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VARIETIES OF GROUPOIDS DETERMINED BY SHORT LINEAR IDENTITIES

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0. INTRODUCTION

In this paper we are going to study the varieties of groupoids that can be defined by a set of linear identities of length $\leqq 6$. An identity $a = b$ is said to be *linear* if every variable occurring in either a or b occurs exactly twice in $a = b$, once on each side. The length of an identity $a = b$ is the number of occurrences of variables in $a = b$ (so that in the case of a linear identity, the length is always na even number).

There are exactly sixteen nonequivalent linear groupoid identities of length $\leqq 6$. They are listed in Section 1; the corresponding varieties are denoted by $V_0, \ldots, V_{15}$. This implies that there are at most 2^{16} groupoid varieties determined by a set of linear identities of length $\leqq 6$. We shall show in Section 2 that the exact number is 56. The purpose of the present paper is to prove this result and, moreover, to investigate some properties of the varieties $V_0, \ldots, V_{15}$.

The first three members of this collection are the varieties of all groupoids, of commutative groupoids and of semigroups. Various properties of these varieties are well known. V_3 is the variety of left permutable groupoids investigated in [3] and V_4 is the variety of left modular groupoids investigated in [4]. As each of the varieties $V_{11}, \ldots, V_{15}$ is dual to one of the varieties $V_0, \ldots, V_{10}$, we shall restrict our attention to the varieties $V_5, \ldots, V_{10}$.

For most of the varieties V_i we shall describe the corresponding free groupoids. We shall be concerned with what can be said about *left* and *right cancellation* and *division groupoids in* V_i. Further, in each case we shall either describe the class of simple groupoids belonging to V_i or show that the class is very large. In most cases the problem of describing simple groupoids in V_i can be reduced to the analogous question for one of the following four varieties: commutative semigroups, left unars, right unars and commutative groupoids. For this reason it will be useful to collect here some information about simple objects in these four varieties.

A commutative semigroup is simple iff it is either a cyclic group of prime order or a two-element semilattice or a two-element semigroup with zero multiplication. A left unar (a groupoid satisfying $xy = xz$) is simple iff it is either a cycle of prime order or a two-element semigroup with zero multiplication or a two-element semigroup of left zeros. Simple right unars can be described analogously. On the other hand, every commutative groupoid can be imbedded into a simple commutative groupoid and so the class of simple commutative groupoids is very large. All these facts are well known and can be easily proved.

Two special relations p_G and q_G which can be defined on any groupoid G will play a role in the sequel. They are defined as follows: $(a, b) \in p_G$ iff $ax = bx$ for all $x \in G$; $(a, b) \in q_G$ iff $xa = xb$ for all $x \in G$.

For a groupoid G and an element $a \in G$ we define two mappings L_a and R_a of G into itself by $L_a(x) = ax$ and $R_a(x) = xa$.

Let us remark that varieties of quasigroups determined by short linear identities were investigated in the papers [1] and [2]. Linear identities were called *balanced* in these papers. In the present paper we choose to use the name "linear", as the name "balanced" is usually reserved for a more general notion: an identity $a = b$ is said to be *balanced* if every variable has the same number of occurrences in a as in b.

1. ONE LINEAR IDENTITY OF LENGTH $\leqq 6$

1.1. Proposition. *Every linear identity of length $\leqq 6$ is equivalent to one of the following sixteen identities:*

(0) $x = x$,	(8) $x \, . \, yz = zy \, . \, x$,
(1) $xy = yx$,	(9) $x \, . \, yz = yz \, . \, x$,
(2) $x \, . \, yz = xy \, . \, z$,	(10) $x \, . \, yz = zx \, . \, y$,
(3) $x \, . \, yz = y \, . \, xz$,	(11) $xy \, . \, z = xz \, . \, y$,
(4) $x \, . \, yz = z \, . \, yx$,	(12) $xy \, . \, z = zy \, . \, x$,
(5) $x \, . \, yz = x \, . \, zy$,	(13) $xy \, . \, z = yx \, . \, z$,
(6) $x \, . \, yz = y \, . \, zx$,	(14) $xy \, . \, z = zx \, . \, y$,
(7) $x \, . \, yz = yx \, . \, z$,	(15) $x \, . \, yz = xz \, . \, y$.

Moreover, the identity (11) *is dual to* (3), (12) *is dual to* (4), (13) *is dual to* (5), (14) *is dual to* (6) *and* (15) *is dual to* (7); *the remaining six identities are self-dual.*

Proof. Clearly, $x \, . \, yz = z \, . \, xy$ is equivalent to (6) and $xy \, . \, z = yz \, . \, x$ is equivalent to (14).

For $i = 0$ to 15 we denote by V_i the variety determined by the equation (i).

V_0 is the variety of all groupoids, V_1 is the variety of commutative groupoids, V_2 is the variety of semigroups, V_3 is the variety of left permutable groupoids and V_4 is the variety of left modular groupoids.

1.2. Proposition. *Let $i, j \in \{0, \ldots, 15\}$. Then $V_i \subseteq V_j$ iff either $i = j$ or $j = 0$ or else $i = 1$ and $j \in \{5, 8, 9, 13\}$. Consequently, the varieties $V_0, \ldots, V_{15}$ are pairwise different.*

Proof. The converse implication is evident. In order to prove the direct implication, we shall construct for every $i \in \{0, \ldots, 15\}$ a groupoid G_i for which it is a matter of routine to verify that G_i belongs to V_i but not to V_j for any $j \in \{1, \ldots, 15\}$ different from i and different from 5, 8, 9, 13 in the case $i = 1$.

G_0	0	1	2
0	0	1	2
1	1	2	1
2	2	0	2

G_1	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

G_2	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

G_3	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	5	3	4
2	1	2	0	4	5	3
3	0	1	2	3	4	5
4	2	0	1	5	3	4
5	1	2	0	4	5	3

G_4	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	2	4	1	3	10	12	14	11	13	5	7	9	6	8
1	4	1	3	0	2	14	11	13	10	12	9	6	8	5	7
2	3	0	2	4	1	13	10	12	14	11	8	5	7	9	6
3	2	4	1	3	0	12	14	11	13	10	7	9	6	8	5
4	1	3	0	2	4	11	13	10	12	14	6	8	5	7	9
5	5	7	9	6	8	0	2	4	1	3	10	12	14	11	13
6	9	6	8	5	7	4	1	3	0	2	14	11	13	10	12
7	8	5	7	9	6	3	0	2	4	1	13	10	12	14	11
8	7	9	6	8	5	2	4	1	3	0	12	14	11	13	10
9	6	8	5	7	9	1	3	0	2	4	11	13	10	12	14
10	10	12	14	11	13	5	7	9	6	8	0	2	4	1	3
11	14	11	13	10	12	9	6	8	5	7	4	1	3	0	2
12	13	10	12	14	11	8	5	7	9	6	3	0	2	4	1
13	12	14	11	13	10	7	9	6	8	5	2	4	1	3	0
14	11	13	10	12	14	6	8	5	7	9	1	3	0	2	4

G_5	0	1	2	3	4	5
0	0	2	1	0	2	1
1	2	1	0	2	1	0
2	1	0	2	1	0	2
3	3	5	4	3	5	4
4	5	4	3	5	4	3
5	4	3	5	4	3	5

G_6	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	4	0	0	7	0	0	0
2	0	0	0	5	0	0	7	0	0
3	0	6	0	0	7	0	0	0	0
4	0	0	0	8	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

G_7	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	6	0
2	0	4	0	5	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	6	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0

G_8	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	6	0	0
2	0	0	0	4	0	0	0
3	0	0	5	0	0	0	0
4	0	0	0	0	0	0	0
5	0	6	0	0	0	0	0
6	0	0	0	0	0	0	0

G_9	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	5	0
2	0	0	0	4	0	0
3	0	0	0	0	0	0
4	0	5	0	0	0	0
5	0	0	0	0	0	0

G_{10}	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	6	0
2	0	0	0	5	0	0	0
3	0	4	0	0	0	0	0
4	0	0	6	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0

The groupoids $G_{11}, G_{12}, G_{13}, G_{14}, G_{15}$ can be defined as the duals of G_3, G_4, G_5, G_6, G_7, respectively.

Let us remark that instead of the groupoids G_3, G_4 and G_5 it would be also possible to take the groupoids G'_3, G'_4, G'_5 with the underlying set $\{0, 1, \ldots, 8\}$ and multiplication defined by $xy = 0$ for all x, y except for the following cases:

$$\text{in } G'_3\text{:}\quad 1\cdot2 = 4\,,\quad 1\cdot3 = 5\,,\quad 1\cdot6 = 7\,,\quad 2\cdot3 = 6\,,\quad 2\cdot5 = 7\,,\quad 4\cdot3 = 8\,;$$

$$\text{in } G'_4\text{:}\quad 1\cdot2 = 4\,,\quad 1\cdot6 = 7\,,\quad 2\cdot1 = 5\,,\quad 2\cdot3 = 6\,,\quad 3\cdot5 = 7\,,\quad 4\cdot3 = 8\,;$$

$$\text{in } G'_5\text{:}\quad 1\cdot2 = 4\,,\quad 1\cdot5 = 7\,,\quad 1\cdot6 = 7\,,\quad 2\cdot3 = 5\,,\quad 3\cdot2 = 6\,,\quad 4\cdot3 = 8\,.$$

The groupoid G'_4 is of smaller cardinality than the quasigroup G_4.

2. SEVERAL LINEAR IDENTITIES OF LENGTH $\leqq 6$

Put

$$V_{16} = V_3 \cap V_4 \cap V_5 \cap V_6\,,$$
$$V_{17} = V_{11} \cap V_{12} \cap V_{13} \cap V_{14}\,,$$
$$V_{18} = V_3 \cap V_{11}\,,$$
$$V_{19} = V_3 \cap V_{12}\,,$$
$$V_{20} = V_3 \cap V_{13}\,,$$
$$V_{21} = V_3 \cap V_{14}\,,$$
$$V_{22} = V_4 \cap V_{11}\,,$$
$$V_{23} = V_4 \cap V_{12}\,,$$

$$V_{24} = V_4 \cap V_{13},$$
$$V_{25} = V_4 \cap V_{14},$$
$$V_{26} = V_5 \cap V_{11},$$
$$V_{27} = V_5 \cap V_{12},$$
$$V_{28} = V_5 \cap V_{13},$$
$$V_{29} = V_5 \cap V_{14},$$
$$V_{30} = V_6 \cap V_{11},$$
$$V_{31} = V_6 \cap V_{12},$$
$$V_{32} = V_6 \cap V_{13},$$
$$V_{33} = V_6 \cap V_{14},$$
$$V_{34} = V_3 \cap V_4 \cap V_5 \cap V_6 \cap V_{11},$$
$$V_{35} = V_3 \cap V_4 \cap V_5 \cap V_6 \cap V_{12},$$
$$V_{36} = V_3 \cap V_4 \cap V_5 \cap V_6 \cap V_{13},$$
$$V_{37} = V_3 \cap V_4 \cap V_5 \cap V_6 \cap V_{14},$$
$$V_{38} = V_3 \cap V_{11} \cap V_{12} \cap V_{13} \cap V_{14},$$
$$V_{39} = V_4 \cap V_{11} \cap V_{12} \cap V_{13} \cap V_{14},$$
$$V_{40} = V_5 \cap V_{11} \cap V_{12} \cap V_{13} \cap V_{14},$$
$$V_{41} = V_6 \cap V_{11} \cap V_{12} \cap V_{13} \cap V_{14},$$
$$V_{42} = V_3 \cap V_4 \cap V_5 \cap V_6 \cap V_{11} \cap V_{12} \cap V_{13} \cap V_{14},$$
$$V_{43} = V_2 \cap V_3 \cap V_7 \cap V_{13},$$
$$V_{44} = V_3 \cap V_8 \cap V_{10} \cap V_{11},$$
$$V_{45} = V_3 \cap V_9 \cap V_{12} \cap V_{15},$$
$$V_{46} = V_2 \cap V_4 \cap V_8 \cap V_{12},$$
$$V_{47} = V_4 \cap V_7 \cap V_9 \cap V_{11},$$
$$V_{48} = V_4 \cap V_{10} \cap V_{13} \cap V_{15},$$
$$V_{49} = V_2 \cap V_5 \cap V_{11} \cap V_{15},$$
$$V_{50} = V_5 \cap V_7 \cap V_{10} \cap V_{12},$$
$$V_{51} = V_5 \cap V_8 \cap V_9 \cap V_{13},$$
$$V_{52} = V_2 \cap V_6 \cap V_9 \cap V_{10} \cap V_{14},$$
$$V_{53} = V_6 \cap V_7 \cap V_8 \cap V_{14} \cap V_{15},$$
$$V_{54} = V_2 \cap \ldots \cap V_{15},$$
$$V_{55} = V_1 \cap \ldots \cap V_{15}.$$

For $i \in \{16, \ldots, 55\}$ denote by S_i the set of the numbers $j \in \{1, \ldots, 15\}$ such that V_j occurs in the intersection defining the variety V_i. For example, $S_{16} = \{3, 4, 5, 6\}$ and $S_{45} = \{3, 9, 12, 15\}$.

For $i \in \{1, \ldots, 15\}$ put $S_i = \{i\}$. Finally, put $S_0 = \emptyset$.

2.1. Proposition. *If $i, j \in \{3, \ldots, 6\}$ and $i \neq j$ then $V_i \cap V_j = V_{16}$. If $i, j \in \in \{11, \ldots, 14\}$ and $i \neq j$ then $V_i \cap V_j = V_{17}$.*

Proof. It is sufficient to prove the first assertion, as the latter is dual. The identities (i) and (j) can be written as $x . yz = p(x) . p(y) p(z)$ and $x . yz = q(x) . q(y) q(z)$ where p, q are two permutations of x, y, z generating the symmetric group on $\{x, y, z\}$. It is easy to see that for any equational theory E the set of the permutations r of $\{x, y, z\}$ for which the identity $x . yz = r(x) . r(y) r(z)$ belongs to E is a group. So, if E contains both (i) and (j) then it contains all of the identities (3), (4), (5), (6).

2.2. Proposition. *The varieties $V_3, \ldots, V_6, V_{11}, \ldots, V_{14}, V_{16}, \ldots, V_{42}$ are pairwise different and this collection of varieties is closed under intersection. If $i, j \in \in \{3, \ldots, 6, 11, \ldots, 14, 16, \ldots, 42\}$ then $V_i \subseteq V_j$ iff $S_j \subseteq S_i$.*

Proof. By 2.1, the collection is closed under intersection. It is enough to prove that $V_i \subseteq V_j$ implies $S_j \subseteq S_i$. From 2.1 it follows easily that if this were not true then we would have either $V_{16} \cap V_i \subseteq V_{17}$ for some $i \in \{11, \ldots, 14\}$ or $V_{17} \cap V_i \subseteq V_{16}$ for some $i \in \{3, \ldots, 6\}$. Since the latter possibility is dual, we shall consider the first only. For every $i \in \{11, \ldots, 14\}$ we shall find a groupoid K_i such that $K_i \in V_{16} \cap V_i$ and $K_i \notin V_{17}$:

K_{11}	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	4	5	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	6	0	0	0
5	0	0	6	0	0	0	0
6	0	0	0	0	0	0	0

K_{12}	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	4	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	5	0	0	0	0
4	0	0	0	6	0	0	0
5	0	6	0	0	0	0	0
6	0	0	0	0	0	0	0

K_{13}	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	4	0	0	0	0
2	0	5	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	6	0	0	0
5	0	0	0	6	0	0	0
6	0	0	0	0	0	0	0

K_{14}	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	4	0	0	0	0	0
2	0	0	0	6	0	0	0	0
3	0	5	0	0	0	0	0	0
4	0	0	0	7	0	0	0	0
5	0	0	7	0	0	0	0	0
6	0	7	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0

2.3. Proposition. *Let $i \in \{43, \ldots, 53\}$ and let j, k be two different numbers from S_i such that the set $\{j, k\}$ is not contained in $\{3, \ldots, 6\} \cup \{11, \ldots, 14\}$. Then $V_i = = V_j \cap V_k$.*

Proof. The result needs just a tedious checking.

2.4. Proposition. *$V_1 \cap V_i = V_{55}$ for any $i \in \{2, \ldots, 15\} \setminus \{5, 8, 9, 13\}$.*

Proof. It is easy.

2.5. Proposition. *The collection $V_0, \ldots, V_{55}$ is closed under intersection.*

Proof. It can be verified easily by using the previous results.

2.6. Theorem. *There are exactly 56 groupoid varieties determined by a set of linear identities of length $\leqq 6$, and namely the varieties $V_0, \ldots, V_{55}$. If $i, j \in \{0, \ldots, 55\}$ then $V_i \subseteq V_j$ iff either $S_j \subseteq S_i$ or else $i = 1$ and $j \in \{5, 8, 9, 13, 28, 51\}$.*

Proof. By 2.5, there are no other groupoid varieties determined by a set of linear identities of length $\leqq 6$ than $V_0, \ldots, V_{55}$. It remains to prove the second assertion, in which only the direct implication is not evident. Using the previous results we easily see that everything will be proved if for every $i \in \{42, \ldots, 53\}$ we construct a groupoid L_i belonging to V_i but not to V_j for any $j \in \{1, \ldots, 15\} \setminus S_i$.

L_{42}	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	4	4	0	0
2	0	4	4	4	0	0
3	0	4	4	4	0	0
4	0	5	5	5	0	0
5	0	0	0	0	0	0

L_{43}	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	4	7	0	0	8	0	0
2	0	5	0	6	0	0	0	8	0
3	0	0	0	0	0	0	0	0	0
4	0	0	0	8	0	0	0	0	0
5	0	0	0	8	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

L_{44}	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	4	0	0	8	0	0
2	0	0	0	6	8	0	0	0	0
3	0	5	7	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	8	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	8	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

L_{45}	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	4	0	6	0
2	0	0	0	5	6	0	0
3	0	0	0	0	0	0	0
4	0	0	6	0	0	0	0
5	0	6	0	0	0	0	0
6	0	0	0	0	0	0	0

L_{46}	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	4	0	0	0	8	0	0
2	0	5	0	6	0	0	0	0	0
3	0	0	7	0	0	8	0	0	0
4	0	0	0	8	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	8	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

L_{48}	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	4	0	0	0	8	0
2	0	5	0	7	0	0	0	0	0
3	0	6	0	0	0	8	0	0	0
4	0	0	8	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	8	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

L_{51}	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	6	6	0
2	0	0	0	4	0	0	0
3	0	0	5	0	0	0	0
4	0	6	0	0	0	0	0
5	0	6	0	0	0	0	0
6	0	0	0	0	0	0	0

L_{52}	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	4	0	0	0	7	0
2	0	0	0	6	0	7	0	0
3	0	5	0	0	7	0	0	0
4	0	0	0	7	0	0	0	0
5	0	0	7	0	0	0	0	0
6	0	7	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0

L_{53}	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	4	6	0	0	0	0	10	0	0
2	0	5	0	8	0	0	0	10	0	0	0
3	0	7	9	0	10	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	10	0	0	0	0	0	0	0
6	0	0	10	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0
9	0	10	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0

The groupoids L_{47}, L_{49}, L_{50} can be defined as the duals of the groupoids L_{45}, L_{43}, L_{48}, respectively.

Let us remark that (as follows from the from the above results) the varieties $V_0, \ldots, V_{15}$ can be determined by one linear identity, the varieties $V_{16}, \ldots, V_{33}$, $V_{43}, \ldots, V_{53}, V_{55}$ by two linear identities, the varieties $V_{34}, \ldots, V_{41}, V_{54}$ by three linear identities and the variety V_{42} by four linear identities.

3. THE VARIETY V_5

V_5 is the variety determined by $x \,.\, yz = x \,.\, zy$.

3.1. Proposition. *Let a, b be two terms; we can express them uniquely in the form $a = (((xa_1)\, a_2) \ldots)\, a_n$, $b = (((yb_1)\, b_2) \ldots)\, b_m$ where x, y are variables, $n, m \geqq 0$ and $a_1, \ldots, a_n, b_1, \ldots, b_m$ are terms. The identity $a = b$ is satisfied in V_5 iff $x = y$, $n = m$, and the identity $a_i = b_i$ is a consequence of the commutative law for any $i \in \{1, \ldots, n\}$.*

Proof. It can be easily proved that if $a = b$ is a consequence of $xy = yx$ then $xa = xb$ is a consequence of $x \,.\, yz = x \,.\, zy$. The direct implication can be proved by induction on the length of a formal proof of $a = b$ from $x \,.\, yz = x \,.\, zy$.

3.2. Proposition. *Let F be a free commutative groupoid over a set X. Define a groupoid G as follows: the underlying set of G is the set of finite sequences $(x, a_1, \ldots, a_n)$ where $x \in X$, $n \geqq 0$ and $a_1, \ldots, a_n \in F$; the multiplication is defined by $(x, a_1, \ldots, a_n)(y, b_1, \ldots, b_m) = (x, a_1, \ldots, a_n, a_{n+1})$ where $a_{n+1} = (((yb_1)\, b_2) \ldots) . . b_m \in F$. Then G is a free V_5-groupoid over X.*

Proof. It follows from 3.1.

3.3. Proposition. *Free V_5-groupoids are right cancellative but not left cancellative.*

Proof. The first assertion follows from 3.2. If x is a free generator of a free V_5-groupoid then it is easy easy to see that $x(x \,.\, xx) = x(xx \,.\, x)$ but $x \,.\, xx \neq xx \,.\, x$.

3.4. Proposition. *Let $G \in V_5$. Then q_G is a congruence of G and the factor G/q_G is commutative.*

Proof. Clearly, q_G is an equivalence. Let $(a, b) \in q_G$ and $c \in G$. Then $ca = cb$ and $d \,.\, ac = d \,.\, ca = d \,.\, cb = d \,.\, bc$ for all $d \in G$, so that $(ca, cb) \in q_G$ and $(ac, bc) \in q_G$. It is evident that G/q_G is commutative.

3.5. Proposition. *Every simple groupoid from V_5 is either commutative or it is a left unar.*

Proof. It follows from 3.4.

3.6. Example. Let f, g be two surjective endomorphisms of an abelian group $(G, +)$ such that $gf = g^2$. For $a, b \in G$ put $ab = f(a) + g(b)$. Then G becomes a groupoid (under the new multiplication) belonging to V_5 and G is a division groupoid. If $f \neq g$ then G is not commutative and G is not left cancellative. For example, we can take a free abelian group $(G, +)$ over the set $\{x_1, x_2, x_3, \ldots\}$ and define f, g by $f(x_1) = f(x_2) = f(x_3) = g(x_1) = g(x_2) = x_1$, $g(x_3) = x_2$ and $f(x_i) = g(x_i) = x_{i-2}$ for $i \geqq 4$.

4. THE VARIETY V_6

V_6 is the variety determined by $x \,.\, yz = y \,.\, zx$.

4.1. Proposition. *Let x, y, u, v be four different variables. An identity $xy \,.\, uv = t$ is satisfied in V_6 iff t is one of the following nine terms: $xy \,.\, uv$, $u(v \,.\, xy)$, $v(xy \,.\, u)$, $u(x \,.\, yv)$, $u(y \,.\, vx)$, $x(yv \,.\, u)$, $yv \,.\, ux$, $y(vx \,.\, u)$, $vx \,.\, uy$. Consequently, V_6 is not contained in the variety of medial groupoids.*

Proof. It is easy.

4.2. Proposition. *Let $G \in V_6$ be either left or right cancellative. Then G is a commutative semigroup.*

Proof. It follows from $x \,.\, xy = x \,.\, yx$ and $xy \,.\, uy = yx \,.\, uy$ that G is commutative; but then G is a commutative semigroup.

4.3. Proposition. *Let $G \in V_6$ be either a left division or a right division groupoid. Then G is an abelian group.*

Proof. It follows from $R_{yz} = L_y L_z$ that G is a division groupoid. In a division groupoid the identity $xy . ux = yx . ux$ implies $xy . z = yx . z$. This means that G belongs to V_{13}, i.e. to the variety dual to V_5. By the dual of 3.4, p_G is a congruence of G and the factor G/p_G is commutative. It follows that G/p_G is an abelian group. Let $a, b \in G$. There exists an element $c \in G$ with $b = bc$. For all $x, y \in G$ we have (using 4.1)

$$xy . b = xy . bc = x(yc . b) = x(yc . bc) = x(y(cc . b)) = \\ = x(y(cc . bc)) = x(y(b(c . cc))) .$$

This implies $(b, b(c . cc)) \in p_G$, since G/p_G is an abelian group. Consequently, $ba = (b(c . cc))\, a$. Also, we get $xy . b = x(y(b(c . cc))) = b(c . cc) . xy$ for all $x, y \in G$ and so $ab = (b(c . cc))\, a$. This shows that $ab = ba$.

4.4. Proposition. *Let $G \in V_6$ and $a \in G$. Then aG is a left ideal of G; if $G = GG$ then aG is an ideal of G.*

Proof. For $b, c, d \in G$ we have $c . ab = a . bc \in aG$ and $ab . cd = a(bd . c) \in aG$.

4.5. Proposition. *Let $G \in V_6$ be simple. Then G is either an abelian group or a left unar or a two-element semilattice.*

Proof. We can assume that $G = GG$ and that G contains at least two elements. Put $I = \{x \in G; xG = G\}$ and $K = \{x \in G; \operatorname{Card}(xG) = 1\}$. By 4.4, G is the disjoint union of I and K. If K is empty then G is a left division groupoid and so an abelian group by 4.3. Let K be nonempty.

Let $a \in K$ and $b \in G$. For any $x = x_1 x_2 \in G$ we have $ab . x = ab . x_1 x_2 = a(bx_2 . x_1) = aa$ and $ba . x = ba . x_1 x_2 = a(x_2 b . x_1) = aa$, so that $ab \in K$ and $ba \in K$. We have proved that K is an ideal of G. Since G is simple, either $K = G$ or $\operatorname{Card}(K) = 1$. If $K = G$ then G is a left unar. Let $K = \{a\}$ for an element a.

It is clear that a is a zero of G. If $b, c \in I$ then L_b, L_c are both surjective, so that the mapping $R_{bc} = L_b L_c$ is also surjective; it follows that $bc \notin K$ and so $bc \in I$. We have proved that I is a subgroupoid of G; but then $(I \times I) \cup \mathrm{id}_G$ is a congruence of G, $\operatorname{Card}(I) = 1$ and G is a two-element semilattice.

5. THE VARIETY V_7

V_7 is the variety determined by $x . yz = yx . z$.

5.1. Proposition. *Let $a = a_1 a_2$, $b = b_1 b_2$ be two terms. The identity $a = b$ is satisfied in V_7 iff it is balanced (every variable has the same number of occurrences in a as in b), the last variable in a_1 coincides with the last variable in b_1 and the last variable in a_2 coincides with the last variable in b_2.*

Proof. It is easy to prove by induction on the length of a formal proof of $a = b$

from $x . yz = yx . z$ that the last variable in a_i coincides with the last variable in b_i (for $i = 1, 2$); clearly, the identity $a = b$ is balanced. Conversely, let $a = b$ be a balanced identity satisfying the above condition. We can easily prove by induction on the length of t that if t is any term which is not a variable then there exist a term u and a variable x such that the identity $t = ux$ is satisfied in V_7. Especially, there exist a variable x and two terms c, d such that the identities $a = cx$ and $b = dx$ are satisfied in V_7. If either c or d is a variable then evidently $c = d$ and everything is clear. Let c, d be not variables. There exist a variable y and two terms e, f such that the identitites $c = ey$ and $d = fy$ are satisfied in V_7. Consequently, the identities $a = ey . x$ and $b = fy . x$ are satisfied in V_7. Clearly, every variable has the same number of occurrences in e as in f. Now, in V_7 we have

$$(u . xy) v = (xu . y) v = y(xu . v) = y(u . xv) = uy . xv = (x . uy) v = (ux . y) v .$$

The identities $(u.xy) v = (ux.y) v$ and $(u.xy) v = (xu.y) v$ imply $a = b$; this is similar to the fact that the associative and commutative laws imply any balanced identity.

5.2. Proposition. *The variety V_7 is not contained in the variety of medial groupoids.*

Proof. If follows from 5.1.

5.3. Proposition. *Let X be a nonempty set. Denote by $(F, +, 0)$ the free commutative monoid over X. Define a groupoid G as follows: its underlying set is the set of the triples (a, x, y) such that $a \in F$, $x \in X \cup \{0\}$, $y \in X$ and if $a \neq 0$ then $x \neq 0$; the multiplication is defined by $(a, x, y)(b, u, v) = (a + x + b + u, y, v)$. Let us identify the elements y of X with the triples $(0, 0, y)$. Then G is a free V_7-groupoid over X. Consequently, free V_7-groupoids are neither left nor right cancellative.*

Proof. It follows from 5.1.

5.4. Proposition. *Let $G \in V_7$ be a right division groupoid. Then G is an abelian group.*

Proof. We have $x . yx = yx . x$ for all $x, y \in G$, so that G is commutative; consequently, G is associative.

5.5. Proposition. *Let $G \in V_7$ be right cancellative. Then G is a commutative semigroup.*

Proof. It follows from $(xu . y) v = (ux . y) v$ that G is commutative.

5.6. Proposition. *Let $G \in V_7$. Then p_G is a congruence of G and the factor G/p_G is a semigroup; if G is a semigroup then G/p_G is commutative.*

Proof. Let $(a, b) \in p_G$ and $c \in G$. Then $ac = bc$ and $ca . d = a . cd = b . cd = cb . d$ for all $d \in G$, so that $(ac, bc) \in p_G$, $(ca, cb) \in p_G$ and p_G is a congruence. It follows from $(u . xy) v = (ux . y) v$ that G/p_G is a semigroup. If G is a semigroup and $a, b \in G$ then $abc = bac$ for all $c \in G$, $(ab, ba) \in p_G$, and G/p_G is commutative.

5.7. Proposition. *Let $G \in V_7$ be simple. Then G is either a right unar or an abelian group or a two-element semilattice.*

Proof. It follows from 5.6.

5.8. Example. Let f be an endomorphism of an abelian group $(G, +)$ such that $f \neq \mathrm{id}_G$ and $f = f^2$. Put $xy = f(x) + y$ for all $x, y \in G$. Then $G \in V_7$ and G is an associative left quasigroup. Moreover, G is not commutative.

6. THE VARIETY V_8

V_8 is the variety determined by $x \,.\, yz = zy \,.\, x$.

6.1. Proposition. *Let $G \in V_8$ be either left or right cancellative. Then G is commutative.*

Proof. We have $ab \,.\, ba = ab \,.\, ab$ for all $a, b \in G$; if G is left cancellative, we get $ab = ba$.

6.2. Proposition. *Free V_8-groupoids are neither left nor right cancellative.*

Proof. It follows from 6.1.

6.3. Proposition. *Let $G \in V_8$ be either a left division or a right division groupoid. Then G is commutative.*

Proof. It follows from $L_xL_y = R_xR_y$ that G is a left division groupoid iff it is a right division groupoid. So, it is sufficient to assume that G is a division groupoid. Let $a, b \in G$. We can write $a = c \,.\, dd = dd \,.\, c$ for some $c, d \in G$ and $a = ae$ for some $e \in G$. We have

$$ab = ae \,.\, b = b \,.\, ea = b \,.\, e(c \,.\, dd) = b((dd \,.\, c)\, e) = b \,.\, ae = ba\,.$$

6.4. Lemma. *Let $G \in V_8$. Then both p_G and q_G are congruences of G; if $G = GG$ then $p_G = q_G$.*

Proof. It is obvious that p_G is an equivalence. If $(a, b) \in p_G$ and $c, d \in G$ then $ac = bc$ and $ca \,.\, d = d \,.\, ac = d \,.\, bc = cb \,.\, d$, so that $(ac, bc) \in p_G$ and $(ca, cb) \in p_G$. Quite similarly, q_G is a congruence. Now let $G = GG$. If $(a, b) \in p_G$ and $c, d \in G$ then $cd \,.\, a = a \,.\, dc = b \,.\, dc = cd \,.\, b$, so that $(a, b) \in q_G$ and $p_G \subseteq q_G$. Similarly, $q_G \subseteq p_G$.

6.5. Lemma. *Let $G \in V_8$ be such that $G = GG$. Define a binary relation r on G by $(a, b) \in r$ iff $ax = xb$ for all $x \in G$. Then $r \cup p_G$ is a congruence of G.*

Proof. If $(a, b) \in r$ then $b \,.\, xy = yx \,.\, b = a \,.\, yx = xy \,.\, a$ for all $x, y \in G$, so that $(b, a) \in r$. If $(a, b) \in r$ and $(b, c) \in r$ then $a \,.\, xy = xy \,.\, b = b \,.\, yx = yx \,.\, c = c \,.\, xy$ for all $x, y \in G$, so that $(a, c) \in p_G$. If $(a, b) \in r$ and $(b, c) \in p_G$ then $a \,.\, xy = xy \,.\, b = b \,.\, yx = c \,.\, yx = xy \,.\, c$ for all $x, y \in G$, so that $(a, c) \in r$. If $(a, b) \in p_G$ and $(b, c) \in r$ then $(a, c) \in r$ follows similarly. If $(a, b) \in r$ and $c \in G$ then $ca \,.\, x =$

$= x \,.\, ac = x \,.\, cb$ and $ac \,.\, x = cb \,.\, x = x \,.\, bc$ for all $x \in G$, so that $(ca, cb) \in r$ and $(ac, bc) \in r$. The rest follows from 6.4.

6.6. Proposition. *Every simple groupoid from V_8 is commutative.*

Proof. Let $G \in V_8$ be simple. If $G \neq GG$ then G is a semigroup with zero multiplication, so that G is commutative. Let $G = GG$. By 6.4 we have $p_G = \mathrm{id}_G$; by 6.5, $r \cup \mathrm{id}_G$ is a congruence of G. Since $(ab, ba) \in r$ for all $a, b \in G$, the factor $G/(r \cup \mathrm{id}_G)$ is commutative. It follows that if $r \cup \mathrm{id}_G = \mathrm{id}_G$ then G is commutative. If $r \cup \mathrm{id}_G = G \times G$ then it is clear that G is commutative.

6.7. Example. Consider the groupoid K with the underlying set $\{0, 1, 2, 3\}$ and multiplication defined by $xy = 0$ for all x, y except for $1 \cdot 2 = 3$. This groupoid K belongs to V_8. The relation r introduced in 6.5 consists of the pairs $(0, 0)$, $(0, 1)$, $(0, 3)$, $(2, 0)$, $(2, 1)$, $(2, 3)$ and is not symmetric. We have $p_K \neq q_K$, as p_K is the congruence identifying 0, 2, 3, while q_K is the congruence identifying 0, 1, 3.

We leave as an open problem to find a nice description of free V_8-groupoids.

7. THE VARIETY V_9

V_9 is the variety determined by $x \,.\, yz = yz \,.\, x$.

7.1. Proposition. Let X be a nonempty set. Denote by $(F, +)$ the free commutative groupoid over the disjoint union $X \cup (X \times X)$. Put $G = F \setminus \{x + y;\ x, y \in X\}$ and, for $a, b \in G$, define an element $ab \in G$ as follows: if $a, b \in X$ then $ab = (a, b)$; in all other cases put $ab = a + b$. Then G is a free V_9-groupoid over X. Consequently, free V_9-groupoids are cancellative.

Proof. It is easy.

7.2. Proposition. If $G \in V_9$ then GG is a commutative groupoid. If $G \in V_9$ is a left division groupoid (or a right division groupoid) then G is commutative. Every simple groupoid from V_9 is commutative.

Proof. It is obvious.

8. THE VARIETY V_{10}

V_{10} is the variety determined by $x \,.\, yz = zx \,.\, y$.

8.1. Proposition. *Every groupoid from V_{10} is medial and satisfies the identity $xy \,.\, uv = uv \,.\, xy$.*

Proof. $xy \,.\, uv = (v \,.\, xy)\, u = (yv \,.\, x)\, u = xu \,.\, yv$ and $xy \,.\, uv = xu \,.\, yv = u(yv \,.\, x) = u(v \,.\, xy) = (xy \,.\, u)\, v = (y \,.\, ux)\, v = ux \,.\, vy = uv \,.\, xy$.

8.2. Proposition. *Let $G \in V_{10}$. If $G = GG$ then G is commutative. Consequently, if G is either a left or a right division groupoid then G is an abelian group.*

Proof. It follows from 8.1.

8.3. Proposition. *Let $G \in V_{10}$. If G is either left or right cancellative then G is a commutative semigroup.*

Proof. A groupoid $G \in V_{10}$ satisfies $xy \,.\, uu = xu \,.\, yu = yu \,.\, xu = yx \,.\, uu$, so that it is commutative if it is right cancellative.

8.4. Proposition. *Free V_{10}-groupoids with at least two free generators are neither left nor right cancellative.*

Proof. It follows from 8.3.

8.5. Proposition. *Every one-generated groupoid from V_{10} is a commutative semigroup.*

Proof. For a variable x put $x^1 = x$, $x^2 = xx$, $x^3 = x \,.\, xx$, $x^4 = x(x \,.\, xx)$, etc. It is easy to prove by induction on the length of a term t that if t contains a single variable x then the identity $t = x^n$, where n is the length of t, is satisfied in V_{10}.

8.6. Proposition. *Every simple groupoid from V_{10} is a commutative semigroup.*

Proof. It follows from 8.1.

9. SURVEY OF SOME PROPERTIES

	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}
FLC	+	+	+	+	−	−	−	−	−	+	−
FRC	+	+	+	+	−	+	−	−	−	+	−
DQ	−	−	+	−	−	−	+	+	−	−	+
DHQ	+	+	+	+	?	−	+	+	+	+	+
ID	+	+	−	−	−	?	−	−	−	−	−
CIQ	+	+	−	−	+	+	+	+	+	−	+
IS	+	+	+	+	−	+	?	?	?	−	−
FIQ	+	+	+	+	−	−	−	−	−	−	−
SIM	e	e	e	e	f	p	f	f	p	p	f
SBV	c	c	c	?	c	c	?	?	c	c	?
MIN	c	c	d	?	d	c	d	d	c	c	d

In this table V_i are the varieties defined above and the abbreviations in the left column stand for the following properties of varieties:

FLC ... free groupoids in the variety are left cancellative;
FRC ... free groupoids are right cancellative;
DQ ... every divisible groupoid in the variety is a quasigroup;
DHQ ... every divisible groupoid is a homomorphic image of a quasigroup in the variety;
ID ... every groupoid can be imbedded into a divisible groupoid in the variety;
CIQ ... every cancellative groupoid can be imbedded into a quasigroup in the variety;
IS ... every groupoid can be imbedded into a groupoid G with $G = GG$;
FIQ ... free groupoids can be imbedded into quasigroups in the variety.

In the line starting with SIM we give an information for each V_i about the class of simple groupoids in V_i; here e stands for "enough simple groupoids" (every groupoid from V_i can be imbedded into a simple groupoid from V_i), p stands for the fact that V_i contains a proper class of pairwise nonisomorphic simple groupoids but there are not enough simple groupoids; and f stands for the fact that all simple groupoids in V_i are finite and there are infinitely many nonisomorphic simple groupoids in V_i. In the lines starting with SBV and MIN we give the number of subvarieties and minimal subvarieties of V_i, respectively; here d stands for $\aleph_0$ and c stands for $2^{\aleph_0}$.

Many interesting properties of varieties, e.g. the properties of the amalgamation type, of the word problem type, the Schreier property, etc. are not included. Certainly, some of them are easy to check, while some can cause certain difficulties.

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