boldsymbol{T}) = S(\boldsymbol{T} \mid E_{\boldsymbol{T}}(F))$.

Corollary 8. *If $E_{\boldsymbol{T}}(F)$ is closed and if $S(\boldsymbol{T}) \subseteq F$ then $E_{\boldsymbol{T}}(F)$ is spectral maximal.*

Proof. Clear, since $\sigma(\boldsymbol{T} \mid E_{\boldsymbol{T}}(F)) \subseteq F$.

Corollary 9. *If $E_{\boldsymbol{T}}(F)$ is closed then F contains the topological boundary of $S(\boldsymbol{T})$.*

Proof. If $E_T(F)$ is closed then $\sigma(T \mid E_T(F))$ is closed as well and by Lemma 7,

$$S(T)^- \subseteq \sigma(T \mid E_T(F)) \subseteq F \cup S(T).$$

Since $\delta(S(T)) \cap S(T) = \emptyset$ it follows that $\delta(S(T) \subseteq F$.

Proposition 10. *Suppose $F \subseteq \mathbb{C}$ is closed and $F_T(F)$ is closed. Then $E_T(F) = X_T(F)$.*

Proof. Let $Y := E_T(F)$. By Lemma 2, $\sigma_s(T \mid Y) = \bigcup_{x \in Y} \sigma_{T|Y}(x)$. Obviously, for $x \in Y$, $\varrho_{T|Y}(x) \subseteq \varrho_T(x)$ so $\sigma_T(x) \subseteq \sigma_s(T \mid Y)$ for every $x \in E_T(F)$. Since $\sigma_s(T \mid Y) \subseteq F$ the Proposition follows.

Note that the example before Lemma 6 then shows that closedness of $X_T(F)$ is not sufficient for spectral maximality, either, in the absence of SVEP.

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