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ADDITIVE RADICALS

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INTRODUCTION

For the basic notions and results from radical theory we refer to V. A. Andrunakievič, J. M. Rjabuchin [3], N. Divinsky [7], F. A. Szász [16], R. Wiegandt [18]. Recall that A. Sulinski, T. Anderson, N. Divinsky [14] showed that for every radical α and for every ideal I of a ring A, $\alpha(I)$ is an ideal of A. Thus we get a mapping α from the lattice $L(A)$ of ideals of A to $L(A)$. It is natural to investigate the relations between α and the lattice operations on $L(A)$.

In this paper we discuss the following problems:

1) To characterize additive radicals α, i.e. $\alpha(I + J) = \alpha(I) + \alpha(J)$ for arbitrary ideals I, J of an arbitrary ring A (F. A. Szász [16], Problem 12).

2) To characterize radicals α such that $\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$ for arbitrary ideals I, J of an arbitrary ring A (F. A. Szász [16], Problem 13).

S. A. Amitsur [1] showed that $\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$ is valid for hereditary radicals. F. A. Szász [15] proved that α is additive if the semisimple class $\mathscr{S}(\alpha)$ of α is homomorphically closed. Note that radicals with homomorphically closed semisimple classes were described by Wiegandt ([18], Theorem 33.11): such radicals are upper radicals determined by finite classes of finite fields, which are closed under subfields.

The main results of this paper are the following statements:

Theorem 2.1. *The following conditions are equivalent:*

1) *α is an additive radical;*

2) *either there exists $n > 1$ such that the semisimple class $\mathscr{S}(\alpha)$ of α satisfies the polynomial identity $x^n - x = 0$, or α is a subidempotent radical and for arbitrary ideals I, J of an arbitrary ring A the equalities $I + J = A$, $\alpha(A) = A$ imply that $\alpha(I) + \alpha(J) = A$.*

Theorem 2.2. *Let α be a hereditary radical. Then the following conditions are equivalent:*

1) *α is an additive radical;*

2) *α induces an endomorphism of the lattice $L(A)$ of ideals of A;*

3) *either there exists $n > 1$ such that the semisimple class $\mathscr{S}(\alpha)$ of α satisfies the polynomial identity $x^n - x = 0$, or α is a subidempotent radical.*

However, we shall give an example of a subidempotent non hereditary radical such that α gives an endomorphism of the lattice $L(A)$. We recall that a ring A is called *idempotent* if $A^2 = A$. A radical α is subidempotent if $A^2 = A$ for every α-radical ring A. In what follows we assume that all classes contain the one element ring. A class $\mathfrak{M}$ is hereditary if $\mathfrak{M}$ is closed under ideals.

We denote by

$\lvert S\rvert$	the cardinality of a set S;
$\mathscr{S}(\alpha)$	the semisimple class of a radical α;
β	the Baer lower nil radical;
$A^\#$	the ring which is constructed starting from A by the adjunction of the unity element;
A^0	the zero-ring with an additive group A, i.e. $ab = 0$ for all $a \in A$, $b \in A$;
A^+	the additive group of A;
$\boldsymbol{Z}$	the ring of integers;
$\boldsymbol{Z}_p$	the cyclic group of order p;
$\boldsymbol{Z}(p^\infty)$	the quasicyclic group;
$\Phi\langle X\rangle$	the free (without unity element) Φ algebra (over a commutative ring Φ with a unity element).

Let $A[x, y]$ be the ring of all polynomials in two variables x, y with coefficients from the ring A, i.e.

$$A[x, y] = \{f(x, y) = \sum_{i,j=0}^{m} a_{ij}x^i y^i \quad \text{for} \quad a_{ij} \in A,\ 0 \leqq i, j \leqq m\}.$$

Let $\deg f = \max\{i + j \mid a_{ij} \neq 0\}$. Note that $x, y \in A[x, y]$ if and only if A has a unity element.

1. SEMISIMPLE CLASSES AND PI-ALGEBRAS

For the basic notion and results on PI-algebras we refer to [10] and [13]. Let Φ be a commutative ring with a unity element, X an infinite set and $\mathfrak{M}$ an abstract class of Φ-algebras. Let $T(X; \mathfrak{M})$ be the intersection of all kernels of Φ-algebra homomorphisms from $\Phi\langle X\rangle$ to algebras of $\mathfrak{M}$. The elements of $T(X; \mathfrak{M})$ are called *polynomial identities* for algebras of the class $\mathfrak{M}$. A polynomial identity is called *proper* if the unity element of Φ appears among its coefficients. The process of linearization (when for every polynomial identity $f(x_1, x_2, \ldots, x_n)$ we get a semilinear identity $\tilde{f}(x_1, x_2, \ldots, x_n)$) is described, for example, in [13]. Note that if one of the coefficients of f is equal to $a \in \Phi$ then at least one of the coefficients of $\tilde{f}$ is also equal to a.

Lemma 1.1. *Let τ be an infinite cardinal, X a set with $|X| = \tau$, $\mathfrak{M}$ an abstract* *class of Φ algebras. Let $|\Phi| \leqq \tau$ and $|A| \leqq \tau$ for all $A \in \mathfrak{M}$. Then $T(X; \mathfrak{M}) = = \bigcap \{I | I \lhd \Phi\langle X\rangle$ and $\Phi\langle X\rangle/I \in \mathfrak{M}\}$.*

Proof. Consider the ideal $M = \bigcap \{I | I \lhd \Phi\langle X\rangle$ and $\Phi\langle X\rangle/I \in \mathfrak{M}\}$. Clearly $T(X; \mathfrak{M}) \subseteq M$. Assume that $T(X; \mathfrak{M}) \neq M$. Let $f = f(x_1, x_2, \ldots, x_n) \in M \setminus T(X; \mathfrak{M})$. Thus f is not a polynomial identity for the class $\mathfrak{M}$, therefore there exist a ring $A \in \mathfrak{M}$ and elements $a_1, a_2, \ldots, a_n \in A$ such that $f(a_1, a_2, \ldots, a_n) \neq 0$. Since $|A| \leqq |X|$ and $|X|$ is an infinite cardinal we may choose elements $x_{s_1}, x_{s_2}, \ldots, x_{s_n} \in \{x_s\}$. Consider a surjection $\varphi\colon \varphi\langle X\rangle \to A$ sending $x_{s_i} \mapsto a_i$, defining φ arbitrarily on the other variables. Since $\varphi\langle X\rangle/\mathrm{Ker}\,\varphi \cong A \in \mathfrak{M}$, we have $\mathrm{Ker}\,\varphi \supseteq M$, which contradicts $f \in M$. Thus $M = T(X; \mathfrak{M})$.

Corollary 1.1. *Let α be a radical in a universal class of Φ algebras, τ an infinite cardinal such that $|\Phi| \leqq \tau$, $\mathscr{S}_\tau(\alpha) = \{A | \alpha(A) = 0$ and $|A| \leqq \tau\}$, X a set and $|X| = \tau$. Then $\alpha(\varphi\langle X\rangle) = T(X; \mathscr{S}_\tau(\alpha))$.*

Proof. Clearly $|\varphi\langle X\rangle| = \tau$. Lemma 1.1 implies that $T(X; \mathscr{S}_\tau(\alpha)) = \bigcap \{I \mid I \lhd \Phi\langle X\rangle$ and $\Phi\langle X\rangle/I \in \mathscr{S}_\tau(\alpha)\} = \bigcap \{I \mid I \lhd \Phi\langle X\rangle$ and $\Phi\langle X\rangle/I \in \mathscr{S}(\alpha)\} = \alpha(\Phi\langle X\rangle)$.

Theorem 1.1. *Let Φ be a principal ideal ring. Let $\mathfrak{M}$ be an abstract hereditary class of Φ algebras closed under subdirect sums. Then either there exists a proper polynomial identity which holds in all algebras of $\mathfrak{M}$ or there exists a proper ideal I of Φ and an infinite set X such that $(\Phi/I)\langle X\rangle \in \mathfrak{M}$.*

Proof. Suppose that the statement of the theorem is not valid for a ring Φ. Consider $H = \{I \mid I \lhd \Phi$ and the theorem is not valid for $\Phi/I\}$. Clearly $0 \in H$ and H is non empty. Since Φ is a principal ideal ring, there is a maximal ideal M in H. Without loss of generality we may assume that the theorem is not valid for the ring Φ but is valid for all proper homomorphic images of Φ.

Let Y be a countable set and F a set of all polynomials in $\Phi\langle Y\rangle$ such that among their coefficients we have the unity element of the ring Φ. Since the theorem is not valid for the ring Φ, for every $f \in F$ there exists $A_f \in \mathfrak{M}$ such that f does not vanish on A_f. Choose an infinite cardinal τ such that $|\Phi| \leqq \tau$ and $|A_f| \leqq \tau$ for all $f \in F$. Let X be a set of cardinality τ, $\mathfrak{M}_\tau = \{A \in \mathfrak{M} | |A| \leqq \tau\}$ Lemma 1.1 implies that $\Phi\langle X\rangle/T(X; \mathfrak{M}_\tau) \in \mathfrak{M}$. By our assumptions $T(X; \mathfrak{M}_\tau) \neq 0$. Let L be the set of all coefficients of the polynomials $g(x) \in T(X; \mathfrak{M}_\tau)$. Using the idea of the proof of Hilbert's Nullstellensatz we will prove that L is an ideal of A. Since the theorem is not valid for the ring Φ we have $\Phi \neq L$. Clearly $L = a\Phi$ for some $0 \neq a \in \Phi$. By the definition of the ideal L we have a polynomial $h \in T(X, \mathfrak{M}_\tau)$ such that one of its coefficients in equal to a and the others are divisible by a. Clearly there exists a polynomial g such that $h = ag$ and one of its coefficients is equal to 1. Let $\mathfrak{M}(L) = = \{A \in \mathfrak{M} | LA = 0\}$. Clearly $\mathfrak{M}(L)$ is an abstract hereditary class of Φ/L algebras closed under subdirect sums. By the assumption the theorem is valid for the ring Φ/L. Furthermore, $\mathfrak{M}(L) \subseteq \mathfrak{M}$ and the theorem is not valid for the ring Φ. Therefore there exists a proper identity $q(x_1, x_2, \ldots, x_m)$ which holds for all $B \in \mathfrak{M}(L)$. Let $g =$

$= g(y_1, y_2, \ldots, y_n)$, $f = q(g(x_{11}, x_{12}, \ldots, x_{1n}), g(x_{21}, x_{22}, \ldots, x_{2n}), \ldots, g(x_{m1}, x_{m2}, \ldots$ $\ldots, x_{mn}))$ where x_{ij}, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ are pairwise distinct variables. Clearly one of the coefficients of f is equal to 1. By the definition of A_f it follows that f does not vanish in A_f. Therefore

$$(*) \qquad q(g(b_{11}, b_{12}, \ldots, b_{1n}), \ldots, g(b_{m1}, b_{m2}, \ldots, b_{mn})) \neq 0$$

for some $b_{ij} \in A_f$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. Since $|A_f| \leqq \tau$ we have $A_f \in \mathfrak{M}_\tau$ and A_f satisfies the polynomial identity $h(y_1, y_2, \ldots, y_n) = 0$. Let $B = \{b \in A_f | \, Lb = 0\}$. Clearly B is an ideal in A_f. Therefore $B \in \mathfrak{M}$ and $B \in \mathfrak{M}(L)$. Now we have $0 = h(b_{i1}, b_{i2}, \ldots, b_{in}) = ag(b_{i1}, b_{i2}, \ldots, b_{in})$. Thus $g(b_{i1}, b_{i2}, \ldots, b_{in}) \in B$. We know that $B \in \mathfrak{M}(L)$. Consequently, B satisfies the polynomial identity q. Therefore $q(g(b_{11}, b_{12}, \ldots, b_{1n}), \ldots, g(b_{m1}, b_{m2}, \ldots, b_{mn})) = 0$. This contradicts $(*)$ and the proof is complete.

Recall that a polynomial $p(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ is said to be *alternating* (in the x's) if $p(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ is a homogeneous polynomial multilinear only in $x_1, x_2, \ldots, x_m$ which vanishes if two of the x_i's are made equal ([2], p. 129). A polynomial $f(x_1, x_2, \ldots, x_n)$ is said to be *central* for A if $f(a, a_2, \ldots, a_n) \in C$, the center of A, for all $a_1, a_2, \ldots, a_n$ in A, and there exist $b_1, b_2, \ldots, b_n$, $d_1, d_2, \ldots, d_n$ in A such that $f(b_1, b_2, \ldots, b_n) \neq f(d_1, d_2, \ldots, d_n)$.

Let A be a prime ring which satisfies a proper polynomial identity $g(x_1, x_2, \ldots, x_m)$ of degree d. Then by the Posner Theorem ([10], Chapter 2, Theorem 5.7) A is a right Goldie ring and its classical ring of quotients $Q(A)$ is a simple Artinian ring, which satisfies a polynomial identity $g(x_1, x_2, \ldots, x_m)$. By the Kaplansky Theorem ([10], Chapter 2, Theorem 1.1) $Q(A)$ is a central simple algebra of dimension n^2 over its center, where $n \leqq \frac{1}{2}d$. Following Amitsur ([2], p. 128) we denote $\operatorname{pid}(A) = n$ (polynomial identity degree). Therefore $\operatorname{pid}(A) \leqq \frac{1}{2}d$. If A is a semiprime ring with a proper polynomial identity of degree d we set $\operatorname{pid}(A) = \max \operatorname{pid}(A/P)$ where P ranges over all prime ideals P of A. Clearly, $\operatorname{pid}(A) \leqq \frac{1}{2}d$. Obviously $\operatorname{pid}(A) = 1$ if and only if A is a commutative ring.

Remark 1.1 ([2], Theorem 3). Let A be a semiprime ring with $\operatorname{pid}(A) = n$. Then A has an alternating central identity $\delta(x_1, x_2, \ldots, x_{n^2}; y_1, \ldots, y_m)$.

2. ADDITIVE RADICALS

Lemma 2.1. *Let α be an additive radical. Then either α is subidempotent or $\alpha \supseteq \beta$.*

Proof. Suppose that α is not subidempotent. Then there exists an α-radical ring A such that $A^2 \neq A$. Let $B = A/A^2$. Note that to prove $\alpha \supseteq \beta$ it is sufficient to show that Z^0 is a radical ring.

Suppose now that $pB \neq B$ for some prime number p. Then the nonzero α-radical ring $\bar{B} = B/pB$ is a vector space over the p-element field F_p and $\bar{B}^2 = 0$. Thus Z_p^0 is a homomorphic image of $\bar{B}$. Therefore Z_p^0 is an α-radical ring. Let $\mathscr{D} = Z^0 + Z^0$,

$I = \{(n, 0) \mid n \in \mathbf{Z}\}$, $J = \{(0, n) \mid n \in \mathbf{Z}\}$, $M = \{(pn, pn) \mid n \in \mathbf{Z}\}$. Clearly M is an ideal of $\mathscr{D}$, $I \cap M = 0$, $J \cap M = 0$. Denote $\bar{\mathscr{D}} = \mathscr{D}/M$, $\bar{I} = (I + M)/M$, $\bar{J} = (J + M)/M$ and $x = (1, 1) + M \in \bar{\mathscr{D}}$. Obviously $\mathscr{D} = I + J$, $\bar{\mathscr{D}} = \bar{I} + \bar{J}$. Therefore $\alpha(\bar{\mathscr{D}}) = \alpha(\bar{I}) + \alpha(\bar{J})$. Since the α-radical ring $\mathbf{Z}_p^0$ and the ideal $\bar{B}$ of $\bar{\mathscr{D}}$ generated by x are isomorphic we have $\alpha(\bar{\mathscr{D}}) \neq 0$. Therefore either $\alpha(\bar{I}) \neq 0$ or $\alpha(\bar{J}) \neq 0$. Let $\alpha(\bar{I}) \neq 0$. Clearly $\bar{I} \cong I \cong \mathbf{Z}^0$. Thus $\alpha(\mathbf{Z}^0) \neq 0$. Since every nonzero ideal of $\mathbf{Z}^0$ is isomorphic to $\mathbf{Z}^0$ we have $\alpha(\mathbf{Z}^0) \cong \mathbf{Z}^0$, i.e. $\mathbf{Z}^0$ is an α-radical ring. Thus $\alpha \supseteq \beta$.

Let us suppose now that $pB = B$ for all prime numbers p. Then B is a divisible abelian group. Since $B^2 = 0$ there exists a prime number p such that $L = \mathbf{Z}(p^\infty)^0$ is a homomorphic image of the α-radical ring B. Clearly $\alpha(L) = L$. Let $U_i = V_i = \mathbf{Z}^0$ for $i = 1, 2, \ldots, n, \ldots$ and $u_i \in \bigoplus_{i=1}^{\infty} U_i$, $u_i(j) = 0$ for $j \neq i$ and $u_i(i) = 1$. Similarly we define v_i for $i = 1, 2, \ldots, n, \ldots$. Consider $\mathscr{D} = (\bigoplus_{i=1}^{\infty} \mathscr{U}_i) \oplus (\bigoplus_{i=1}^{\infty} V_i)$, $I = \{(u, 0) \mid u \in \bigoplus_{i=1}^{\infty} U_i)$, $J = \{(0, v),\ v \in \bigoplus_{i=1}^{\infty} V_i\}$, $x_i = (u_i, v_i)$ for $i = 1, 2, \ldots, n, \ldots$. Let M be the subgroup of $\mathscr{D}^+$ generated by the elements $px_1, x_i - px_{i+1}$ for $i = 1, 2, \ldots, n, \ldots$. Clearly $M \cap J = M \cap I = 0$. Let $\bar{\mathscr{D}} = \mathscr{D}/M$, $\bar{I} = (I + M)/M$, $\bar{J} = (J + M)/M$. It is clear that the subgroup G of $\bar{\mathscr{D}}^+$ generated by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \ldots$ is isomorphic to $\mathbf{Z}(p^\infty)$. Since $\bar{\mathscr{D}}^2 = 0$ we see that G is an ideal of $\bar{\mathscr{D}}^+$ and $G \cong \mathbf{Z}(p^\infty)^0$. Therefore $\alpha(\bar{\mathscr{D}}) \supseteq G \neq 0$. Moreover, $\bar{\mathscr{D}} = \bar{I} + \bar{J}$ and $0 \neq \alpha(\bar{\mathscr{D}}) = \alpha(\bar{I}) + \alpha(\bar{J})$. Thus either $\alpha(\bar{I}) \neq 0$ or $\alpha(\bar{J}) \neq 0$. Let $\alpha(\bar{I}) \neq 0$. Since $\bar{I} \cong I$ we have $\alpha(I) \neq 0$. However, $\alpha(I)^+$ is a subgroup of the free abelian group I. Therefore $\alpha(I)^+$ is also a free abelian group. Since $I^2 = 0$ we have $\alpha(I)^2 = 0$. Consequently, $\mathbf{Z}^0$ is a homomorphic image of $\alpha(I)$, i.e. $\mathbf{Z}^0$ is an α-radical ring. Thus $\alpha \supseteq \beta$.

Lemma 2.2. *Let α be an additive nonsubidempotent radical, R a commutative ring with unity element, X an infinite set and $R\langle X\rangle$ a free R-algebra. Then $\alpha(R\langle X\rangle) \neq 0$.*

Proof. Let $\{x, y\}$ be a two-element set, $x \notin X$, $y \notin X$, $Y = X \cup \{x, y\}$, $A = R\langle Y\rangle$, $B = R\langle X \cup \{x\}\rangle \subseteq A$ $\mathscr{D} = R\langle X \cup \{y\}\rangle \subseteq A$. Let I be the ideal of B generated by x, J the ideal of $\mathscr{D}$ generated by y, M the ideal of A generated by $\{(x + y)\, xa$, $(x + y)\, ax$, $ax(x + y)$, $xa(x + y)$, $(x + y)\, ya$, $(x + y)\, ay$, $ay(x + y)$, $ya(x + y) \mid a \in A^\#\}$. We show that $M \cap I = 0$. Consider the mapping φ from A to A such that $\varphi(x) = x$, $\varphi(y) = -x$, $\varphi(z) = z$ for $z \in X$. Clearly $\varphi(b) = b$ for all $b \in B$ and $\varphi(M) = 0$. Therefore $I \cap M = \varphi(I \cap M) = 0$. Similarly $J \cap M = 0$. Clearly $x + y \notin M$. Let $\bar{A} = A/M$, $\bar{I} = (I + M)/M$, $\bar{J} = (J + M)/M$, $\bar{a} = a + M$ for $a \in A$. By the definition of the ideal M we have $\bar{y}\bar{x}\bar{a} = -\bar{x}^2\bar{a}$, $\bar{y}\bar{a}\bar{x} = -\bar{x}\bar{a}\bar{x}$, $\bar{a}\bar{x}\bar{y} = -\bar{a}\bar{x}^2$, $\bar{x}\bar{a}\bar{y} = -\bar{x}\bar{a}\bar{x}$, $\bar{x}\bar{y}\bar{a} = -\bar{y}^2\bar{a}$, $\bar{x}\bar{a}\bar{y} = -\bar{y}\bar{a}\bar{y}$, $\bar{a}\bar{y}\bar{x} = -\bar{a}\bar{y}^2$, $\bar{y}\bar{a}\bar{x} = -\bar{y}\bar{a}\bar{y}$ for $\alpha \in A^\#$. Therefore $\bar{J}\bar{I} \subseteq \bar{J}$, $\bar{I}\bar{J} \subseteq \bar{I}$, $\bar{I}\bar{J} \subseteq \bar{J}$ and $\bar{J}\bar{I} \subseteq \bar{J}$. Thus $\bar{I} + \bar{J}$ is a subring of $\bar{A}$ and $\bar{I}, \bar{J}$ are ideals of $\bar{I} + \bar{J}$. By the definition of M, $\bar{x} + \bar{y} \neq 0$ and $(\bar{x} + \bar{y})(\bar{I} + \bar{J}) = 0$. Since $\bar{x} \in \bar{I}$, $\bar{y} \in \bar{J}$ we have $\bar{x} + \bar{y} \in \beta(\bar{I} + \bar{J})$. By Lemma 2.1, $\alpha(\bar{I} + \bar{J}) \neq 0$. Therefore either $\alpha(\bar{I}) \neq 0$ or $\alpha(\bar{J}) \neq 0$. We may assume that $\alpha(\bar{I}) \neq 0$. Since $I \cap M = 0$ we have $\bar{I} \cong I$. Consequently, $\alpha(I) = 0$. By the Andersen-Divinsky-

Sulinski Lemma ([14] or [3], Chapter 2, § 4, Proposition 1) it follows that $\alpha(R\langle X \cup \{x\}\rangle) \neq 0$. Since $|X| = \infty$ we have $R\langle X\rangle \cong R\langle X \cup \{x\}\rangle$. Therefore $\alpha(R\langle X\rangle) \neq 0$.

Corollary 2.1. *Let α be an additive nonsubidempotent radical. Then the semisimple class $\mathcal{S}(\alpha) = \{A \mid \alpha(A) = 0\}$ satisfies a proper polynomial identity with integer coefficients.*

Proof. By Theorem 1.1 the semisimple class $\mathcal{S}(\alpha)$ either satisfies a proper polynomial identity or there exists a proper ideal I of $\boldsymbol{Z}$ and an infinite set X such that $(\boldsymbol{Z}/I)(X) \in \mathcal{S}(\alpha)$. However, the second case contradicts Lemma 2.2.

We recall that a ring A is called *reduced* if it is without non zero nilpotent elements.

Lemma 2.3. *Let α be an additive nonsubidempotent radical. Then every ring in the semisimple class $\mathcal{S}(\alpha) = \{A \mid \alpha(A) = 0\}$ is reduced.*

Proof. Suppose that there exists $A \in \mathcal{S}(\alpha)$ such that $a^2 = 0$ for some $0 \neq a \in A$. Consider

$$B = \prod_{i=1}^{\infty} A_i\,, \quad L = \sum_{i=1}^{\infty} A_i\,,$$

where $A_i = A$ for all $i = 1, 2, \ldots, \mathcal{D} = B[x, y]$. We may assume that A_i are subrings of B. Clearly, $A_i \lhd B$ for $i = 1, 2, \ldots$. Let b be an element of B such that its all co-ordinates are equal to a, let M be an ideal of $\mathcal{D}$ generated by $L(x - 1) \cup L(y - 1)$. Since $b \notin L$, we have $b(x - y) \notin M$. Consider $\bar{\mathcal{D}} = \mathcal{D}/M$, $\bar{A}_i = (A_i + M)/M$, $\bar{L} = (L + M)/M$, $\bar{z} = z + M$ for all $z \in \mathcal{D}$. Let I be the subring of $\bar{\mathcal{D}}$ generated by $\bar{L} \cup \{\bar{b}\bar{x}\}$, J the subring of $\bar{\mathcal{D}}$ generated by $\bar{L} \cup \{\bar{b}\bar{y}\}$. Since $b^2 = 0$ and $u\bar{x} = u$ for all $u \in \bar{L}$ we can represent every element $z \in I$ as a sum $z = n\bar{b}\bar{x} + l$, where $n \in \boldsymbol{Z}$, $l \in \bar{L}$. Similarly, every element $z \in J$ can be represented as a sum $z = n\bar{b}\bar{y} + l$, where $n \in \boldsymbol{Z}$, $l \in \bar{L}$. Clearly $IJ \subseteq \bar{L}$ and $JI \subseteq \bar{L}$. Therefore $I + J$ is a subring of $\bar{\mathcal{D}}$, $I \lhd I + J$ and $J \lhd I + J$. Consider the homomorphism from $\mathcal{D}$ to B such that

$$\varphi\Big(\sum_{i,j=0}^{m} b_{ij}x^i y^i\Big) = \sum_{i,j=0}^{m} b_{ij}$$

for all

$$\sum_{i,j=0}^{m} b_{ij}x^i y^i \in \mathcal{D}\,.$$

Obviously $\varphi(M) = 0$. Consequently, there is a homomorphism $\psi\colon \bar{\mathcal{D}} \to B$ such that $\psi(d + M) = \varphi(d)$. Clearly, $\operatorname{Ker} \psi \cap I = 0$ and $\psi(I) \supseteq L$. It is well known that every subring of B which contains L is a subdirect sum of rings A_i, $i = 1, 2, \ldots$. Therefore I is the subdirect sum of the α-semisimple rings A_i. Hence $\alpha(I) = 0$. Similarly $\alpha(J) = 0$. By our assumption $\alpha(I + J) = \alpha(I) + \alpha(J) = 0$. Since $bx - by \notin M$ we have $\bar{b}\bar{x} - \bar{b}\bar{y} \neq 0$. Then it follows from $\bar{b}^2 = 0$ and $z\bar{x} = z\bar{y} = z$ for all $z \in \bar{L}$ that $(I + J)(\bar{b}\bar{x} - \bar{b}\bar{y}) = 0$. Therefore $\beta(I + J) \neq 0$. By Lemma 1.1, $\alpha \supseteq \beta$. This contradicts the equality $\alpha(I + J) = 0$.

Lemma 2.4. *Let α be an additive nonsubidempotent radical, A a nonzero α-semisimple ring, $A[x]$ the ring of all polynomials with coefficients from A. Then $\alpha(A[x]) \neq 0$.*

Proof. Suppose $\alpha(A[x]) = 0$. Let $B = A[x, y]$, $I = A[x]\, x \lhd A[x] \subseteq B$, $J = A[y]\, y \lhd A[y] \subseteq B$, $M = Bx(x + y) + By(x + y)$.

By the Anderson-Divinsky-Sulinski Lemma ([14], Theorem 1.7) we have $\alpha(I) = 0 = \alpha(J)$. Clearly, $M \lhd B$ and $a(x + y) \notin M$ for all $0 \neq a \in A$. Consider $\bar{B} = B/M$, $\bar{I} = (I + M)/M$, $\bar{J} = (J + M)/M$, $\bar{z} = z + M$ for all $z \in B$. Since $\bar{b}\bar{x}\bar{y} = -\bar{b}\bar{x}^2$ for all $b \in B$ we have $\bar{I}\bar{J} \subseteq \bar{I}$ and $\bar{J}\bar{I} \subseteq \bar{I}$. Similarly, $\bar{I}\bar{J} \subseteq \bar{J}$ and $\bar{J}\bar{I} \subseteq \bar{J}$. Therefore $\bar{J} + \bar{I}$ is a subring of B, $\bar{I} \lhd \bar{I} + \bar{J}$ and $\bar{J} \lhd \bar{I} + \bar{J}$.

Now we show that $I \cap M = 0$. Consider the homomorphism φ from B to B given by $\varphi(f(x, y)) = f(x, -x)$ for all $f(x, y) \in B$. Clearly $\varphi(z) = z$ for all $z \in I$ and $\varphi(M) = 0$. Therefore $I \cap M = \varphi(I \cap M) = 0$. Thus $I \cong \bar{I}$ and $\alpha(\bar{I}) = 0$. Similarly $\alpha(\bar{J}) = 0$. By our assumption $\alpha(\bar{I} + \bar{J}) = \alpha(\bar{I}) + \alpha(\bar{J}) = 0$. Since $a(x + y) \notin M$ for all $0 \neq a \in A$, we have $\bar{a}\bar{x} + \bar{a}\bar{y} \neq 0$ for all $0 \neq a \in A$. By the definition of M we obtain $(\bar{I} + \bar{J})(\bar{a}\bar{x} + \bar{a}\bar{y}) = 0$. Hence $\beta(\bar{I} + \bar{J}) \neq 0$. Lemma 1.1 implies that $\alpha \supseteq \beta$, which contradicts the equality $\alpha(\bar{I} + \bar{J}) = 0$.

Lemma 2.5. *Let α be an additive non subidempotent radical. Then every ring in the semisimple class is commutative.*

Proof. By Corollary 2.1 it follows that $\mathcal{S}(\alpha)$ satisfies a proper polynomial identity. It is sufficient to prove that pid $(A) = 1$ for all $A \in \mathcal{S}(\alpha)$. Assume that pid $(A) = n > 1$ for some $A \in \mathcal{S}(\alpha)$. By Remark 1.1 A has an alternating (in $x_1, x_2, \ldots, x_{n^2}$) central polynomial $\delta(x_1, x_2, \ldots, x_{n^2}, y_1, y_2, \ldots, y_m)$. Let B be an ideal of A generated by the set $T = \{\delta(a_1, a_2, \ldots, a_{n^2}, b_1, b_2, \ldots, b_m) \mid a_1, a_2, \ldots, \ldots, a_{n^2}, b_1, b_2, \ldots, b_m \in A\}$. Clearly $B \neq 0$. By the Anderson-Divinsky-Sulinski Lemma it follows that $B \in \mathcal{S}(\alpha)$. Let $B[x]$ be the ring of all polynomials with coefficients from B, let φ_t be the homomorphism from $B[x]$ to B such that $\varphi_t(f(x)) = f(t)$ for all $f(b) \in B[x]$ where $t \in T$. Clearly φ is surjective. Lemma 2.4 implies that $\alpha(B[x]) \neq 0$. Since $\alpha(B[x]) \subseteq \bigcap_{t \in T} \operatorname{Ker} \varphi_t$ there exists $0 \neq f(x) = d_0x^q + d_1x^{q-1} + \ldots + d_q \in B[x]$ such that $f(t) = 0$ for all $t \in T$. Since A is reduced and B is the ideal generated by T we have $d_0t \neq 0$ for some $t \in T$ (see Lemma 2.3). Consider $S = \{d_0t, (d_0)^2, \ldots, \ldots, (d_0t)^n, \ldots\}$. Then there exists a prime ideal Q of A such that $Q \cap S = \emptyset$. Moreover, there exists a minimal prime ideal P of A such that $P \cap S = \emptyset$. It is well known that $\bar{A} = A/P$ contains no nonzero divisors ([3], Chapter 4, § 2, Theorem 1). Since $\bar{A}$ satisfies a polynomial identity we conclude that $\bar{A}$ has a nontrivial center C ([2], Theorem 3).

Assume now that $|C| < \infty$. Thus C is a finite commutative ring without nonzero divisors. Therefore C is a field. By ([13], Corollary 1.6.28) $\bar{A}$ is a semisimple ring and by Kapalansky's Theorem ([13], Theorem 1.5.16) it follows that $\bar{A}$ is finite dimensional over its center C and $\bar{A}$ is isomorphic to the matrix ring over the division ring. Since $\bar{A}$ is a domain we have that $\bar{A}$ is a division ring. The inequality $|C| < \infty$

implies that $|\bar{A}| < \infty$. Apply Wedderburn's Theorem on finite division rings to obtain $\bar{A} = C$. Consider $\bar{a} = a + P$ for all $a \in A$. By the definition of P it follows that $\bar{t} \neq 0$, where $t = \delta(a_1, a_2, \ldots, a_{n^2}, \; b_1, b_2, \ldots, b_m)$ for some $a_1, a_2, \ldots, a_{n^2}$, $b_1, b_2, \ldots, b_m \in A$. Since $\bar{A} = C$ and $\delta(x_1, x_2, \ldots, x_{n^2}, \; y_1, y_2, \ldots, y_m)$ is a central alternating in $x_1, x_2, \ldots, x_{n^2}$ polynomial we have $\bar{t} = \delta(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n^2}, \bar{b}_1, \bar{b}_2, \bar{b}_m) = \bar{a}_1\bar{a}_2\delta(1, 1, \bar{a}_3, \ldots, \bar{a}_{n^2}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m)$. This contradicts the inequality $\bar{t} \neq 0$.

Now we may assume that $|C| = \infty$. Consider $\bar{T} = \{\bar{b} \mid b \in T\}$. We have $c\bar{t} = \delta(c\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n^2}, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m)$ for $c \in C$ and $t = \delta(a_1, a_2, \ldots, a_{n^2}, b_1, b_2, \ldots, b_m)$. Therefore $c\bar{t} \in \bar{T}$ and $|\bar{T}| = \infty$. Let $t_1, t_2, \ldots, t_q \in \bar{T}$ be pairwise distinct elements of B. The definition of $f(x) = d_0x^q + d_1x^{q-1} + \ldots + d_q$ yields

$$\begin{aligned} \bar{d}_0t_1^q + \bar{d}_1t_1^{q-1} + \ldots + \bar{d}_q &= 0\,, \\ \bar{d}_0t_2^q + \bar{d}_1t_2^{q-1} + \ldots + \bar{d}_q &= 0\,, \\ &\ldots \\ \bar{d}_0t_q^q + \bar{d}_1t_q^{q-1} + \ldots + \bar{d}_q &= 0\,. \end{aligned} \tag{1}$$

The determinant of the system (1) is Vandermond's determinant. Since $\bar{A}$ has a classical ring of quotients $\mathscr{D}$ which is a division ring ([13], Theorem 1.7.9), the center C of $\bar{A}$ is contained in the center of $\mathscr{D}$ and $\bar{T} \subseteq C$, we have that the system(1) has no nontrivial solution in the division ring $\mathscr{D}$. Therefore $\bar{d}_0 = 0$. Thus $d_0 \in P$. Moreover, $d_0t \in P$. This contradicts the relation $P \cap S = \emptyset$. Therefore pid $(A) = 1$ and A is a commutative ring.

Lemma 2.6. *Let A be a commutative ring without divisors of zero, $f(x_1, x_2, \ldots, x_n)$ a polynomial of degree m with relatively prime integer coefficients and constant term 0. Suppose that $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in A$. Then A is a finite field and $|A| \leqq m$.*

Proof. Let K be the quotient field of A. Without loss of generality we may assume that $f(x_1, x_2, \ldots, x_n)$ is an arbitrary nonzero polynomial of $K[x_1, x_2, \ldots, x_n]$ (i.e. $f(x_1, x_2, \ldots, x_n)$ may have a nonzero constant term). We prove our statement by induction on n. If $n = 1$ our statement follows from the fact that a polynomial of degree m cannot have more than m roots in the field. Suppose now that the statement holds for polynomials in $n - 1$ variables. Then $f(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{t} f_i(x_1, x_2, \ldots \ldots, x_{n-1})\, x_n^i$ where $t \leqq m$, $f_t \neq 0$ and $\deg f_t \leqq m$. If $f_t(a_1, a_2, \ldots, a_{n-1}) = 0$ for all $a_1, a_2, \ldots, a_{n-1} \in A$ then our statement holds by the induction hypothesis. Assume now that $f_t(a_1, a_2, \ldots, a_{n-1}) \neq 0$ for some $a_1, a_2, \ldots, a_{n-1} \in A$. Then $g(x_n) = f(a_1, a_2, \ldots, a_{n-1}, x_n) \in K[x_n]$ is a nonzero polynomial and $\deg g(x_n) \leqq m$. Since $g(a) = 0$ for all $a \in A$ we have $|A| \leqq m$. Hence Lemma 2.6 is proved.

Remark 2.1. *Let A be a ring satisfying the polynomial identity $x^n - x = 0$ and let $m = t(n - 1) + 1$ for some natural number t.*

Then

a) *a^{n-1} is an idempotent element of A for all $a \in A$;*

b) $a^m - a = 0$ *for all* $a \in A$;

c) *A is a regular ring.*

Proof. Since $a^n - a = 0$ for all $a \in A$ we have $a(a^{n-2})\, a = a$. Thus A is a regular ring and $a^{n-1} = a^{n-2}a$ is an idempotent element of A. Furthermore, $a^m = (a^{n-1})^t\, a = a^{n-1}a = a^n = a$. Thus $a^m - a = 0$ for all $a \in A$.

Remark 2.2. *Let* $\{F_i, i \in I\}$ *be a family of fields such that* $|F_i| \leqq n$ *for all* $i \in I$. *Then every field* F_i, $i \in I$ *satisfies a polynomial identity* $x^m - x = 0$ *where* $m = n! + 1$.

Proof. Consider $z = |F_i|$. Clearly $x^z - x = 0$ for all $x \in F_i$. Since $z \leqq n$, $z - 1$ divides $n!$. By Remark 2.1 we have $x^m - x = 0$ for all $x \in F_i$, $i \in I$.

Lemma 2.7. *Let A be an essential ideal of B, E(A) the set of all idempotents of A. Suppose that the ring A satisfies the polynomial identity* $x^n - x = 0$, *where* $n > 1$. *Then*

a) *every idempotent of A is a central idempotent of B*;

b) *is a subdirect sum of rings eB*, $e \in E(A)$;

c) *B satisfies the polynomial identity* $x^n - x = 0$.

Proof. By Theorem 1 ([9], Chapter X, § 1) it follows that A is a commutative ring. Choose $b \in B$, $e \in E(A)$. Since $be, eb \in A$ we have $eb = e^2b = e(eb) = (eb)\, e = e(be) = (be)\, e = be$. Thus e is a central idempotent of B. Clearly B can be mapped homomorphically onto eB for every $e \in E(A)$. Now it is sufficient to show that for every nonzero $b \in B$ there exists an idempotent $e \in E(A)$ such that $eb \neq 0$. By Remark 2.1 A is regular. Hence $\beta(A) = 0$. Since $\beta(A) = \beta(B) \cap A$ and A is an essential ideal of B, we have $\beta(B) = 0$. Consider $\mathcal{D} = \{b \in B \mid bA = 0\}$. Then $(\mathcal{D} \cap A)^2 \subseteq \mathcal{D}(A) = 0$. Thus $\mathcal{D} \cap A = 0$ and $\mathcal{D} = 0$. Choose $0 \neq b \in B$. Then we have $ba \neq 0$ for some $a \in A$. By Remark 2.1 $e = a^{n-1}$ is an idempotent of A. Finally, $(be)\, a = ba^n = ba \neq 0$. Thus $be \neq 0$.

Corollary 2.2. *If the semisimple class* $\mathcal{S}(\alpha)$ *satisfies the polynomial identity* $x^n - x = 0$ *where* $h > 1$ *then* α *is a supernilpotent radical.*

Proof. By Remark 2.1 it follows that every ring in the class $\mathcal{S}(\alpha)$ is regular. Therefore $\mathcal{S}(\alpha)$ is a class of semisimple rings and $\alpha \supseteq \beta$. Suppose now that A is an essential ideal of B and $A \in \mathcal{S}(\alpha)$. Since $eB \lhd A$ for all $e \in E(A)$, we have $eB \in \mathcal{S}(\alpha)$ (see [14]). By Lemma 2.7 $B \in \mathcal{S}(\alpha)$. Hence the class $\mathcal{S}(\alpha)$ is essentially closed. Thus α is a hereditary radical (see [4], [11] or [3], Chapter 3, § 1, Theorem 1). Clearly α is a supernilpotent radical.

Lemma 2.8. *Let* α *be an additive nonsubidempotent radical. Then the semisimple class* $\mathcal{S}(\alpha)$ *satisfies the polynomial identity* $x^n - x = 0$, *where* $n > 1$.

Proof. By Lemma 2.3 and Lemma 2.5 every ring in the class $\mathcal{S}(\alpha)$ is semiprime and commutative. Let us suppose that the statement of the lemma is not valid. Then for every natural number m there exists a ring $A_m \in \mathcal{S}(\alpha)$ such that $x^v - x = 0$

is not an identity of A_m, where $v = m! + 1$. Let X be an infinite set such that $|X| \geqq A_m$ for all $m = 1, 2, \dots$ and A a ring of polynomials in X with integer coefficients and the constant term 0. Choose $x \notin X$. Clearly $A[x] \cong A$. Thus Lemma 2.4 implies that $\alpha(A) \neq 0$. For every $0 \neq g(x_1, x_2, \dots, x_n) \in \alpha(A)$ we have $g(x_1, x_2, \dots, x_n) = lf(x_1, x_2, \dots, x_n)$ where all coefficients of $f(x_1, x_2, \dots, x_n)$ are relatively prime and l is the greatest common divisor of all coefficients of $g(x_1, x_2, \dots, x_n)$. Consider $a_1, a_2, \dots, a_n \in A_m$. Clearly there is a homomorphism $\varphi: A \to A_m$ such that $\varphi(x_i) = a_i$ for all $i = 1, 2, \dots, n$. Since $g \in \alpha(A)$ and $A_m \in \mathcal{S}(\alpha)$ we have $lf(a_1, a_2, \dots, a_n) = \varphi(g) = 0$. Let B_m be an ideal of A_m generated by $\{f(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in A_m\}$ and $\mathcal{D}_m = \{a \in A_m \mid aB_m = 0\}$. Then $lB_m = 0$ and $(\mathcal{D}_m \cap B_m)^2 = \mathcal{D}_m B_m = 0$. Since A_m is a semiprime ring we have $\mathcal{D}_m \cap \mathcal{B}_m = 0$. Consider $b_1, b_2, \dots, b_n \in \mathcal{D}_m$. Obviously $f(b_1, b_2, \dots, b_n) \in B_m \cap \mathcal{D}_m = 0$. Thus $f(b_1, b_2, \dots, b_n) = 0$ for all $b_1, b_2, \dots, b_n \in \mathcal{D}_m$. Let P be a prime ideal of $\mathcal{D}_m$. Clearly $f(b_1, b_2, \dots, b_n) = 0$ for all $b_1, b_2, \dots, b_n \in \mathcal{D}_m/P$. By Lemma 2.6 $|\mathcal{D}_m/P| \leqq \deg f$. Since $\mathcal{D}_m$ is a semiprime ring, $\mathcal{D}_m$ is a subdirect product $\mathcal{U}$ of finite fields $\mathcal{D}_m/P$ for all prime ideals P of $\mathcal{D}_m$. Let $t = \deg f$ and $r = t! + 1$. By Remark 2.2 $u^r - u = 0$ for all $u \in \mathcal{U}$. Thus $x^r - x = 0$ is an identity of $\mathcal{D}_m$. Let $m > r$ and $v = m! + 1$. By Remark 2.2 $x^v - x = 0$ is an identity of $\mathcal{D}_m$.

Let us suppose that $x^v - x = 0$ is an identity of B_m. Since $B_m \cap \mathcal{D}_m = 0$, $B_m + \mathcal{D}_m$ is a direct sum of ideals B_m and $\mathcal{D}_m$. Therefore $x^v - x = 0$ is an identity of $B_m + \mathcal{D}_m$. We claim that $B_m + \mathcal{D}_m$ is an essential ideal of A_m. Let $L \lhd A_m$ and $L \cap (B_m + \mathcal{D}_m) = 0$. Then $LB_m \subseteq L \cap (B_m + \mathcal{D}_m) = 0$ and $L \subseteq \mathcal{D}_m$. Therefore $L = L \cap (B_m + \mathcal{D}_m) = 0$. Thus $B_m + \mathcal{D}_m$ is an essential ideal of A_m. Lemma 2.7 implies that $x^v - x = 0$ is an identity of A_m. This contradicts the assumption that $x^v - x = 0$ is not an identity of A_m. Thus $x^v - x = 0$ is not an identity of B_m.

Let $p_1, p_2, \dots, p_k$ be all pairwise distinct prime divisors of l and $B_m(p_i) = \{b \in B_m \mid p_i b = 0\}$. Since B_m is semiprime and $lB_m = 0$ we have $B_m = \oplus_{i=1}^{k} B_m(p_i)$. Let $v(m) = m! + 1$. By the preceding there exists a prime divisor p of l and a sequence of natural numbers $r < m_1 < m_2 < \dots < m_q < \dots$ such that $B_m(p)$ does not satisfy the identity $x^{v(m_i)} - x = 0$. Let B be the ring of all polynomials in X with coefficients from the field $F_p = Z/(p)$ and zero constant term. The proof of $\alpha(B) \neq 0$ is similar to the proof of $\alpha(A) \neq 0$. Clearly for every polynomial $0 \neq h(x_1, x_2, \dots, x_n) \in \alpha(B)$ and for every elements $b_1, b_2, \dots, b_n \in B_m(p)$ we have $h(b_1, b_2, \dots, b_n) = 0$. Let $0 \neq q(x_1, x_2, \dots, x_n) \in \alpha(B)$. Following the proof of Lemma 2.6, $|B_m(p)/Q| \leqq \deg q$ for all prime ideals Q of $B_m(p)$. Therefore $B_m(p)$ satisfies the polynomial identity $x^u - x = 0$ where $u = (\deg q)! + 1$. If $m_i > u$ then $B_m(p)$ satisfies the polynomial identity $x^{v(m_i)} - x = 0$ (see Remark 2.1). This contradicts the fact that $B_m(p)$ does not satisfy the identity $x^{v(m_i)} - x = 0$. Lemma 2.8 is proved.

Proof of Theorem 2.1.

1) $\Rightarrow$ 2). If the radical is not a subidempotent additive radical then the semisimple class $\mathcal{S}(\alpha)$ satisfies the polynomial identity $x^n - x = 0$ for some $n > 1$ (see Lemma 2.8). Suppose now that α is a subidempotent additive radical. Let I, J be ideals of

the α-radical ring A such that $I + J = A$. Then $A = \alpha(A) = \alpha(I + J) = \alpha(I) + \alpha(J)$. Thus 1) $\Rightarrow$ 2) is proved.

2) $\Rightarrow$ 1). Suppose now that the semisimple class $\mathscr{S}(\alpha)$ satisfies the polynomial identity $x^n - x = 0$ for some $n > 1$. Corrolary 2.2 implies that α is a supernilpotent radical. Let I, J be ideals of A. Then $\alpha(I) = \alpha(A) \cap I$ and $\alpha(J) = \alpha(A) \cap J$. Consider $Q = \alpha(I) + \alpha(J)$, $\bar{A} = A/Q$, $\bar{I} = I/(Q \cap I)$, $\bar{J} = J/(Q \cap J)$. Then $Q \cap I = \alpha(I)$ and $Q \cap J = \alpha(J)$. Clearly $\bar{I} \lhd \bar{A}$, $\bar{J} \lhd \bar{A}$ and $\alpha(\bar{I}) = \alpha(\bar{J}) = 0$. Since $\bar{I}, \bar{J} \in \mathscr{S}(\alpha)$, $\bar{I}, \bar{J}$ are regular rings. Therefore $B = \bar{I} + \bar{J}$ is regular ([8], Proposition 1.5). Thus $\beta(B) = 0$. Since α is a supernilpotent radical and $\bar{I}, \bar{J} \in \mathscr{S}(\alpha)$ we have $0 = \alpha(\bar{I}) = \alpha(B) \cap \bar{I}$ and $0 = \alpha(\bar{J}) = \alpha(B) \cap \bar{J}$. Therefore $\alpha(B)\bar{I} = 0 = \alpha(B)\bar{J}$. Clearly $\alpha(B)(\bar{I} + \bar{J}) = \alpha(B)B = 0$ and $\alpha(B) \subseteq \beta(B) = 0$. Thus $\alpha(B) = 0$. Since $B = (I + J)/Q$ we have $\alpha(I + J) \subseteq Q = \alpha(I) + \alpha(J)$. Clearly $\alpha(I) + \alpha(J) \subseteq \alpha(I + J)$. Therefore $\alpha(I) + \alpha(J) = \alpha(I + J)$.

Let us suppose that α is a subidempotent radical. Then $\alpha(I + J) = (\alpha(I + J))^2 \subseteq \alpha(I + J)(I + J) = \alpha(I + J)I + \alpha(I + J)J$. Since $\alpha(I + J)I \subseteq \alpha(I + J)$ and $\alpha(I + J)J \subseteq \alpha(I + J)$ we have $\alpha(I + J) = \alpha(I + J)I + \alpha(I + J)J$. Consider $B = \alpha(I + J)$, $L = \alpha(I + J)I$ and $M = \alpha(I + J)J$. Then $\alpha(B) = B$ and $B = L + M$. By assumption $B = \alpha(L) + \alpha(M)$. Clearly $L \lhd I$ and $M \lhd J$. Therefore $\alpha(L) \subseteq \alpha(I)$ and $\alpha(M) \subseteq \alpha(J)$. Thus $\alpha(I + J) = B = \alpha(L) + \alpha(M) \subseteq \alpha(I) + \alpha(J)$. Obviously $\alpha(I) + \alpha(J) \subseteq \alpha(I + J)$. Therefore $\alpha(I + J) = \alpha(I) + \alpha(J)$. Theorem 2.1 is proved.

Proof of Theorem 2.2 immediately follows from Theorem 2.2 and [1].

Lemma 2.9. *For an arbitrary radical α and for an arbitrary ring A the following conditions are equivalent*:

1) $\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$ *for arbitrary ideals I, J of A*;

2) *if I and J are α-radical ideals of A such that $I + J = A$ and $I \cap J \neq 0$ then* $\alpha(I \cap J) \neq 0$.

Proof. Clearly it is sufficient to show 2) $\Rightarrow$ 1).

2) $\Rightarrow$ 1). Let $M, N \lhd B$. Since $M \cap N \lhd B$ we have $\alpha(M \cap N) \lhd B$ ([14] or [3], Chapter 2, § 4, Proposition 1). Therefore $\alpha(M \cap N) \subseteq \alpha(M)$, $\alpha(M \cap N) \subseteq \alpha(N)$ and $\alpha(M \cap N) \subseteq \alpha(M) \cap \alpha(N)$. Suppose now that $\alpha(M) \cap \alpha(N) \neq 0$. Consider $I = \alpha(M)$, $J = \alpha(N)$ and $A = I + J$. Obviously $\alpha(M \cap N) = \alpha(I \cap J)$. Consider the homomorphism $\pi: A \to A/\alpha(I \cap J)$ such that $\pi(a) = a + \alpha(I \cap J)$. Since $\alpha(I \cap J) \subseteq I$ and $\alpha(I \cap J) \subseteq J$ we have $\pi(I) + \pi(J) = \pi(A)$, $\alpha(\pi(I)) = \pi(I)$, $\alpha(\pi(J)) = \pi(J)$, $\alpha(\pi(I) \cap \pi(J)) = \alpha(\pi(I \cap J)) = \alpha(I \cap J/\alpha(I \cap J)) = 0$. By assumption $\pi(I) \cap \pi(J) = 0$. Therefore $\pi(I \cap J) = 0$. Thus $I \cap J = \alpha(I \cap J)$. Consequently $\alpha(M) \cap \alpha(N) = I \cap J = \alpha(I \cap J) = \alpha(M \cap N)$.

Lemma 2.9 is proved.

Lemma 2.10. *Let $\mathfrak{M}$ be a homomorphically closed class of rings with a unity element and let $\alpha = \mathscr{L}\mathfrak{M}$ be the lower radical generated by the class $\mathfrak{M}$. Then for every α-radical ring A we have*

1) *A has a nonzero central idempotent e such that $eM \in \mathfrak{M}$*;

2) *if a homomorphic image $\bar{A}$ of A is directly irreducible then $\bar{A} \in \mathfrak{M}$.*

Proof. 1) By ([14], Lemma 1) there exists an ascending chain of subrings $0 \neq A_0 \lhd A_1 \lhd A_2 \lhd \ldots \lhd A_n \lhd A$ such that $A_0 \in \mathfrak{M}$. Let e be a unit of A_0. Then e is a central idempotent and $A_0 = eA$ ([6], Lemma 4). Thus $eA \in \mathfrak{M}$.

2) By the preceding there exists a nonzero central idempotent $u \in \bar{A}$ such that $u\bar{A} \in \mathfrak{M}$. Since $\bar{A} = u\bar{A} + (1 - u)\bar{A}$ and $\bar{A}$ is directly irreducible we have $\bar{A} = u\bar{A}$. Thus $\bar{A} \in \mathfrak{M}$.

Proposition 2.1. *Let $\mathfrak{M}$ be a homomorphically closed class of rings with a unity element and let $\alpha = \mathscr{L}\mathfrak{M}$ be the lower radical generated by the class $\mathfrak{M}$. Then*

1) $\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$ *for arbitrary ideals I, J of an arbitrary ring A;*

2) *if the radical α is hereditary then every directly irreducible ring in $\mathfrak{M}$ is simple.*

Proof. 2) Let A be a directly irreducible ring, $0 \neq I \lhd A$ and $\alpha(I) = I$. Lemma 2.10 implies that there exists a central idempotent $e \in I$ such that $0 = eI \in \mathfrak{M}$. For every $a \in A$ we have ea, $ae \in I$ and $ea = e^2a = e(ea) = (ea)e = e(ae) = (ae)e = ae$. Thus e is a central idempotent of A. Clearly $I \supseteq eA$, $eA \supseteq eI \supseteq eA$ and $eA = eI \in \mathfrak{M}$. Hence every α-radical ideal I of A contains a nonzero central idempotent e such that $eA = eI \in \mathfrak{M}$. Since A is directly irreducible we have $e = 1$ and $I = A$. Thus no directly irreducible ring contains proper α-radical ideals. By assumption α is hereditary. Therefore we have proved that any directly irreducible ring in $\mathfrak{M}$ is simple.

1) Let $I, J \lhd A$, $I + J = A$, $\alpha(I) = I$, $\alpha(J) = J$, $I \cap J \neq 0$. By Lemma 2.9 it is sufficient to prove that $\alpha(I \cap J) \neq 0$. By Zorn's lemma there exists an ideal M maximal with respect to $M \subseteq I$, $M \cap J = 0$. Consider the homomorphism $\pi\colon A \to A/M$ such that $\pi(a) = a + M$. Then $\alpha(\pi(I)) = \pi(I)$, $\alpha(\pi(J)) = \pi(J)$, $\pi(I) + \pi(J) = \pi(a)$. Since $M \subseteq I$ we have $(I + M) \cap (J + M) = I \cap (J + M) = I \cap J + M$ and $\pi(I) \cap \pi(J) = \pi(I \cap J)$. We claim that $\pi(I) \cap \pi(J) = \pi(I \cap J) \neq 0$. Suppose that $\pi(I \cap J) = 0$. Then $I \cap J \subseteq M$ and $I \cap J = I \cap J \cap M = M \cap J = 0$. Thus $\pi(I) \cap \pi(J) \neq 0$. Lemma 2.10 implies that there exists a central idempotent $e \in \pi(I)$ such that $0 \neq e\,\pi(I) \in \mathfrak{M}$. By the above e is a central idempotent of $\pi(A)$ and $e\,\pi(I) = e\,\pi(A) \in \mathfrak{M}$. Since e is a central idempotent we have a homomorphism from the α-radical ring $\pi(J)$ to $e\,\pi(A)$ such that $\varphi(a) = ea$. Clearly either $e\,\pi(J) = 0$ or there exists a central idempotent $v \in \pi(A)$ such that $0 \neq ve\,\pi(J) \in \mathfrak{M}$. Suppose now that $e\,\pi(J) = 0$. Then $0 = \pi(J) = e\,\pi(A) \cap \pi(J)$ and $M = \pi^{-1}(e\,\pi(A) \cap \pi(J)) = \pi^{-1}(e\,\pi(A) \cap \pi^{-1}(\pi(J)) = \pi^{-1}(e\,\pi(A)) \cap (J + M) = \pi^{-1}(e\,\pi(A) \cap J) + M$. Therefore $\pi^{-1}(e\,\pi(A)) \cap J \subseteq M \cap J = 0$. This contradicts the relation $\pi^{-1}(e\,\pi(A)) \not\supseteq M$. So there exists a central idempotent v such that $0 \neq ve\,\pi(J) \in \mathfrak{M}$. Clearly $ve\,\pi(J) \lhd \pi(A)$ and $ve\,\pi(J) \subseteq \pi(I) \cap \pi(J)$. Therefore $0 \neq \alpha(\pi(I) \cap \pi(J)) = \alpha(\pi(I \cap J))$. Since $M \cap J = 0$ we conclude that the rings $I \cap J$ and $\pi(I \cap J)$ are isomorphic. Thus $\alpha(I \cap J) \neq 0$. Proposition 2.1 is proved.

Lemma 2.11. *For a subidempotent radical* α *the following conditions are equivalent*:

1) α *is an additive radical*;

2) *if* I, J *are ideals of an arbitrary* α*-radical ring* A *such that* $I + J = A$ *and* $\alpha(I) = 0$ *then* $J = A$.

Proof. It is sufficient to prove 2) $\Rightarrow$ 1). Let $M, N \lhd B, B = M + N$ and $\alpha(B) = B$. We shall show that $B = \alpha(M) + \alpha(N)$. Consider the homomorphism $\pi: B \to B/\alpha(N)$ such that $\pi(b) = b + \alpha(N)$. Then $\alpha(\pi(N)) = \alpha(N/\alpha(N)) = 0$, $\alpha(\pi(B)) = \pi(B)$ and $\pi(N) + \pi(M) = \pi(B)$. By assumption $\pi(M) = \pi(B)$. Therefore $\alpha(N) + M = B$. Consider the homomorphism $\varphi: B \to B/\alpha(M)$ such that $\pi(b) = b + \alpha(M)$. Then $\alpha(\varphi(M)) = 0$, $\alpha(\varphi(B)) = \varphi(B)$ and $\varphi(M) + \varphi(\alpha(N)) = \varphi(B)$. By assumption $\varphi(\alpha(N)) = B$. Therefore $\alpha(N) + \alpha(M) = B$. Theorem 2.1 implies that α is additive.

Lemma 2.12. *Let* $\mathfrak{M}$ *be a homomorphically closed class of rings with a unity element, let* $\alpha = \mathscr{L}\mathfrak{M}$ *be the lower radical generated by the class* $\mathfrak{M}$. *Suppose that no subdirectly irreducible ring in* $\mathfrak{M}$ *contains nontrivial idempotents. Then for every* α*-radical ring* A *we have*

1) *every idempotent in* A *is central*;

2) *for every* $a \in A$ *there exists an idempotent* $e \in A$ *such that* $ea = a$.

Proof. 1) Let $0 \neq e = e^2 \in A$ and $ea - ae \neq 0$ for some $a \in A$. By Zorn's lemma there exists an ideal M maximal with respect to $ea - ae \notin M$. Consider $\bar{A} = A/M$ and the homomorphism $\pi: A \to \bar{A}$ such that $\pi(a) = a + M$. Clearly $\bar{A}$ is subdirectly irreducible and $\pi(e)\,\pi(a) - \pi(a)\,\pi(e) \neq 0$. Lemma 2.10 implies that $\bar{A} \in \mathfrak{M}$. By assumption $\pi(e) = 1$. Thus $\pi(e)\,\pi(a) - \pi(a)\,\pi(e) = 0$. This contradicts the inequality $\pi(e)\,\pi(a) - \pi(a)\,\pi(e) \neq 0$.

2) By ([8], Lemma 6.9) it is sufficient to show that every homomorphic image $\bar{A} = \varphi(A)$ of A contains an idempotent e such that $e\,\varphi(a) = \varphi(a)$. Lemma 2.10 implies that $\bar{A} \in \mathfrak{M}$. Therefore $\bar{A} \ni 1$. Clearly $1\,\varphi(a) = \varphi(a)$. Thus the ring A contains an idempotent e such that $ea = a$ ([8], Lemma 6.9).

Theorem 2.3. *Let* $\mathfrak{M}$ *be a homomorphically closed class of rings with a unity element and let* $\alpha = \mathscr{L}\mathfrak{M}$ *be the lower radical generated by the class* $\mathfrak{M}$. *Suppose that every directly irreducible ring* B *in* $\mathfrak{M}$ *fulfils the following conditions*:

a) *either* B *is simple or the set of proper ideals of* B *contains a greatest ideal*;

b) B *does not contain nontrivial idempotents.*

Then

1) *for an arbitrary ring* A, α *induces an endomorphism of the lattice* $L(A)$ *of ideals of* A;

2) *if the radical* α *is hereditary then every ring in* $\mathfrak{M}$ *is simple.*

Proof. By Proposition 2.1 and Lemma 2.11 it is sufficient to prove that for arbitrary ideals I, J of the ring A the equalities $I + J = A$, $\alpha(I) = 0$, $\alpha(A) = A$ imply that $J = A$. Suppose that $J \neq A$. Then there exists $a \in A$ such that $a \notin J$. Consider $E = \{e \in A \mid e^2 = e\}$ and $T = \{u \in E \mid ua \in J\}$. By Lemma 2.12 E is a set of central idempotents of A and $ea = a$ for some $e \in E$. Clearly $0 \in T$, $e \notin T$. For arbitrary $u \in T$, $w \in T$, $v \in E$ we have $u + w - uw \in T$ and $uv \in T$. Consider the set $P = \{e - ue \mid u \in T\}$. Clearly $P \subseteq E$, $e \in P$ and $xy \in P$ for arbitrary $x, y \in P$. Moreover, $P \cap T = \emptyset$. Indeed, suppose that $e - ue \in T$ for some $u \in T$; then $(e - ue)\, a \in J$. Therefore $ea - uea = a - ua \in J$. Since $u \in T$ then $ua \in J$. This contradicts $a \notin J$. By Zorn's lemma there exists a subset $S \subseteq E$ maximal with respect to $P \subseteq S$, $S \cap T = \emptyset$ and $xy \in S$ for arbitrary $x, y \in S$. Consider $M = \{b \in A \mid xb = 0$ for some $x \in S\}$. Clearly $M \lhd A$ and $T \subseteq M$. Let $\bar{A} = A/M$ and let π be a homomorphism from A to $\bar{A}$ such that $\bar{b} = \pi(b) = b + M$. Obviously $(1 - e)\, A \subseteq M$. Therefore $\bar{e}$ is a unity element of $\bar{A}$. Consider $\bar{I} = \pi(I)$, $\bar{J} = \pi(J)$. We shall show that $\bar{a} \notin \bar{J}$. Suppose that $\bar{a} \in \bar{J}$. Then $v(a - b) = 0$ for some $v \in S$ and $b \in J$. Therefore $va = vb \in J$ and $v \in T$. Since $S \cap T = \emptyset$ we have $\bar{a} \notin \bar{J}$. Thus $\bar{A} \neq \bar{J}$.

Suppose now that $\bar{A} = \bar{I}$. Then $\bar{e} \in \bar{I}$ and $v(e - b) = 0$ for some $v \in S$ and $b \in I$. Therefore $ve = vb \in I$. Since $ve \in S$ we have $ve \neq 0$. Moreover, ve is a central idempotent of the α-radical ring A. Therefore veA is an α-radical ring. Since $veA \lhd I$ we have a contradiction with $\alpha(I) = 0$. Thus $\bar{I} \neq \bar{A}$.

Now we show that $\bar{A} \in \mathfrak{M}$. Since $\alpha(\bar{A}) = \bar{A}$ we have $w\bar{A} \in \mathfrak{M}$ for some nonzero central idempotent w of $\bar{A}$ (see Lemma 2.10). Let $w = \pi(b)$ where $b \in A$. Then $b^2 - b \in M$ and $v(b^2 - b) = 0$ for some $v \in S$. Thus $(vb)^2 = vb$. Clearly $(1 - v)\, A \subseteq M$. Therefore $\pi(v)$ is a unity element of $\bar{A}$ and $\pi(vb) = \pi(b) = w \neq 0$. Thus $vb \notin M$ and $x(vb) \neq 0$ for all $x \in S$. Since $T \subseteq M$ we have $vb \notin T$. The maximality of S yields $vb \in S$. Therefore $\pi(vb) = \pi(v)$. Thus $w = \pi(vb)$ is a unity element of $\bar{A}$ and $\bar{A} \in \mathfrak{M}$. In the same way it is possible to show that $\bar{A}$ is directly irreducible. The assumptions together with $\bar{I} \neq \bar{A}$ and $\bar{J} \neq \bar{A}$ yield $\bar{I} + \bar{J} \neq \bar{A}$. This contradicts $I + J = A$. Thus $J = A$ and Theorem 2.3 is proved.

Corollary 2.3. *Let $\mathfrak{M}$ be a homomorphically closed class of rings generated by the ring of integers and let $\alpha = \mathscr{L}\mathfrak{M}$ be the lower radical generated by the class $\mathfrak{M}$. Then*

1) *α is a subidempotent radical;*
2) *α is not additive;*
3) *$\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$ for arbitrary ideals I, J of an arbitrary ring A.*

Proof. By Proposition 2.1 it is sufficient to prove that the radical α is not additive. Clearly $\alpha(\mathbf{Z}) = \mathbf{Z}$. Lemma 2.10 implies that $\mathbf{Z}$ does not contain proper α-radical ideals. Let p, q be two distinct prime numbers. Then $\alpha(p\mathbf{Z}) = \alpha(q\mathbf{Z}) = 0$ and $\mathbf{Z} = \alpha(\mathbf{Z}) = \alpha(p\mathbf{Z} + q\mathbf{Z}) \neq \alpha(p\mathbf{Z}) + \alpha(q\mathbf{Z}) = 0$. Thus α is not additive.

Corollary 2.4. *Let p be a prime number, $A = Z/p^2Z$, $\mathfrak{M}$ a homomorphically closed class of rings generated by A and $\alpha = \mathscr{L}\mathfrak{M}$ the lower radical generated by $\mathfrak{M}$. Then*

1) α *is a subidempotent radical*;

2) α *is not a hereditary radical*;

3) *for an arbitrary ring* B, α *induces an endomorphism of the lattice* $L(B)$ *of ideals of* B.

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