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LATTICES OF ORTHOGONAL THEORIES

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INTRODUCTION

In this paper we continue the study of associative rings by means of lattices of orthogonal theories. The general concept of an orthogonal theory was introduced in [11]. There are at least two important groups of examples of this concept. First, the torsion theories (= orthogonal theories of the Hom bifunctor), introduced by Dickson ([4]) and now widely used in the theory of rings and modules (see [2], [13] etc.). Secondly, the cotorsion theories (= orthogonal theories of the Ext bifunctor), introduced by Salce ([14]) and now applied to the study of extensions in various module categories (see e.g. [15]). Particular results relating the structure of an associative ring to the structure of the lattice of orthogonal theories of the Hom, Ext and tensor product bifunctors were obtained in [8], [2], [15], [16], [17], [12] and [18], respectively. Recall also analogous results on lattices of hereditary torsion theories and lattices of torsions (see e.g. [13]).

The paper is divided into two sections. In the first, for a given ring R and a non-negative integer n, we study a semilattice embedding relating the lattice of orthogonal theories of Tor_R^n to that of Ext_R^n. In the second, we concentrate on cotorsion theories and continue the work of [16] and [17]. Using another approach to the Ext bifunctor we extend results of Eklof ([5], [6]) and apply them to the examination of cotorsion theories over von Neumann regular rings in the universe of constructible sets.

PRELIMINARIES

In the following, an ordinal is identified with the set of its predecessors and a cardinal is an ordinal which is not equipotent with any of its predecessors. Let c be a regular uncountable cardinal. A subset $C \subseteq c$ is a cub if (i) $X \subseteq C$ and $\sup X < c$ implies $\sup X \in C$, and (ii) $\sup C = c$. A subset $E \subseteq c$ is said to be *stationary* if $E \cap C \neq \emptyset$ for any cub C. Let M be a set of cardinality $\leqq c$. A sequence $(M_a \mid a < c)$ is a c-filtration of the set M provided (i) $M = \bigcup_{a<c} M_a$, (ii) $a < b < c$ implies $M_a \subseteq M_b$, (iii) $\operatorname{card}(M_a) < c$ for each $a < c$, and (iv) $M_b = \bigcup_{a<b} M_a$ for each limit ordinal $b < c$.

In the following, all rings are associative with unit. The ring of integers is denoted by Z. For a ring R, the category of unitary left and right R-modules is denoted by R-mod and mod-R, respectively. Homomorphisms in R-mod are written as acting on the right. A unitary left R-module is simply called a *module*. The sum and the direct sum of modules is denoted by Σ and $\dot{\Sigma}$, respectively. Let M be a module. If c is a cardinal, $c \geqq 1$, then $M^{(c)}$ and M^c denote the direct sum and the direct product, respectively, of c copies of M. Further, $I(M)$ denotes the injective hull of M. If c is an infinite cardinal, then M is c-projective if every submodule of M which is generated by less than c elements is projective. Further, gen(M) denotes the minimum of cardinalities of the generating sets of the module M. If $x \in M$, then Ann (x) denotes the left annihilator of x in R. If c is a regular uncountable cardinal, then a sequence of modules $(M_a \mid a < c)$ is a c-filtration of the module M if the conditions (i), (ii) and (iv) of the definition of the c-filtration of the set M are satisfied and, moreover, $M_0 = 0$, gen$(M_a) < c$ for each $a < c$, and M_a is a submodule of M_b for each $a < b < c$.

Let R be a ring. If n is a non-negative integer, then Ext^n_R and Tor^n_R denote the n-th derived bifunctor of the Hom_R bifunctor and of the tensor product bifunctor, respectively. Further, R is said to be *completely reducible* if the module R is semisimple. For further details and terminology, the reader is referred to [1], [3] and [6].

1. A SEMILATTICE EMBEDDING

1.1. Definition. Let F be a bifunctor from R-mod (or mod-R) to Z-mod. For a non-empty class A, let $A^\perp = \{Y \mid F(X, Y) = 0 \;\forall X \in A\}$ and ${}^\perp A = \{X \mid F(X, Y) = 0 \;\forall Y \in A\}$ be the orthogonal complements of A with respect to F. A pair of non-empty classes (A, B) is an *orthogonal theory* if $A = {}^\perp B$ and $B = A^\perp$.

For a non-empty class A put Gen $(A) = ({}^\perp(A^\perp), A^\perp)$ and Cog $(A) = ({}^\perp A, ({}^\perp A)^\perp)$. Then Gen (A) and Cog (A) are orthogonal theories of F. Denote by OT the class of all orthogonal theories of F. Analogously to the particular case of torsion theories (F = Hom), we define a partial order $\leqq$ on OT as follows. If $(A, B), (C, D) \in OT$, then $(A, B) \leqq (C, D)$ iff $A \subseteq C$ (iff $D \subseteq B$). It is easy to see that $\leqq$ is a lattice order and $L = (OT, \inf, \sup, 0, 1)$ is a complete lattice, where $0 = \text{Gen}(\{0_R\})$, $1 = \text{Cog}(\{0_R\})$ and for a subclass $C \subseteq OT$, $C = \{(A_a, B_a) \mid a \in I\}$ we have

$$\sup C = \text{Gen}\Big(\bigcup_{a\in I} A_a\Big) = \text{Cog}\Big(\bigcap_{a\in I} B_a\Big) = \Big({}^\perp\Big(\bigcap_{a\in I} B_a\Big), \bigcap_{a\in I} B_a\Big) \tag{1}$$

and

$$\inf C = \text{Cog}\Big(\bigcup_{a\in I} B_a\Big) = \text{Gen}\Big(\bigcap_{a\in I} A_a\Big) = \Big(\bigcap_{a\in I} A_a, \Big(\bigcap_{a\in I} A_a\Big)^\perp\Big). \tag{2}$$

Note that our terminology is not standard since OT is a class and not a set. But in most cases OT can be represented by a set and hence L is in fact a lattice in the usual sense. In some cases (e.g. for $R = Z$ and $F = \text{Hom}_Z$) OT has no small cardinality (see e.g. [10]) and L is a lattice in the "conditional" sense, i.e. the assertions concerning L are understood as the appropriate assertions concerning the elements of L. In the following, $E_{n,R}$ and $T_{n,R}$ denote the class of all orthogonal theories of the Ext_R^n and Tor_R^n bifunctors, respectively. The operator $\perp$ for $F = \mathrm{Ext}_R^n$ and $F = \mathrm{Tor}_R^n$ is denoted by $\perp_n$ and $+_n$, respectively. For $F = \mathrm{Ext}_R^n$ we denote Gen and Cog by Gen_n and Cog_n, respectively.

1.2. Lemma. *Let R and S be Morita equivalent rings. Let F_1 and F_2 be additive functors realizing the equivalence of R-mod to S-mod and mod-R to mod-S, respectively. For $\emptyset \neq A \subseteq R\text{-mod}$, $\emptyset \neq B \subseteq \text{mod-}R$ put $F_1(A) = \{F_1(X) \mid X \in A\}$ and $F_2(B) = \{F_2(X) \mid X \in B\}$. Then, for each non-negative integer n, the mappings $e_n\colon E_{n,R} \to E_{n,S}$ and $t_n\colon T_{n,R} \to T_{n,S}$ defined by $e_n((A, B)) = (F_1(A), F_1(B))$ and $t_n((A, B)) = (F_2(A), F_1(B))$, respectively, are lattice isomorphisms.*

Proof. Using the definition of Ext_R^n via injective resolutions and [1, Propositions 21.2 and 21.6] we see that $\mathrm{Ext}_R^n(M, N) \simeq \mathrm{Ext}_S^n(F_1(M), F_1(N))$ for all $M, N \in R\text{-mod}$. Hence, the assertion concerning e_n holds. Let U be an injective cogenerator in Z-mod. By [3, Proposition VI.5.1] we have $\mathrm{Tor}_R^n(M, N) = 0$ iff $\mathrm{Ext}_R^n(N, \mathrm{Hom}_Z(M, U)) = 0$ iff $\mathrm{Ext}_R^n(F_1(N), F_1(\mathrm{Hom}_Z(M, U))) = 0$. By [1, Proposition 20.6, Theorem 22.1 and Corollary 22.3] we see that this is equivalent to $\mathrm{Ext}_R^n(F_1(N), \mathrm{Hom}_Z(F_2(M), U)) = 0$ and hence to $\mathrm{Tor}_R^n(F_2(M), F_1(N)) = 0$, and the rest is clear.

1.3. Theorem. *Let R be a ring and n a non-negative integer. Then the complete upper semilattice $(T_{n,R}, \sup_{\mathrm{Tor}_R^n})$ is isomorphic to a subsemilattice in the complete lower semilattice $(E_{n,R}, \inf_{\mathrm{Ext}_R^n})$. The complete semilattice embedding is given by the mapping $f_n\colon T_{n,R} \to E_{n,R}$, $(A, B) f_n = (B, B^{\perp_n})$. In particular $0_{\mathrm{Tor}_R^n} f_n = 1_{\mathrm{Ext}_R^n}$*

Proof. Let $(A, B) \in T_{n,R}$ and let U be an injective cogenerator in Z-mod. Denote by H the class of all modules of the form $\mathrm{Hom}_Z(X, U)$, $X \in A$. Then $B \subseteq {}^{\perp_n}H$, as by [3, Proposition VI.5.1], $\mathrm{Ext}_R^n(Y, \mathrm{Hom}_Z(X, U)) \simeq \mathrm{Hom}_Z(\mathrm{Tor}_R^n(X, Y), U) = 0$ for each $Y \in B$. Moreover, for $W \in {}^{\perp_n}H$ we have $\mathrm{Hom}_Z(\mathrm{Tor}_R^n(X, W), U^c) = 0$ for any $X \in A$ and any cardinal c, whence $B = {}^{\perp_n}H$. Thus $(B, B^{\perp_n}) = \mathrm{Cog}_n(H)$. Now, using (1) and (2), we easily see that f_n maps suprema in $L_{\mathrm{Tor}_R^n}$ to infima in $L_{\mathrm{Ext}_R^n}$ and the rest is clear.

1.4. Remark. If f_n is onto, then clearly f_n is a lattice antiisomorphism. In general, f_n need not be onto (see 1.5 or 1.8). Moreover, f_n need not be a lattice antihomomorphism, since it need not map infima in $L_{\mathrm{Tor}_R^n}$ to suprema in $L_{\mathrm{Ext}_R^n}$ — see 1.8 (iii). Note that special cases of 1.3 were applied in [15, Proposition 2.2] and [16, Proposition I.1].

1.5. Proposition. *Let R be a ring and n a non-negative integer. Then $1_{\mathrm{Tor}_R^n} f_n = 0_{\mathrm{Ext}_R^n}$ iff for each module M,* w.gl. dim $(M) \leqq n - 1$ *implies* gl. dim $(M) \leqq n - 1$. *In particular: always $1_{\otimes_R} f_0 = 0_{\mathrm{Hom}_R}$, and $1_{\mathrm{Tor}_R} f_1 = 0_{\mathrm{Ext}_R}$ iff R is left perfect.*

Proof. Easy.

The following proposition gives a criterion for elements of $E_{n,R}$ to belong to $\operatorname{Im} f_n$.

1.6. Proposition. *Let R be a ring, n a non-negative integer, U an injective cogenerator in Z-mod and $(A, B) \in E_{n,R}$. Then $(A, B) \in \operatorname{Im} f_n$ iff ${}^{\perp_n}\{X \in B \mid \exists Y \in \text{mod-}R: X = \operatorname{Hom}_Z(Y, U)\} \subseteq A$.*

Proof. By 1.3, $(A, B) \in \operatorname{Im} f_n$ iff $({}^{+_n}A)^{+_n} = A$. By [3, Proposition VI.5.1] it is easy to see that $W \in ({}^{+_n}A)^{+_n}$ iff $\operatorname{Tor}_R^n(Y, W) = 0$ for each $Y \in \text{mod-}R$ such that $\operatorname{Hom}_Z(Y, U) \in B$ iff $W \in {}^{\perp_n}\{X \in B \mid \exists Y \in \text{mod-}R: X = \operatorname{Hom}_Z(Y, U)\}$, q.e.d.

1.7. Proposition. *Let R be a ring and $\emptyset \neq D \subseteq R\text{-mod}$. Assume there are a ring S and an injective cogenerator $U \in S\text{-mod}$ such that $D \subseteq \{X \in R\text{-mod} \mid \exists Y \in S\text{-mod-}R: X = \operatorname{Hom}_S(Y, U)\}$. Then $\operatorname{Cog}_n(D) \in \operatorname{Im} f_n$ for each non-negative integer n.*

Proof. Analogous to the proof of 1.6.

Denote by F, C, S, N and I the class of all finitely generated, countably generated, singular, non-singular and injective modules, respectively. The following theorem provides us with some illustrative examples.

1.8. Theorem. (i) *Let R be a commutative QF-ring. Then $\operatorname{Cog}_n(A) \in \operatorname{Im} f_n$ for each non-negative integer n and each $\emptyset \neq A \subseteq F$.*

(ii) *Assume that either R is a completely reducible ring or R is a full matrix ring over a commutative local artinian P.I.R. or R is Morita equivalent to an upper triangular matrix ring of degree two over a division ring* (see [15, Remark 6.2]). *Then for any non-negative integer n, f_n is a lattice antiisomorphism of $L_{\operatorname{Tor}_R^n}$ onto $L_{\operatorname{Ext}_R^n}$.*

(iii) *Let R be a P.I.D. which is not a field. Then, for $\emptyset \neq A \subseteq F$, $\operatorname{Cog}_0(A) \in \operatorname{Im} f_0$ iff $A \nsubseteq F \cap N$. If R is not local, then there are $(B, C), (D, E) \in \operatorname{Im} f_0$ such that $\sup_{\operatorname{Hom}_R}\{(B, C), (D, E)\} \notin \operatorname{Im} f_0$, i.e. f_0 does not map infima in $L_{\otimes_R}$ to suprema in L_{Hom_R}. If R is a complete local ring, then $\operatorname{Cog}_1(A) \in \operatorname{Im} f_1$ for each $\emptyset \neq A \subseteq F$. If R is not complete and local then, for $\emptyset \neq A \subseteq F$, $\operatorname{Cog}_1(A) \in \operatorname{Im} f_1$ iff $A \subseteq F \cap S$. Anyway, $\operatorname{Im} f_n = E_{n,R} = 0$ provided $n \geqq 2$.*

(iv) *Let R be a simple countable non-completely reducible von Neumann regular ring* (see [9]). *Then, for each $A \subseteq C$ with $A \neq \emptyset$, $\{0\}$, we have $\operatorname{Cog}_0(A) \notin \operatorname{Im} f_0$ and $\operatorname{Cog}_1(A) \notin \operatorname{Im} f_1$. For $n \geqq 2$ we have $\operatorname{Im} f_n = E_{n,R} = 0$.*

Proof. (i) By 1.7, for $S = U = R$ and $D = A$.

(ii) The assertion concerning completely reducible rings is easy. If R is a commutative local artinian P.I.R., then every module is a direct sum of cyclic modules and (i) and 1.2 apply. If R is an upper triangular matrix ring of degree two over a division ring, then R is a perfect hereditary T-ring (see [15, Proposition 5.2], and hence the assertion holds for $n \geqq 1$. Moreover, every module is isomorphic to a direct sum of direct powers of modules X, Y and $I(Y)$, where X, Y are representatives of the class of all simple modules, X is injective and Y is projective (see [15, Proposition 5.1]). It is easy to see that $E_{0,R} = \operatorname{Im} f_0 = \{0_{\mathrm{Hom}_R}, \mathrm{Cog}_0(X), \mathrm{Cog}_0(Y), \mathrm{Cog}_0(I(Y)), 1_{\mathrm{Hom}_R}\}$, and 1.2 applies.

(iii) Denote by K the quotient field of R. If $\emptyset \neq A \subseteq F \cap S$, then, using 1.7 for $S = R$, $U = K/R$ and $D = A$, we get $\mathrm{Cog}_n(A) \in \operatorname{Im} f_n$ for each $n \geqq 0$. If $A \nsubseteq F \cap N$, then ${}^{\perp 0}A = {}^{\perp 0}(A \cap S)$, as each left ideal is isomorphic to R. Hence $\mathrm{Cog}_0(A) \in \operatorname{Im} f_0$. If $\emptyset \neq A \subseteq F \cap N$, then ${}^{\perp 0}A = {}^{\perp 0}\{R\} \supseteq S \cup I$. Thus ${}^{+0}({}^{\perp 0}\{R\}) \subseteq S \cap I$ and it follows easily that ${}^{+0}({}^{\perp 0}\{R\}) = \{0\}$. Thus, by 1.3, $\mathrm{Cog}_0(A) \notin \operatorname{Im} f_0$. Assume R is not local and J_1, J_2 are simple modules such that $J_1 \not\simeq J_2$. For $i = 1, 2$ put $H_i = \mathrm{Hom}_R(I(J_i), K/R)$. Then, for $i = 1, 2$, we have $\mathrm{Cog}_0(H_i) \in \operatorname{Im} f_0$, as $X \in {}^{\perp 0}\{H_i\}$ iff $X \in \{I(J_i)\}^{+0}$. Further, $R \subseteq H_i$ for $i = 1, 2$, whence $({}^{\perp 0}\{R\})^{\perp 0} \subseteq ({}^{\perp 0}\{H_1\})^{\perp 0} \cap ({}^{\perp 0}\{H_2\})^{\perp 0}$ and $\sup_{\mathrm{Hom}_R, i=1,2}\{\mathrm{Cog}_0(H_i)\} \leqq \mathrm{Cog}_0(R)$. On the other hand, for $i = 1, 2$, using the results of [7, Ch. X], we get ${}^{+0}(\{I(J_i)\}^{+0}) = \{X \in \text{mod-}R \mid \exists c\colon X \simeq I(J_i)^{(c)}\}$, whence $\inf_{\otimes_R, i=1,2}\{\mathrm{Gen}(I(J_i))\} = (\{0\}, R\text{-mod})$. Hence f_0 does not map infima in $L_{\otimes_R}$ to suprema in L_{Hom_R}. If R is a complete local ring and $\emptyset \neq A \subseteq F$, then by [6, Theorem 9.1] $\mathrm{Ext}_R(K, R) = 0$ and $\mathrm{Cog}_1(A) = \mathrm{Cog}_1(A \cap S) \in \operatorname{Im} f_1$, as ${}^{\perp 1}\{R\} = N$. If R is not complete and local and $A \nsubseteq F \cap S$, then $\mathrm{Ext}_R(K, R) \neq 0$ and $N \nsubseteq {}^{\perp 1}A$, whence $\mathrm{Cog}_1(A) \notin \operatorname{Im} f_1$ by 1.3, and the rest is clear.

(iv) We have ${}^{\perp 0}A \supseteq I$, whence ${}^{+0}({}^{\perp 0}A) = 0$ and $\mathrm{Cog}_0(A) \notin \operatorname{Im} f_0$. By the regularity of R we get $T_{1,R} = \{(\text{mod-}R, R\text{-mod})\}$. Hence, if $\emptyset \neq B \subseteq R\text{-mod}$ and $\mathrm{Cog}_1(B) \in \operatorname{Im} f_1$, then by 1.3 $B \subseteq I$ and $B \cap C = \{0\}$. The final assertion follows from the fact that R is left hereditary.

2. A DIAMOND PRINCIPLE AND COTORSION THEORIES

There are at least three different approaches to extensions of groups and modules. Namely, extensions may be regarded either as systems of factors or as short exact sequences or as certain homomorphisms (see e.g. [7, Ch. IX]). The approach via systems of factors is used more in the general group theory. Nevertheless, it proved to be useful in the study of vanishing of Ext in abelian groups under the assumption of the weak diamond (see [6, Ch. 3 and Ch. 4]). On the other hand, using the definition of Ext via short exact sequences, and assuming the diamond, Eklof proved a theorem on the vanishing of Ext for modules over countable left hereditary rings ([5, Theorem 1.5]). Here, in 2.2, we obtain an analogous result even for rings of cardinality $\leqq \aleph_1$, assuming a diamond principle close to the weak diamond. This is possible since we start with the homological definition of Ext. It enables us to use another proof, replacing the cardinality assumption on the ring by a cardinality assumption on an injective hull of a module. Finally, we apply our result to the study of cotorsion theories over von Neumann regular rings.

Throughout this section, c denotes a regular uncountable cardinal and E a stationary subset of c. As usual, we denote by $\diamondsuit_c(E)$ and $\Phi_c(E)$ the diamond and the weak diamond, respectively (see [6]). We shall need the following diamond principle

$D_c(E)$, in which partitions from the weak diamond are replaced by partitions into a varying finite number of parts: $D_c(E)$: "Let A be a set of cardinality c and $(A_a \mid a < c)$ a c-filtration of the set A. Assume that for each $a \in E$ there are a natural number $p_a \geqq 2$ and a partition P_a of $\operatorname{Exp}(A_a)$ into p_a parts, i.e. P_a: $\operatorname{Exp}(A_a) \to p_a$. Then there exists a function $\varphi: E \to \aleph_0$ such that $\varphi(a) \in p_a$ for each $a \in E$ and the set $S_X = \{a \in E \mid P_a(X \cap A_a) = \varphi(a)\}$ is stationary in c for each $X \subseteq A$". Clearly, $\diamond_c(E)$ implies $D_c(E)$, and $D_c(E)$ implies $\Phi_c(E)$. Hence, under the assumption of the axiom of constructibility ($V = L$), $D_c(E)$ holds for any regular uncountable cardinal c and any stationary set $E \subseteq c$.

2.1. Lemma. *Assume $D_c(E)$. Let R be a ring such that* $\operatorname{card}(R) \leqq c$. *Let M be a c-projective module with* $\operatorname{gen}(M) = c$ *and N a module with* $\operatorname{gen}(I(N)) \leqq c$. *Assume there is a c-filtration $(C_a \mid a < c)$ of the module M such that* $E = \{a < c \mid \operatorname{Ext}_R(C_{a+1}/C_a, N) \neq 0\}$. *Then* $\operatorname{Ext}_R(M, N) \neq 0$.

Proof. Clearly, there is a c-filtration $(A_a \mid a < c)$ of the set c, and for each $a < c$ an element $m_a \in M$, such that $C_a = \sum_{b \in A_a} Rm_b$. Let $(B_a \mid a < c)$ be a c-filtration of the Z-module $I(N)$. Denote by ϱ the embedding of N into $I(N)$, by π the canonical projection of $I(N)$ onto $I(N)/N\varrho$ and, for $a < c$, by v_a the inclusion of C_a into C_{a+1}. For $a \in E$ put $X_a = \operatorname{Hom}_R(C_a, N)$ and $Y_a = v_a \operatorname{Hom}_R(C_{a+1}, N)$, and take a fixed $f_a \in X_a - Y_a$. Denote by r_a the order of $f_a + Y_a$ in the group $\operatorname{Ext}_R(C_{a+1}/C_a, N) = X_a/Y_a$. If $r_a = \infty$, put $p_a = 2$. Otherwise, put $p_a = r_a$. Define an equivalence relation $\sim$ on the set of all mappings from A_a to B_a as follows: $u \sim v$ iff there are an integer n and $y \in Y_a$ such that $v = u + n(f_a \mid A_a) + y \mid A_a$. Note that if $r_a = \infty$ then n is unique, and if $r_a < \infty$, then n is unique modulo p_a. Let P_a: $\operatorname{Exp}(A_a \times B_a) \to p_a$ be any partition such that, for all u, v with $u \sim v$, the following two conditions are satisfied: (1) if $r_a = \infty$ then $P_a(u) = P_a(v)$, (2) if $r_a < \infty$ then $P_a(u) = P_a(v)$ iff n is divisble by p_a. Let $\varphi: E \to \aleph_0$ be the function corresponding by $D_c(E)$ to $P_a, p_a, a < c$. To prove $\operatorname{Ext}_R(M, N) \neq 0$, we have to construct $g \in \operatorname{Hom}_R(M, I(N)/N\varrho) - \operatorname{Hom}_R(M, I(N))\,\pi$. By induction on $a < c$ we construct a sequence of homomorphisms g_a: $C_a \to I(N)/N\varrho$ such that $g_{a+1} \mid C_a = g_a$ for each $a < c$ and $g_a = \bigcup_{b<a} g_b$ for each limit ordinal $a < c$. Put $g_0 = 0$. Assume g_a is defined for $a < c$. We distinguish the following two cases: (I) $a \in E$ and there is $f \in \operatorname{Hom}_R(C_{a+1}, I(N))$ such that $f \mid A_a \subseteq A_a \times B_a$ and $P_a(v_a f) = \varphi(a)$ and $g_a = v_a f\pi$, (II) = not (I). In the case (II), the projectivity of C_a yields the existence of $h_a \in \operatorname{Hom}_R(C_a, I(N))$ such that $g_a = h_a\pi$. The injectivity of $I(N)$ yields the existence of $h_{a+1} \in \operatorname{Hom}_R(C_{a+1}, I(N))$ such that $v_a h_{a+1} = h_a$. Put $g_{a+1} = h_{a+1}\pi$. Then clearly $v_a g_{a+1} = g_a$. In the case (I), take a fixed f satisfying the conditions of (I) and use the injectivity of $I(N)$ to get $h_a \in \operatorname{Hom}_R(C_{a+1}, I(N))$ such that $v_a h_a = v_a f - f_a$. Put $g_{a+1} = h_a\pi$. Then $v_a g_{a+1} = v_a f\pi - f_a\pi = g_a$. Finally, put $g = \bigcup_{a<c} g_a$. Clearly, $g \in \operatorname{Hom}_R(M, I(N)/N\varrho)$. Assume the existence of $h' \in \operatorname{Hom}_R(M, I(N))$ such that $g = h'\pi$. Clearly, $\{a < c \mid h' \mid A_a \subseteq A_a \times B_a\}$ is a cub in c. Hence there is $a \in E$

such that $v_a h \subseteq A_a \times B_a$, $g_a = v_a h \pi$ and $P_a(v_a h) = \varphi(a)$ where $h = h' \mid C_{a+1}$. Hence the case (I) occurs and $(h - h_a)\pi = 0$ whence $y_a = v_a(h - h_a) \in Y_a$. Finally, $v_a f = f_a + v_a h - y_a$ and thus $P_a(v_a f) = P_a(v_a h + f_a - y_a) \neq P_a(v_a h)$, a contradiction.

2.2. Lemma. *Assume $D_c(E)$ for all stationary $E \subseteq c$. Let R be a left hereditary ring such that* $\operatorname{card}(R) \leqq c$. *Let M be a c-projective module with* $\operatorname{gen}(M) = c$ *and N a module with* $\operatorname{gen}(I(N)) \leqq c$. *Then* $\operatorname{Ext}_R(M, N) = 0$ *iff the module M has a c-filtration $(C_a \mid a < c)$ such that* $\operatorname{Ext}_R(C_{a+1}/C_a, N) = 0$ *for all $a < c$.*

Proof. The same as for [5, Theorem 1.5], only the use of [5, Lemma 1.4] is replaced by the use of 2.1.

2.3. Theorem. *Assume $V = L$. Let R be a von Neumann regular ring such that each left ideal is countably generated and* $\operatorname{card}(R) \leqq \aleph_1$. *Let N be a module such that* $\operatorname{gen}(N) \leqq \aleph_1$ *and* $\operatorname{Ext}_R(M, N) \neq 0$ *for all finitely generated non-projective modules M. Then* $\operatorname{Ext}_R(M, N) \neq 0$ *for all non-projective modules M.*

Proof. By [16, Lemma III.3] we have $\operatorname{Ext}_R(M, N) \neq 0$ for all non-projective modules M with $\operatorname{gen}(M) \leqq \aleph_0$. Further, by [16, Lemma II.2] the module $P = N^{\aleph_0}/N^{(\aleph_0)}$ is injective. Clearly, the homomorphism $g \in \operatorname{Hom}_R(N, P)$ defined by $xg = (x \mid i < \aleph_0) + N^{(\aleph_0)}$, $x \in N$, is injective. Since $2^{\aleph_0} = \aleph_1$, $\operatorname{card}(P) \leqq \aleph_1$ and $\operatorname{gen}(I(N)) \leqq \aleph_1$. Now, the assertion follows by induction from 2.2 and [19, Theorem 5].

2.4. Corollary. *Assume $V = L$. Let R be as in 2.3. Let (A, B) be a cotorsion theory cogenerated by a class of $\aleph_1$-generated modules. Then A contains a non-projective module iff A contains a non-projective finitely generated module.*

2.5. Lemma. *Let R be a ring. Let X, Y, P and N be modules such that P is projective and X, Y are R-independent submodules of P, and* $\operatorname{Ext}_R(P/(X \dotplus Y), N) = 0$. *Then* $\operatorname{Ext}_R(P/X, N) = 0$.

Proof. Easy.

Let $0 < n < \aleph_0$. For $0 \leqq j < n$ denote by π_j the j-th canonical projection of $R^{(n)}$ to R. The following theorem shows that the premise of 2.3 concerning the module N can be tested in much weaker form.

2.6. Theorem. *Let R be a left hereditary von Neumann regular ring. Let N be a module with* $\operatorname{Ext}_R(M, N) \neq 0$ *for all modules M such that there are a positive integer n, orthogonal idempotents e_i, $i < \aleph_0$ in R and elements $x_i \in R^{(n)}$, $i < \aleph_0$, such that (1) $x_i \pi_0 = e_i$ for each $i < \aleph_0$, (2) the elements $x_i \pi_j$, $i < \aleph_0$ are R-independent for each $j < n$, (3)* $\operatorname{Ann}(x_i \pi_j) = R(1 - e_i)$ *for each $i < \aleph_0$ and $j < n$ and (4) $M \simeq R^{(n)}/\dot{\sum}_{i<\aleph_0} Rx_i$. Then* $\operatorname{Ext}_R(M, N) \neq 0$ *for all countably generated non-projective modules M.*

Proof. Let M be a finitely generated non-projective module. Using induction on gen (M) we prove that $\operatorname{Ext}_R(M, N) \neq 0$. For gen $(M) = 1$, this follows immediately from the premises (1) and (4), and from [9, Proposition 2.14]. Let gen $(M) = n > 1$ and $M = R^{(n)}/I$. By [1, Proposition 26.2] and [9, Proposition 2.11], there are a cardinal $c \geqq \aleph_0$ and elements $x_i \in R^{(n)}$, $i < c$, such that $I = \dot{\sum\limits_{i<c}} Rx_i$. By 2.5 we may assume that $c = \aleph_0$. Assume there are infinitely many indices $i < \aleph_0$ such that there exist $j_1, j_2 < n$ with $\operatorname{Ann}(x_i\pi_{j_1}) \neq \operatorname{Ann}(x_i\pi_{j_2})$. Then there is a set $\emptyset \neq S \subsetneqq \{0, \ldots, n-1\}$ such that I has a direct summand $I' \subseteq \dot{\sum\limits_{j\in S}} R^{(n)}\pi_j$ such that gen $(I') = \aleph_0$ and the assertion follows from the inductive premise and 2.5. Hence we may assume that for each $i < \aleph_0$ there is a non-zero idempotent $e_i \in R$ such that $\operatorname{Ann}(x_i\pi_j) = R(1 - e_i)$ for all $j < n$. Moreover, by [9, Proposition 2.14] we may assume that $x_i\pi_0 = e_i$ for all $i < \aleph_0$ and that e_i, $i < \aleph_0$ are orthogonal idempotents. Further, assume there is an index $j < n$ such that no infinite subset of $\{x_i\pi_j \mid i < \aleph_0\}$ is independent. Then there exist finite sets A_i, $i < \aleph_0$ such that card $(A_i) \geqq 2$ for each $i < \aleph_0$; $A_{i_1} \cap A_{i_2} = \emptyset$, $\bigcup\limits_{i<\aleph_0} A_i = \aleph_0$ for each $i_1, i_2 < \aleph_0$, $i_1 \neq i_2$; and for each $i < \aleph_0$ and each $a \in A_i$ there is $r_a \in R$ with $r_a x_a \pi_j \neq 0$ and $\sum\limits_{a\in A_i} r_a x_a \pi_j = 0$. Put $I' = \dot{\sum\limits_{i<\aleph_0}} R(\sum\limits_{b\in A_i} r_b x_b)$. Then gen $(I') = \aleph_0$ and I' is a direct summand of I, and the assertion follows from the inductive premise and 2.5. Thus, we may also assume (2). Finally, [16, Lemma III.3] applies.

2.7. Remark. If R is a simple von Neumann regular ring with card $(R) = \aleph_0$, then the premises of 2.6 are satisfied for any module N with gen $(N) \leqq \aleph_0$ (see [16, Theorem III.4]). Hence 2.3 applies: there are no non-trivial cotorsion theories cogenerated by countably generated modules provided $V = L$ (see [16, Theorem III.6]). Moreover, if R is not completely reducible, then the previous assertion is independent of ZFC + GCH (see [17, Theorem 2.4]). Nevertheless, in this case there is a module N for which the premises of 2.6 fail (see [17, Theorem 1.5]). As for rings of cardinality $\aleph_1$, 2.3 and 2.6 indicate the role of finitely generated modules for tests of the vanishing of Ext. Nevertheless, the validity of results analogous to those for card $(R) = \aleph_0$ remains an open problem.

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