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Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 4, 674–680

Persistent URL: http://dml.cz/dmlcz/102343

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

MAL'CEV CONDITIONS FOR DIRECTLY DECOMPOSABLE COMPATIBLE RELATIONS

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(Received November 11, 1987)

1. PRELIMINARIES

Mal'cev conditions for varieties having directly decomposable tolerances and directly decomposable reflexive compatible relations were given independently in [9] and [4]. Varieties having directly decomposable tolerance classes and directly decomposable relation classes were studied in [3]. The aim of this paper is to show that:

(i) the direct decomposability of tolerances (reflexive compatible relations) coincides with the direct decomposability of tolerance classes (relation classes, respectively) in varieties of algebras;

(ii) Mal'cev conditions from [9], [4] can be replaced by simpler ones;

(iii) all the above mentioned properties of tolerances and reflexive compatible relations in a variety V can be considered only on the square $F_V(2) \times F_V(2)$ of the V-free algebra $F_V(2)$ over two free generators.

To make this paper selfcontained we recall some definitions:

Definition 1. Let A, B be algebras of the same type. The kernels Π_A, Π_B of the canonical projections $pr_A: A \times B \to A$, $pr_B: A \times B \to B$, respectively, are called *factor congruences* on $A \times B$. A binary relation R on $A \times B$ is called a *subfactor relation* whenever $R \subseteq \Pi_A$ or $R \subseteq \Pi_B$.

Definition 2. Let R be a reflexive binary relation on a set A and let $a \in A$. Then the subset $[a] R = \{x \in A;\ \langle x, a\rangle \in R\}$ is called a *relation class* of R. In particular $[a] T$ is called a *tolerance class* provided T is a tolerance on A.

Definition 3. Let A, B be algebras of the same type. The product $A \times B$ has *directly decomposable relations* (*relation classes*) if every relation R (relation class C) on $A \times B$ is a product of its projections $\langle pr_A, pr_A\rangle R$ and $\langle pr_B, pr_B\rangle R$ ($pr_A C$ and $pr_B C$, respectively).

A variety of algebras V has some of the properties listed above whenever for every $A, B \in V$, $A \times B$ has the respective property.

In what follows, by a relation on an algebra A we mean a *compatible relation* on A, i.e. a subalgebra of $A \times A$. It is well known and frequently used that for any subset $M \subseteq A \times A$ the least tolerance $T(M)$ and the least reflexive relation $R(M)$ on A containing M exist. The functional descriptions of $T(M)$ and $R(M)$ are adopted from [1].

The symbol $\boldsymbol{c}^{\rightarrow}$ stands for the finite sequence $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m$.

2. DIRECTLY DECOMPOSABLE TOLERANCES

Theorem 1. *For a variety V the following conditions are equivalent:*

(1) *V has directly decomposable subfactor tolerances;*

(2) *there exist binary terms $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_n$ and a $(2 + n)$-ary term $\boldsymbol{r}$ such that the identities*

$$x = \boldsymbol{r}(x, y, \boldsymbol{c}^{\rightarrow}(x, y)),$$
$$y = \boldsymbol{r}(x, x, \boldsymbol{d}^{\rightarrow}(x, y)),$$
$$y = \boldsymbol{r}(y, x, \boldsymbol{c}^{\rightarrow}(x, y))$$

hold in V.

Proof. (1) $\Rightarrow$ (2): Consider the principal tolerance $T(\langle x, x\rangle, \langle y, x\rangle)$ on the square $F_V(2) \times F_V(2)$ of the V-free algebra $F_V(2)$ over free generators x and y. Since $\langle\langle x, x\rangle, \langle y, x\rangle\rangle \in T(\langle x, x\rangle, \langle y, x\rangle)$ the hypothesis yields $\langle\langle x, y\rangle, \langle y, y\rangle\rangle \in T(\langle x, x\rangle, \langle y, x\rangle)$. Using the functional description of $T(\langle x, x\rangle, \langle y, x\rangle)$, see [1], the identites (2) readily follow.

(2) $\Rightarrow$ (1): Let T be a subfactor tolerance on $A \times B \in V$, say $T \subseteq \Pi_A$. Assuming the identities (2) we find

$$x' = \boldsymbol{r}(x, x, \boldsymbol{d}^{\rightarrow}(x, x')),$$
$$y = \boldsymbol{r}(y, z, \boldsymbol{c}^{\rightarrow}(y, z)),$$
$$x' = \boldsymbol{r}(x, x, \boldsymbol{d}^{\rightarrow}(x, x')),$$
$$z = \boldsymbol{r}(z, y, \boldsymbol{c}^{\rightarrow}(y, z)),$$

i.e. $\langle\langle x, y\rangle, \langle x, z\rangle\rangle \in T$ implies $\langle\langle x', y\rangle, \langle x', z\rangle\rangle \in T$ for any $x, x' \in A$, $y, z \in B$. The proof is complete.

Remark 1. The Mal'cev condition for varieties having directly decomposable subfactor congruences was given by J. Hagemann [8].

Lemma 1. *Let A, B be algebras of the same type. For any tolerance class $[\langle z_1, z_2\rangle]\, T$ on $A \times B$ the following conditions are equivalent:*

(1) *$[\langle z_1, z_2\rangle]\, T$ is directly decomposable;*

(2) (i) *$\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$ implies $\langle x, z_2\rangle, \langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\, T$;*

(ii) *$\langle x, z_2\rangle, \langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\, T$ imply $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): Let $\langle a, b\rangle, \langle a', b'\rangle \in [\langle z_1, z_2\rangle]\, T$. Then $\langle a, z_2\rangle, \langle z_1, b'\rangle \in [\langle z_1, z_2\rangle]\, T$,

by (2)(i). So $\langle a, b'\rangle \in [\langle z_1, z_2\rangle]\, T$, by (2)(ii). The last argument establishes the direct decomposability of the tolerance class $[\langle z_1, z_2\rangle]\, T$.

Lemma 2. *Let A, B be algebras of the same type. The following conditions are equivalent:*

(1) *$A \times B$ has directly decomposable tolerances;*

(2) *$A \times B$ has directly decomposable subfactor tolerances and directly decomposable tolerance classes.*

Proof. Only the implication (2) $\Rightarrow$ (1) is nontrivial: Let T be a tolerance on $A \times B$ and let $\langle\langle a, b\rangle, \langle c, d\rangle\rangle, \langle\langle a', b'\rangle, \langle c', d'\rangle\rangle \in T$. Since $\langle\langle a, b\rangle, \langle c, d\rangle\rangle, \langle\langle c, d\rangle, \langle c, d\rangle\rangle \in T$, the direct decomposability of tolerance classes yields $\langle\langle a, d\rangle, \langle c, d\rangle\rangle \in \in T$. Then also $\langle\langle a, d'\rangle, \langle c, d'\rangle\rangle \in T$ by the direct decomposability of subfactor tolerances. Analogously from $\langle\langle c', d'\rangle, \langle c', d'\rangle\rangle, \langle\langle a', b'\rangle, \langle c', d'\rangle\rangle \in T$ we find $\langle\langle c', b'\rangle, \langle c', d'\rangle\rangle \in T$ and, further, $\langle\langle c, b'\rangle, \langle c, d'\rangle\rangle \in T$. Altogether $\langle\langle a, d'\rangle, \langle c, d'\rangle\rangle, \langle\langle c, b'\rangle, \langle c, d'\rangle\rangle \in T$ which implies $\langle\langle a, b'\rangle, \langle c, d'\rangle\rangle \in T$. This proves the direct decomposability of T, see [2; Thm 1, p. 227].

Theorem 2. *For a variety V the following conditions are equivalent:*

(1) *V has directly decomposable tolerances;*

(2) *V has directly decomposable tolerance classes;*

(3) *there exist ternary terms $p_1, \ldots, p_n, q_1, \ldots, q_n$ and a $(4 + n)$-ary term s such that the identities*

$$\begin{aligned} x &= s(x, y, z, z, \vec{p}(x, y, z)), \\ y &= s(x, y, z, z, \vec{q}(x, y, z)), \\ z &= s(z, z, x, y, \vec{p}(x, y, z)), \\ z &= s(z, z, x, y, \vec{q}(x, y, z)) \end{aligned}$$

hold in V;

(4) *there exist binary terms $f_1, \ldots, f_{n+2}, g_1, \ldots g_{n+2}, h_1, \ldots, h_n, k_1, \ldots, k_n$ and $(4 + n)$-ary terms s_1, s_2 such that the identities*

$$\left.\begin{aligned} x &= s_1(x, y, \vec{f}(x, y)) \\ x &= s_1(y, x, \vec{g}(x, y)) \\ y &= s_1(y, x, \vec{f}(x, y)) \\ x &= s_1(x, y, \vec{g}(x, y)) \end{aligned}\right\} (\Sigma_1)$$

$$\left.\begin{aligned} x &= s_2(x, y, y, y, \vec{h}(x, y)) \\ y &= s_2(x, x, y, x, \vec{k}(x, y)) \\ y &= s_2(y, x, y, y, \vec{h}(x, y)) \\ x &= s_2(x, x, x, y, \vec{k}(x, y)) \end{aligned}\right\} (\Sigma_2)$$

hold in V.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) was already shown in [3; Thm 4, p. 400].

(3) ⇒ (4): Setting $z = y$ in the identities (3) we find that

$$
\begin{aligned}
x &= \mathbf{s}(x, y, y, y, \mathbf{p}^{\rightarrow}(x, y, y)), \\
y &= \mathbf{s}(x, y, y, y, \mathbf{q}^{\rightarrow}(x, y, y)), \\
y &= \mathbf{s}(y, y, x, y, \mathbf{p}^{\rightarrow}(x, y, y)), \\
y &= \mathbf{s}(y, y, x, y, \mathbf{q}^{\rightarrow}(x, y, y)).
\end{aligned}
$$

Interchange the variables x and y in the second and the fourth identities. Then

$$
\begin{aligned}
x &= \mathbf{s}(x, y, y, y, \mathbf{p}^{\rightarrow}(x, y, y)), \\
x &= \mathbf{s}(y, x, x, x, \mathbf{q}^{\rightarrow}(y, x, x)), \\
y &= \mathbf{s}(y, y, x, y, \mathbf{p}^{\rightarrow}(x, y, y)), \\
x &= \mathbf{s}(x, x, y, x, \mathbf{q}^{\rightarrow}(y, x, x)).
\end{aligned}
$$

Defining

$$
\begin{aligned}
\mathbf{s}_1(a, b, w^{\rightarrow}) &= \mathbf{s}(a, w_{n+1}, b, w_{n+2}, w_1, \ldots, w_n), \\
\mathbf{f}^{\rightarrow}(x, y) &= \mathbf{p}_1(x, y, y), \ldots, \mathbf{p}_n(x, y, y), y, y, \quad \text{and} \\
\mathbf{g}^{\rightarrow}(x, y) &= \mathbf{q}_1(y, x, x), \ldots, \mathbf{q}_n(y, x, x), x, x
\end{aligned}
$$

we get the identities (Σ_1).

Setting $z = y$ in the first and the third identities (3) and $z = x$ in the remaining ones we obtain

$$
\begin{aligned}
x &= \mathbf{s}(x, y, y, y, \mathbf{p}^{\rightarrow}(x, y, y)), \\
y &= \mathbf{s}(x, y, x, x, \mathbf{q}^{\rightarrow}(x, y, x)), \\
y &= \mathbf{s}(y, y, x, y, \mathbf{p}^{\rightarrow}(x, y, y)), \\
x &= \mathbf{s}(x, x, x, y, \mathbf{q}^{\rightarrow}(x, y, x)).
\end{aligned}
$$

Now the identities (Σ_2) follow for

$$
\begin{aligned}
\mathbf{s}_2(a, b, c, d, w^{\rightarrow}) &= \mathbf{s}(a, c, b, d, w^{\rightarrow}), \\
\mathbf{h}^{\rightarrow}(x, y) &= \mathbf{p}^{\rightarrow}(x, y, y), \quad \text{and} \\
\mathbf{k}^{\rightarrow}(x, y) &= \mathbf{q}^{\rightarrow}(x, y, x).
\end{aligned}
$$

(4) ⇒ (1): The identities (Σ_2) ensure the direct decomposability of subfactor tolerances. Defining

$$
\begin{aligned}
\mathbf{r}(a, b, w^{\rightarrow}) &= \mathbf{s}_2(a, b, w_{n+1}, w_{n+2}, w_1, \ldots, w_n), \\
\mathbf{c}^{\rightarrow}(x, y) &= \mathbf{h}_1(x, y), \ldots, \mathbf{h}_n(x, y), y, y, \quad \text{and} \\
\mathbf{d}^{\rightarrow}(x, y) &= \mathbf{k}_1(x, y), \ldots, \mathbf{k}_n(x, y), y, x
\end{aligned}
$$

we get the identities from Theorem 1 (2).

Further, the identities (Σ_1) yield

$$
\begin{aligned}
x &= \mathbf{s}_1(x, z_1, \mathbf{f}^{\rightarrow}(x, z_1)), \\
z_2 &= \mathbf{s}_1(y, z_2, \mathbf{g}^{\rightarrow}(z_2, y)), \\
z_1 &= \mathbf{s}_1(z_1, x, \mathbf{f}^{\rightarrow}(x, z_1)). \\
z_2 &= \mathbf{s}_1(z_2, y, \mathbf{g}^{\rightarrow}(z_2, y)),
\end{aligned}
$$

which means that $\langle x, z_2\rangle \in [\langle z_1, z_2\rangle]\, T$ whenever $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$. Similarly

$$\begin{aligned} y &= s_1(y, z_2, f^{\to}(y, z_2))\,, \\ z_1 &= s_1(x, z_1, g^{\to}(z_1, x))\,, \\ z_2 &= s_1(z_2, y, f^{\to}(y, z_2))\,, \\ z_1 &= s_1(z_1, x, g^{\to}(z_1, x)) \end{aligned}$$

follow from the identities (Σ_1). This establishes that $\langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\, T$ whenever $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$.

Finally, we use again the identities (Σ_2). One easily checks that

$$\begin{aligned} x &= s_2(x, z_1, z_1, z_1, h^{\to}(x, z_1))\,, \\ y &= s_2(z_2, z_2, y, z_2, k^{\to}(z_2, y))\,, \\ z_1 &= s_2(z_1, x, z_1, z_1, h^{\to}(x, z_1))\,, \\ z_2 &= s_2(z_2, z_2, z_2, y, k^{\to}(z_2, y))\,, \end{aligned}$$

which proves that $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$ is a consequence of $\langle x, z_2\rangle, \langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\, T$. Lemma 1 and Lemma 2 complete the proof.

Corollary 1. *For a variety V the following conditions are equivalent*:

(1) *V has directly decomposable tolerances*;

(2) *$F_V(2) \times F_V(2)$ has directly decomposable tolerances.*

3. DIRECTLY DECOMPOSABLE REFLEXIVE RELATIONS

In this section we generalize the above results to reflexive relations. Since the proofs of Theorems 3, 4 are very similar to those of Theorems 1, 2 we omit the details.

Theorem 3. *For a variety V the following conditions are equivalent*:

(1) *V has directly decomposable subfactor reflexive relations*;

(2) *there exist binary terms $c_1, \ldots, c_n, d_1, \ldots, d_n$ and a $(1 + n)$-ary term u such that the identities*

$$\begin{aligned} x &= u(x, c^{\to}(x, y))\,, \\ y &= u(x, d^{\to}(x, y))\,, \\ y &= u(y, c^{\to}(x, y)) \end{aligned}$$

hold in V

Proof. Apply [1] and the proof of Theorem 1.

Theorem 4. *For a variety V the following conditions are equivalent*:

(1) *V has directly decomposable reflexive relations*;

(2) *V has directly decomposable relation classes*;

(3) *there exist ternary terms $p_1, \ldots, p_n, q_1, \ldots, q_n$ and a $(2 + n)$-ary term v such*

that the identities

$$x = v(x, y, p^{\to}(x, y, z)),$$
$$y = v(x, y, q^{\to}(x, y, z)),$$
$$z = v(z, z, p^{\to}(x, y, z)),$$
$$z = v(z, z, q^{\to}(x, y, z))$$

hold in V;

(4) *there exist binary terms $f_1, \dots, f_{n+1}$, $g_1, \dots, g_{n+1}$, $h_1, \dots, h_n$, $k_1, \dots, k_n$ and $(2 + n)$-ary terms v_1, v_2 such that the identities*

$$x = v_1(x, f^{\to}(x, y)),$$
$$x = v_1(y, g^{\to}(x, y)),$$
$$y = v_1(y, f^{\to}(x, y)),$$
$$x = v_1(x, g^{\to}(x, y)),$$
$$x = v_2(x, y, h^{\to}(x, y)),$$
$$y = v_2(x, y, k^{\to}(x, y)),$$
$$y = v_2(y, y, h^{\to}(x, y)),$$
$$x = v_2(x, x, k^{\to}(x, y))$$

hold in V.

Proof. (1) $\Rightarrow$ (2) is trivial.

The implication (2) $\Rightarrow$ (3) was already proved in [3; Thm 5, pp. 400–401].

The rest of the proof follows the same lines as in the proof of Theorem 2.

Corollary 2. *For a variety V the following conditions are equivalent:*

(1) *V has directly decomposable reflexive relations;*

(2) *$F_V(\mathbf{2}) \times F_V(\mathbf{2})$ has directly decomposable reflexive relations.*

Example 1. The variety L of all lattices satisfies all the above identities. This follows directly from the fact that $F_L(\mathbf{2}) \cong \mathbf{2} \times \mathbf{2}$.

4. CONCLUSION

The Mal'cev condition for varieties having *directly decomposable congruences* was given by G. A. Fraser and A. Horn in [6]. Using the method exhibited in Section 2 of this paper one easily checks that also the direct decomposability of congruences in varieties can be considered only on the square $F_V(\mathbf{2}) \times F_V(\mathbf{2})$. The simplification of the original Fraser-Horn identities is shown in [5].

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