Czechoslovak Mathematical Journal

Božena Mihalíková

Asymptotic properties of solutions of differential systems with deviating argument

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 4, 559–577

Persistent URL: http://dml.cz/dmlcz/102329

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENT

Božena Mihalíková, Košice

(Received March 3, 1986)

We consider systems of differential equations with a deviating argument of the form

$$\text{(A)}\qquad \begin{aligned} &(p_i(t)\, x_i'(t))' = a_i(t)\, x_{i+1}(t)\,, \quad i = 1, \ldots, n-2\,,\\ &(p_{n-1}(t)\, x_{n-1}'(t))' = a_{n-1}(t)\, g(x_n(\tau_n(t)))\,,\\ &(p_n(t)\, x_n'(t))' = f(t, x_1(\tau_1(t)))\,, \end{aligned}$$

where the following conditions are always assumed:

(a) $0 < p_i(t) \in C[a; \infty)$, $\int^\infty (\mathrm{d}s/p_i(s)) = \infty$, $i = 1, \ldots, n$;

(b) $0 \leqq a_i(t) \in C[a; \infty)$, $i = 1, \ldots, n-1$ and a_i's are not identically zero on any subinterval of $[a; \infty)$;

$$\int^\infty a_i(s)\, \mathrm{d}s = \infty\,, \quad i = 1, \ldots, n-1\,;$$

(c) $g(u) \in C(-\infty; \infty)$, $|g(u)| \leqq K|u|^\beta$ for $0 < \beta \leqq 1$, $0 < K = \text{const.}$;

(d) $f(t, v) \in C([a; \infty) \times (-\infty; \infty))$ and $|f(t, v)| \leqq \omega(t, |v|)$ for $(t, x) \in [a; \infty) \times \times (-\infty; \infty)$, where $\omega(t, z) \in C([a; \infty) \times [0; \infty))$ is nondecreasing in z;

(e) $\tau_i(t) \in C[a; \infty)$, $\lim_{t\to\infty} \tau_i(t) = \infty$, $i = 1, n$ and $\tau_n(t) \leqq t$, $\tau^*(t) = \max\{\tau_1(t), t\}$ for $t \geqq a$.

System (A) is called *superlinear* or *sublinear* according to whether $\omega(t, z)/z$ is nondecreasing or nonincreasing in z for $z > 0$.

The objective of this paper is to study the asymptotic behavior of solutions of system (A). We are particularly interested in obtaining an information about a growth or a decay of oscillatory solutions as well as of nonoscillatory ones. Hereafter the term "solution" will mean a solution $(x_1(t), \ldots, x_n(t))$ of (A) which exists on some ray $[T_x; \infty) \in [a; \infty)$ and satisfies

$$\sup\Big\{\sum_{i=1}^n |x_i(t)| : t \geqq T\Big\} > 0 \quad \text{for any} \quad T \geqq T_x\,.$$

Such a solution is said to be *oscillatory* (*weakly oscillatory*) if each component (at least one component) has arbitrarily large zeros. A solution is said to be *non-*

oscillatory (*weakly nonoscillatory*) if each component (at least one component) is eventually of constant sign.

Some aspects of the asymptotic behavior of solutions of two-dimensional differential systems with deviating arguments are studied in papers Kitamura and Kusano [1–3]. In the present paper we proceed further in this direction by extending the theory developed in [1] to the systems of the form (A).

Let $i_k \in \{1, 2, \ldots, 2n-1\}$, $1 \leqq k \leqq 2n-1$ and $t, s \in [a; \infty)$. We define

$$I_0 = 1 \quad I_k(t, s;\ y_{i_k}, \ldots, y_{i_1}) = \int_s^t y_{i_k}(x)\, I_{k-1}(x, s;\ y_{i_{k-1}}, \ldots, y_{i_1})\, dx\,.$$

For $a \leqq s \leqq t$, $1 \leqq i \leqq n-1$, $0 \leqq j \leqq n-i-1$ we introduce the notation

$$Q_0(t, s) = P_0^i(t, s) = 1\,, \quad P_1^n(t, s) = I_1\left(t, s; \frac{1}{p_n}\right)$$

$$P_{2j+1}^i(t, s) = I_{2j+1}\left(t, s; \frac{1}{p_i}, a_i, \frac{1}{p_{i+1}}, a_{i+1}, \ldots, a_{i+j-1}, \frac{1}{p_{i+j}}\right),$$

$$P_{2j+2}^i(t, s) = I_{2j+2}\left(t, s; \frac{1}{p_i}, a_i, \frac{1}{p_{i+1}}, a_{i+1}, \ldots, \frac{1}{p_{i+j}}, a_{i+j}\right),$$

$$P_{2(n-i)+1}^i(t, s) = I_{2(n-i)}\left(t, s; \frac{1}{p_i}, a_i, \frac{1}{p_{i+1}}, a_{i+1}, \ldots, \frac{1}{p_{n-1}}, a_{n-1}(P_1^n(\tau_n))^\beta\right),$$

$$Q_{2i-2}(t, s) = I_{2i-2}\left(t, s; \frac{1}{p_{n-1}}, a_{n-2}, \frac{1}{p_{n-2}}, \ldots, \frac{1}{p_{n-i+1}}, a_{n-i}\right),$$

$$Q_{2i-1}(t, s) = I_{2i-1}\left(t, s; \frac{1}{p_{n-1}}, a_{n-2}, \frac{1}{p_{n-2}}, \ldots, a_{n-i}, \frac{1}{p_{n-i}}\right);$$

$P_k^i(t, a) = P_k^i(t)$, $1 \leqq i \leqq n$, $1 \leqq k \leqq 2(n-i)+1$; $Q_k(t, a) = Q_k(t)$, $1 \leqq k \leqq 2n-3$; $R_k(t) = P_{2n-k}^1(t)/P_{2n-k-1}^1(t)$, $1 \leqq k \leqq 2n-1$.

Lemma 1. *Let* (a), (b) *be valid. Then*

1) $\lim_{t\to\infty} P_k^i(t) = \infty$, $i = 1, \ldots, n$, $k = 1, \ldots, 2(n-i)+1$;
$\lim_{t\to\infty} Q_k(t) = \infty$, $k = 1, \ldots, 2n-3$;

2) $\lim_{t\to\infty} (P_k^i(t)/P_j^i(t)) = 0$, $i = 1, \ldots, n$, $k < j$, $k, j = 1, \ldots, 2(n-i)+1$;

3) *for* $t_0 > a$ *there are constants* $\alpha_{kj} > 0$ [$\beta_{kj} > 0$] *such that for* $t \geqq t_0$, $1 \leqq k < j \leqq 2(n-i)+1$, $i = 1, \ldots, n$ *we have* $\alpha_{kj}P_k^i(t) < P_j^i(t)$ [*for* $1 \leqq k < j \leqq 2n-3$ *we have* $\beta_{kj}\, Q_k(t) < Q_j(t)$].

Proof of Lemma 1 may be found in [6].

Lemma 2. Let $x(t) = (x_1(t), \ldots, x_n(t))$ be solution of (A) on the interval $[a; \infty)$.

Then the following relations hold for $a \leqq s < t$:

$$(1) \qquad |x_i(t)| \leqq \sum_{j=0}^{k-i-1} |x_{j+i}(s)|\, P^i_{2j}(t) + \sum_{j=0}^{k-i-1} |p_{i+j}(s)\, x'_{j+i}(s)|\, P^i_{2j+1}(t) +$$

$$+ I_{2(k-i)}\left(t, s; \frac{1}{p_i}, a_i, \frac{1}{p_{i+1}}, a_{i+1}, \ldots, \frac{1}{p_{k-1}}, a_{k-1}|x_k|\right),$$

$$1 \leqq i \leqq n-2, \quad i+1 \leqq k \leqq n-1;$$

$$(2) \qquad |x_i(t)| \leqq \sum_{j=0}^{n-i-1} |x_{j+i}(s)|\, P^i_{2j}(t) + \sum_{j=0}^{n-i-1} |p_{i+j}(s)\, x'_{j+i}(s)|\, P^i_{2j+1}(t) +$$

$$+ KI_{2(n-i)}\left(t, s; \frac{1}{p_i}, a_i, \frac{1}{p_{i+1}}, a_{i+1}, \ldots, \frac{1}{p_{n-1}}, a_{n-1}|x_n|^\beta\right), \quad 1 \leqq i \leqq n-1.$$

Proof. Let us integrate the first $(n-1)$ equations of (A) from $s \geqq a$ to $t > s$ and transform them to

$$(3) \qquad |x_i(t)| \leqq |x_i(s)| + |p_i(s)\, x'_i(s)|\, P^i_1(t) +$$

$$+ \int_s^t \frac{1}{p_i(v)} \int_s^v a_i(u)\, |x_{i+1}(u)|\, du\, dv, \quad 1 \leqq i \leqq n-2,$$

$$(4) \qquad |x_{n-1}(t)| \leqq |x_{n-1}(s)| + |p_{n-1}(s)\, x'_{n-1}(s)|\, P^{n-1}_1(t) +$$

$$+ K \int_s^t \frac{1}{p_{n-1}(v)} \int_s^v a_{n-1}(u)\, |x_n(\tau_n(u))|\, du\, dv,$$

which are the inequalities (1) (with $1 \leqq i \leqq n-2$, $k = i+1$) and (2) (with $i = n-1$). Substituting successively for $|x_{i+1}(t)|$ in (3) we have the inequalities (1), (2).

Lemma 3. *Suppose that either* (A) *is sublinear and*

$$(5) \qquad \int^\infty P^n_1(\tau^*(t))\, \omega(t, cP^1_{2n-2}(\tau_1(t)))\, dt < \infty \quad \textit{for all} \quad c > 0,$$

or (A) *is superlinear and*

$$(6) \qquad \int^\infty P^n_1(t)\, \omega(t, cP^1_{2n-1}(\tau_1(t)))\, dt < \infty \quad \textit{for all} \quad c > 0.$$

If $(x_1(t), \ldots, x_n(t))$ *is a solution of* (A) *such that* $x_i(t) = o(P^i_{2n-2i+1}(t))$ *as* $t \to \infty$, $i = 1, \ldots, n$ *then* $x_i(t) = O(P^i_{2n-2i}(t))$ *as* $t \to \infty$, $i = 1, \ldots, n$.

Proof. Let $(x_1(t), \ldots, x_n(t))$ be a solution of (A) defined on $[a; \infty)$ such that $x_i(t) = o(P^i_{2n-2i+1}(t))$ as $t \to \infty$, $i = 1, \ldots, n$, and let $t_0 \geqq a$ be such that $\min(\tau_1(t), \tau_n(t), t) > a$ for $t \geqq t_0$. Because $x_n(t) = o(P^n_1(t))$ and $P^n_1(t) \to \infty$ as $t \to \infty$, there exists a $T \geqq t_0$ such that $|x_n(t)| \leqq P^n_1(t)$, $P^n_1(t) \geqq 1$ for $t \geqq T$ and $p_n(t)\, x'_n(t) = o(1)$ as $t \to \infty$. Now, the n-th equation of (A) implies

$$(7) \qquad |x_n(t)| \leqq P^n_1(T) + \int_T^t \frac{1}{p_n(s)} \int_s^\infty \omega(u, |x_1(\tau_1(u))|)\, du\, ds, \quad t \geqq T.$$

Combining (7), (2) (with $i = n-1$, $s = T$) and (1) (with $i = 1$, $k = n-1$, $s = T$)

we get

$$(8)\qquad |x_1(t)| \leqq \sum_{i=1}^{n-1} |x_i(T)|\, P^1_{2i-2}(t) + \sum_{i=1}^{n-1} |p_i(T)\, x_i'(T)|\, P^1_{2i-1}(t) +$$

$$+ KP^n_1(T)\, P^1_{2n-2}(t) + KP^1_{2n-2}(t) \int_T^t \frac{1}{P_n(u)} \int_u^\infty \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s\, \mathrm{d}u\,, \quad t \geqq T,$$

or by Lemma 1

$$(9)\qquad \frac{|x_1(t)|}{P^1_{2n-2}(t)} \leqq c + K \textstyle\int_T^t P^n_1(s)\, \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s + KP^n_1(t) \int_t^\infty \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s\,,$$

$$t \geqq T, \quad c > 0\,, \quad \text{a constant.}$$

Next, let us assume that (A) is sublinear and (5) holds. We shall show that $x_1(t) = O(P^1_{2n-2}(t))$ as $t \to \infty$ in this case. Suppose the contrary. We can choose T_1, T_2 such that $T < T_1 < T_2 < T_3$, $T_0 = \inf_{s \geqq T_2} (\min (\tau_1(s), \tau_n(s), s)) \geqq T_1$,

$$\frac{|x_1(T_0)|}{P^1_{2n-2}(T_0)} \geqq 1\,, \quad \sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P^1_{2n-2}(s)} = \sup_{T_2 \leqq s \leqq t} \frac{|x_1(s)|}{P^1_{2n-2}(s)}\,, \quad t \geqq T_2\,,$$

$$c + K \textstyle\int_T^{T_2} \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, \mathrm{d}s \leqq \dfrac{1}{4} \dfrac{|x_1(T_3)|}{P^1_{2n-2}(T_3)}$$

and

$$\textstyle\int_{T_2}^\infty P^n_1(\tau^*(s))\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s \leqq \dfrac{1}{4K}\,.$$

Let us define

$$(10)\qquad v(t) = \sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P^1_{2n-2}(s)}\,, \quad t \geqq T_0\,.$$

Using (10) and the sublinearity of (A) we obtain from (9)

$$(11)\qquad \frac{3}{4} \frac{v(t)}{P^n_1(t)} \leqq \frac{K}{P^n_1(t)} \int_{T_2}^t v(\tau_1(s))\, P^n_1(s)\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s +$$

$$+ K \textstyle\int_t^\infty v(\tau_1(s))\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s\,, \quad t \geqq T_2\,.$$

For each $t \geqq T_2$ let T_t, J_t denote the sets

$$(12)\qquad I_t = \{s \in [T_2; \infty),\ \tau_1(s) \leqq t\}\,, \quad J_t = \{s \in [T_2; \infty);\ \tau_1(s) > t\}\,.$$

Since $v(\tau_1(s)) \leqq v(t)$ for $s \in I_t$, $v(\tau_1(s))/P^n_1(\tau_1(s)) \leqq \sup_{\sigma \geqq t} v(\sigma)/P^n_1(\sigma)$ for $s \in J_t$, the right-hand side of (11) is bounded from above by

$$\frac{v(t)\, K}{P^1_n(t)} \left[\textstyle\int_{I_t \cap [T_2;t]} P^n_1(s)\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s + \int_{I_t \cap [t;\infty)} \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s\right] +$$

$$+ K \sup_{s \geqq t} \frac{v(s)}{P^n_1(s)} \left[\frac{1}{P^n_1(t)} \textstyle\int_{J_t \cap [T_2;t]} P^n_1(\tau_1(s))\, P^n_1(s)\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s +\right.$$

$$\left. + \textstyle\int_{J_t \cap [t;\infty)} P^n_1(\tau_1(s))\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s\right] \leqq \dfrac{K\, v(t)}{P^n_1(t)} \int_{T_2}^\infty P^n_1(s)\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s +$$

$$+ K \sup_{s \geqq t} \frac{v(s)}{P^n_1(s)} \textstyle\int_{T_2}^\infty P^n_1(\tau_1(s))\, \omega(s, P^1_{2n-2}(\tau_1(s)))\, \mathrm{d}s \leqq \frac{1}{4} \sup_{s \geqq t} \dfrac{v(s)}{P^n_1(s)} + \dfrac{1}{4} \dfrac{v(t)}{P^n_1(t)}\,, \quad t \geqq T_3\,.$$

We conclude that $0 < \sup_{s \geqq t} (v(s)/P_1^n(s)) \leqq \frac{1}{2} \sup_{s \geqq t} (v(s)/P_1^n(s))$, which is a contradiction. So $x_1(t) = O(P_{2n-2}^1(t))$ as $t \to \infty$ and (5) implies that

$$(13) \qquad \left| \int_T^\infty \frac{1}{p_n(s)} \int_s^\infty f(u, x_1(\tau_1(u)))\, du\, ds \right| \leqq \int_T^\infty P_1^n(s)\, \omega(s, cP_{2n-2}^1(\tau_1(s)))\, ds\,.$$

Therefore, (7) implies that $x_n(t) = O(1)$ as $t \to \infty$ and then there exist a $T_1 \geqq T$ and a positive constant L_n such that $x_n(t) \leqq L_n$, $x_n(\tau_n(t)) \leqq L_n$ for $t \geqq T_1$. Substituting this inequality in (2) (with $s = T_1$, $1 \leqq i \leqq n - 1$) we show, by Lemma 1, that $x_i(t) = O(P_{2n-2i}^i(t))$ as $t \to \infty$, $i = 1, \ldots, n - 1$.

Now we assume that (A) is superlinear and (6) holds. We shall show again that $x_1(t) = O(P_{2n-2}^1(t))$ as $t \to \infty$. Dividing (8) by $P_{2n-1}^1(t)$ we obtain, by Lemma 1,

$$(14) \qquad \frac{|x_1(t)|}{P_{2n-1}^1(t)} \leqq C + K \frac{P_{2n-2}^1(t)}{P_{2n-1}^1(t)} \int_T^\infty P_1^n(s)\, \omega(s, |x_1(\tau_1(s))|)\, ds\,, \quad t \geqq T$$

where C is a positive constant. As $x_1(t) = o(P_{2n-1}^1(t))$ as $t \to \infty$, we can choose a $T_1 \geqq T$ such that $T_0 = \inf_{s \geqq T_1} (\min(\tau_1(s), \tau_n(s), s) \geqq T$,

$$|x_1(t)| \leqq P_{2n-1}^1(t)\,, \quad P_{2n-1}^1(t) \geqq P_{2n-2}^1(t) \quad \text{for} \quad t \geqq T_0$$

and

$$\int_{T_1}^\infty P_1^n(s)\, \omega(s, P_{2n-1}^1(\tau_1(s)))\, ds \leqq \frac{1}{4K}.$$

Now, we define

$$u(t) = \sup_{s \geqq t} \frac{|x_1(s)|}{P_{2n-1}^1(s)}\,, \quad t \geqq T_0\,.$$

Using the superlinearity of (A) we can derive the following inequality from (14):

$$(15) \quad \frac{P_{2n-1}^1(t)}{P_{2n-2}^1(t)}\, u(t) \leqq D + K \int_{T_1}^\infty u(\tau_1(s))\, P_1^n(s)\, \omega(s, P_{2n-1}^1(\tau_1(s)))\, ds\,, \quad t \geqq T_1$$

where $D = C + K \int_T^{T_1} P_1^n(s)\, \omega(s, |x_1(\tau_1(s))|)\, ds$. Since

$$\frac{P_{2n-1}^1(\tau_1(s))}{P_{2n-2}^1(\tau_1(s))}\, u(\tau_1(s)) \leqq \sup_{T_0 \leqq \sigma \leqq t} \frac{u(\sigma)\, P_{2n-1}^1(\sigma)}{P_{2n-2}^1(\sigma)} \quad \text{for} \quad s \in I_t\,,$$

$$u(\tau_1(s)) \leqq u(t) \quad \text{for} \quad s \in J_t\,,$$

where I_t and J_t are defined in (12), the right-hand side of (15) is bounded from above by

$$D + K \sup_{T_0 \leqq s \leqq t} \frac{u(s)\, P_{2n-1}^1(s)}{P_{2n-2}^1(s)} \int_{I_t \cap [T_1; t]} P_1^n(z)\, \frac{P_{2n-2}^1(\tau_1(z))}{P_{2n-1}^1(\tau_1(z))}\, \omega(z, P_{2n-1}^1(\tau_1(z)))\, dz\, +$$

$$+\, K \sup_{T_0 \leqq s \leqq t} \frac{u(s)\, P_{2n-1}^1(s)}{P_{2n-2}^1(s)} \int_{I_t \cap [t; \infty)} P_1^n(z)\, \frac{P_{2n-2}^1(\tau_1(z))}{P_{2n-1}^1(\tau_1(z))}\, \omega(z, P_{2n-1}^1(\tau_1(z)))\, dz\, +$$

$$+\, K\, u(t) \int_{J_t \cap [T_1; t]} P_1^n(z)\, \omega(z, P_{2n-1}^1(\tau_1(z)))\, dz + K\, u(t) \int_{J_t \cap [t; \infty)} P_1^n(z)\, \omega(z, P_{2n-1}^1(\tau_1(z)))\, dz \leqq$$

$$\leqq D + K \sup_{T_0 \leqq s \leqq t} \frac{u(s)\, P^1_{2n-1}(s)}{P^1_{2n-2}(s)} \int_{T_1}^{\infty} P^n_1(z)\, \omega(z, P^1_{2n-1}(\tau_1(z)))\, dz +$$

$$+ K\, u(t) \frac{P^1_{2n-1}(t)}{P^1_{2n-2}(t)} \int_{T_1}^{\infty} P^n_1(z)\, \omega(z, P^1_{2n-1}(\tau_1(z)))\, dz \leqq D_1 + \tfrac{1}{2} \sup_{T_1 \leqq s \leqq t} \frac{u(s)\, P^1_{2n-1}(s)}{P^1_{2n-2}(s)}$$

$$\text{for} \quad t \geqq T_1\,, \quad \text{where} \quad D_1 = D + \tfrac{1}{4} \sup_{T_0 \leqq s \leqq T_1} \frac{u(s)\, P^1_{2n-1}(s)}{P^1_{2n-2}(s)}\,.$$

It follows that

$$\frac{|x_1(t)|}{P^1_{2n-2}(t)} \leqq \sup_{T_1 \leqq s \leqq t} \frac{u(s)\, P^1_{2n-1}(s)}{P^1_{2n-2}(s)} \leqq 2D_1\,, \quad t \geqq T_1\,,$$

which means $x_1(t) = O(P^1_{2n-2}(t))$ as $t \to \infty$ and according to (6) we have that the left-hand side of (13) converges. Continuing as in the corresponding part of the proof of Lemma 2 we get $x_i(t) = O(P^i_{2n-2i}(t))$ as $t \to \infty$, $i = 1, \ldots, n$. The proof of Lemma 3 is complete.

Lemma 4. *Let* $1 \leqq k < 2n - 1$. *Suppose that* (A) *is sublinear and*

(16) $\int^{\infty} Q_{k-1}(t)\, a_{n-1}(t) \left(\int_{\tau_n(t)}^{\infty} \omega(s, P^1_{2n-k-2}(\tau_1(s)))\, R_{k+1}(\tau_1(s))\, P^n_1(s)\, ds\right)^{\beta} dt < \infty\,.$

If $(x_1(t), \ldots, x_n(t))$ *is a solution of* (A) *such that*

(17) $x_i(t) = o(P^i_{2n-2i-k+1}(t))\,,\ x_j(t) = o(1)$ *as* $t \to \infty\,,\ 1 \leqq i \leqq n - \left[\frac{k}{2}\right] < j \leqq n$

then

(18) $$x_i(t) = O(P^i_{2n-2i-k}(t))\,, \quad x_j(t) = O(1) \quad \text{as} \quad t \to \infty\,,$$
$$1 \leqq i \leqq n - \left[\frac{k+1}{2}\right] < j \leqq n\,.$$

Proof. Let $(x_1(t), \ldots, x_n(t))$ be a solution of (A) on $[t_0; \infty)$ satisfying (17). Take a $T \geqq t_0$ such that $\min(\tau_1(t), \tau_n(t), t) \geqq t_0$ for $t \geqq T$. Let us consider the two possible cases.

a) k is even. Let m be such that $k = 2m$. Using the properties of the components of the solution we easily see that $p_i(t)\, x_i'(t) = o(1)$ as $t \to \infty$, $i = n - m, \ldots, n$, and from the last $(n - m)$ equations of (A) we get for $t \geqq T$

(19) $$x_n(t) = \int_t^{\infty} \frac{1}{p_n(s)} \int_s^{\infty} f(u, x_1(\tau_1(u)))\, du\, ds\,,$$

(20) $$x_{n-1}(t) = \int_t^{\infty} \frac{1}{p_{n-1}(s)} \int_s^{\infty} a_{n-1}(u)\, g(x_n(\tau_n(u)))\, du\, ds\,,$$

(21) $$x_i(t) = \int_t^{\infty} \frac{1}{p_i(s)} \int_s^{\infty} a_i(u)\, x_{i+1}(u)\, du\, ds\,, \quad n - m + 1 \leqq i < n - 1\,,$$

(22) $$p_{n-m}(t)\, x'_{n-m}(t) = - \int_t^{\infty} a_{n-m}(u)\, x_{n-m+1}(u)\, du\,,$$

which implies

(23) $$|x_n(t)| \leqq \int_t^\infty P_1^n(s)\, \omega(s, |x_1(\tau_1(s))|)\, ds\,,$$

(24) $$|x_i(t)| \leqq K \int_t^\infty Q_{2n-2i-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P_1^n(s)\, ds\right)^\beta du\,,$$
$$n - m < i < n - 1\,,$$

(25) $$|x_{n-m}(t)| \leqq |x_{n-m}(T)| +$$
$$+ K \int_T^\infty Q_{2m-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P_1^n(s)\, ds\right)^\beta du\,.$$

Substituting (25) in (1) (with $i = 1$, $k = n - m$, $s = T$) we have

(26) $$|x_1(t)| \leqq \sum_{i=1}^{n-m} |x_i(T)|\, P_{2i-2}^1(t) + \sum_{i=1}^{n-m-1} |p_i(T)\, x_i'(T)|\, P_{2i-1}^1(t) +$$
$$+ K P_{2(n-m-1)}^1(t) \int_T^\infty Q_{2m-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P_1^n(s)\, ds\right)^\beta du\,, \quad t \geqq T\,.$$

By Lemma 1 there exists a positive constant c_{n-m} such that (26) yields

(27) $$\frac{|x_1(t)|}{P_{2(n-m-1)}^1(t)} \leqq c_{n-m} +$$
$$+ K \int_T^\infty Q_{2m-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P_1^n(s)\, ds\right)^\beta du\,, \quad t \geqq T\,.$$

Now we shall show that $x_1(t) = O(P_{2(n-m-1)}^1(t))$ as $t \to \infty$. Suppose the contrary. The proof is an easy modification of that of Lemma 3 when (A) is sublinear. We choose T_1, T_2, T_3 such that $T < T_1 < T_2 < T_3$, $T_0 = \inf_{s \geqq T_2} (\min (\tau_1(s), \tau_n(s), s)) \geqq T_1$ $|x_1(T_0)| \geqq P_{2(n-m-1)}^1(T_0)$,

$$\sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P_{2(n-m-1)}^1(s)} = \sup_{T_2 \leqq s \leqq t} \frac{|x_1(s)|}{P_{2(n-m-1)}^1(s)}\,,$$

$$c_{n-m} + K \int_T^{T_2} Q_{2m-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty \omega(s, P_{2(n-m-1)}^1(\tau_1(s)))\, P_1^n(s)\, ds\right)^\beta du \leqq \frac{|x_1(T_3)|}{4 P_{2(n-m-2)}^1(T_3)}$$

and

(28) $$\int_{T_2}^\infty Q_{2m-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty \omega(s, P_{2(n-m-1)}^1(\tau_1(s)))\, R_{2m+1}(\tau_1(s)) P_1^n(s)\, ds\right)^\beta du \leqq \frac{1}{4K}\,.$$

Let us define

$$v(t) = \sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P_{2(n-m-1)}^1(s)}\,, \quad t \geqq T_0\,.$$

Using the properties of the function $v(t)$ and the sublinearity of (A) we obtain from (27)

$$\tfrac{3}{4} v^\beta(t) \leqq \tfrac{3}{4} v(t) \leqq K \int_{T_2}^t Q_{2m-1}(u)\, a_{n-1}(u)\, \left(\int_{\tau_n(u)}^\infty v(\tau_1(s))\, \omega(s, P_{2(n-m-1)}^1(\tau_1(s)))\,.\right.$$
$$\left.. \, P_1^n(s)\, ds\right)^\beta du\,, \quad t \geqq T_3\,,$$

Integrating the last inequality over the sets (12) and using the properties of the function $v(t)$ and (28) we have

$$\frac{1}{2} v^\beta(t) \leqq \frac{1}{4}\left(\sup_{s \geqq t} \frac{v(s)}{R_{2m+1}(s)}\right)^\beta$$

or

$$0 < \left(\frac{v(t)}{R_{2m+1}(t)}\right)^\beta \leqq \frac{1}{2}\left(\frac{1}{R_{2m+1}(t)}\right)^\beta \left(\sup_{s \geqq t} \frac{v(s)}{R_{2m+1}(s)}\right)^\beta \leqq \frac{1}{2}\left(\sup_{s \geqq t} \frac{v(s)}{R_{2m+1}(s)}\right)^\beta, \quad t \geqq T_3$$

which is a contradiction. Therefore, we must have $x_1(t) = O(P^1_{2(n-m-1)}(t)$ as $t \to \infty$ By virtue of this fact and (16), the right-hand sides of (23)–(25) are bounded from above and so $x_i(t) = O(1)$ as $t \to \infty$, $n - m \leqq i \leqq n$. Since $x_{n-m}(t)$ is a bounded function, we see by (1) (with $k = n - m$, $s = T_2$) that $x_i(t) = O(P^i_{2(n-m-i)}(t))$ as $t \to \infty$, $1 \leqq i < n - m$.

b) k is odd. Let m be such that $k = 2m + 1$. Taking into account the properties (17) of the components of the solution and the last $(n - m)$ equations of (A) we get the inequalities (23), (24) and

$$(29) \qquad |x_{n-m}(t)| \leqq K \int_t^\infty Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, ds\right)^\beta du\,, \quad t \geqq T.$$

Combining (29) and (1) (with $i = 1$, $k = n - m$, $s = T$) we obtain

$$(30) \qquad |x_1(t)| \leqq \sum_{i=1}^{n-m-1} |x_i(T)|\, P^1_{2i-2}(t) + \sum_{i=1}^{n-m-1} |p_i(T)\, x'_i(T)|\, P^1_{2i-1}(t) +$$
$$+ K P^1_{2n-2m-3}(t) \int_T^\infty Q_{2m}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, ds\right)^\beta du\,, \quad t \geqq T,$$

and by Lemma 1

$$\frac{|x_1(t)|}{P^1_{2n-2m-3}(t)} \leqq D + K \int_T^\infty Q_{2m}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, ds\right)^\beta du\,, \quad t \geqq T,$$

where D is a positive constant.

Defining

$$v(t) = \sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P^1_{2n-2m-3}(s)}$$

and applying the same type of argument that was used to prove the case a), we conclude from the last inequality that $x_1(t) = O(P^1_{2n-2m-3}(t))$ as $t \to \infty$. By virtue of this fact and (16) the right-hand sides of (23), (24), (29) are bounded and hence $x_i(t) = O(1)$ as $t \to \infty$, $n - m - 1 < i \leqq n$. Further, using (19), (20), (21) we have from $(n - m - 1)$st equation of (A)

$$|p_{n-m-1}(t)\, x'_{n-m-1}(t)| \leqq |p_{n-m-1}(T)\, x'_{n-m-1}(T)| +$$
$$+ \int_T^\infty Q_{2m}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, ds\right)^\beta du\,,$$

which implies, by (16), that $p_{n-m-1}(t)\, x'_{n-m-1}(t) = O(1)$ as $t \to \infty$. But then $x_{n-m-1}(t) = O(P_1^{n-m-1}(t))$ as $t \to \infty$, which means that there exist a $T_1 \geqq T$ and a positive constant L such that $|x_{n-m-1}(t)| \leqq L P_1^{n-m-1}(t)$ for $t = T_1$. By this fact and

Lemma 1 we get from (1) (with $s = T_1, k = n - m - 1$) that $x_i(t) = O(P^i_{2(n-m-i)-1}(t))$ as $t \to \infty$, $1 \leqq i \leqq n - m - 1$. The proof is complete.

Lemma 5. *Let $1 \leqq k < 2n - 1$. Suppose that* (A) *is superlinear and*

$$(31) \qquad \int^\infty Q_{k-1}(t)\, a_{n-1}(t) \left(\int_{\tau_n(t)}^\infty \omega(s, P^1_{2n-k-1}(\tau_1(s)))\, P^n_1(s)\, \mathrm{d}s\right)^\beta \mathrm{d}t < \infty\,.$$

If $(x_1(t), \ldots, x_n(t))$ is a solution of (A) *satisfying* (17) *then* (18) *holds.*

Proof. Let $(x_1(t), \ldots, x_n(t))$ be a solution of (A) on $[t_0; \infty)$ satisfying (17). Pick a $T \geqq t_0$ such that $\min(\tau_1(t), \tau_n(t), t) \geqq t_0$ for $t \geqq T$. Let us consider the following two cases.

a) Let k be even, i.e. $k = 2m$. Proceeding in the same way as in the proof of Lemma 4 we get the inequality (26) and this yields

$$(32) \qquad \frac{|x_1(t)|}{P^1_{2n-2m-1}(t)} \leqq c_m + K \frac{P^1_{2n-2m-2}(t)}{P^1_{2n-2m-1}(t)} \cdot$$
$$\cdot \int_{T_1}^\infty Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, \mathrm{d}s\right)^\beta \mathrm{d}u\,, \quad t \geqq T_1$$

where T_1 is sufficiently large, c_m is a positive constant. We shall show that $x_1(t) = O(P^1_{2n-2m-2}(t))$ as $t \to \infty$. We know that $x_1(t) = o(P^1_{2n-2m-1}(t))$ as $t \to \infty$ and (31) holds. We can choose a $T_2 \geqq T_1$ such that $T_0 = \inf_{s \geqq T_2} (\min(\tau_1(s), \tau_n(s), s)) \geqq T_1$,

$$|x_1(t)| \leqq P^1_{2n-2m-1}(t)\,, \quad P^1_{2n-2m-2}(t) \leqq P^1_{2n-2m-1}(t) \quad \text{for} \quad t \geqq T_2$$

and

$$\int_{T_2}^\infty Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, P^1_{2n-2m-1}(\tau_1(s)))\, P^n_1(s)\, \mathrm{d}s\right)^\beta \mathrm{d}u \leqq \frac{1}{4K}\,.$$

Now, we define

$$v(t) = \sup_{s \geqq t} \frac{|x_1(s)|}{P^1_{2n-2m-1}(s)}\,, \quad t \geqq T_0\,.$$

Using the superlinearity of (A) we can derive from (32) the inequality

$$(33) \qquad \frac{P^1_{2n-2m-1}(t)}{P^1_{2n-2m-2}(t)}\, v(t) \leqq d_m +$$
$$+ K \int_{T_2}^\infty Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty v(\tau_1(s)))\, \omega(s, P^1_{2n-2m-1}(\tau_1(s)))\, P^n_1(s)\, \mathrm{d}s\right)^\beta \mathrm{d}u\,,$$
$$t \geqq T_2\,,$$

where

$$d_m = c_m + K \int_{T_1}^{T_2} Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, \mathrm{d}s\right)^\beta \mathrm{d}u\,.$$

Since

$$\frac{P^1_{2n-2m-1}(\tau_1(s))}{P^1_{2n-2m-2}(\tau_1(s))}\, v(\tau_1(s)) \leqq \sup_{T_0 \leqq \sigma \leqq t} \left[\frac{P^1_{2n-2m-1}(\sigma)}{P^1_{2n-2m-2}(\sigma)}\, v(\sigma)\right], \quad s \in I_t$$

$$v(\tau_1(s)) \leqq v(t)\,, \quad s \in J_t\,,$$

where I_t, J_t are defined in (12), the right-hand side of (33) is bounded from above by

$$d_m + K\left[\sup_{T_0 \leqq s \leqq t} \frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} v(s)\right]^\beta \cdot$$

$$\cdot \int_{I_t \cap [T_2;\infty)} Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, P^1_{2n-2m-1}(\tau_1(s)))\, P^n_1(s)\, ds\right)^\beta du +$$

$$+ K\, v(t)^\beta \int_{J_t \cap [T_2;\infty)} Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, P^1_{2n-2m-1}(\tau_1(s)))\, P^n_1(s)\, ds\right)^\beta du \leqq$$

$$\leqq d_m + \frac{1}{4}\left[\sup_{T_0 \leqq s \leqq t} \frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} v(s)\right]^\beta + \frac{1}{4}\left[v(t) \frac{P^1_{2n-2m-1}(t)}{P^1_{2n-2m-2}(t)}\right]^\beta \leqq$$

$$\leqq d_m + \frac{1}{2}\left[\sup_{T_0 \leqq s \leqq t} \frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} v(s)\right]^\beta, \quad t \geqq T_2,$$

and also

$$\sup_{T_0 \leqq s \leqq t} \left[\frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} v(s)\right] \leqq d^*_m + \left[\sup_{T_0 \leqq s \leqq t} \frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} v(s)\right]^\beta, \quad t \geqq T_2$$

where

$$d^*_m = d_m + \sup_{T_0 \leqq s \leqq T_2} \left[\frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} \iota(s)\right].$$

It follows that

$$\frac{|x_1(t)|}{P^1_{2n-2m-2}(t)} \leqq \sup_{T_0 \leqq s \leqq t} \left[\frac{P^1_{2n-2m-1}(s)}{P^1_{2n-2m-2}(s)} v(s)\right] \leqq L, \quad t \geqq T_2,$$

L is a positive constant, which means $x_1(t) = O(P^1_{2n-2m-2}(t))$ as $t \to \infty$. Proceeding further as in the case a) of the proof of Lemma 4 we arrive at the conclusion that $(x_1(t), \ldots, x_n(t))$ satisfies (18).

b) Let k be odd, i.e. $k = 2m + 1$. Similarly as in the case b) of the proof of Lemma 4 we get (30), which by Lemma 1 yields

$$\frac{|x_1(t)|}{P^1_{2n-2m-2}(t)} \leqq D + K \frac{P^1_{2n-2m-3}(t)}{P^1_{2n-2m-2}(t)} \int_T^\infty Q_{2m}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^\infty \omega(s, |x_1(\tau_1(s))|)\, P^n_1(s)\, ds\right)^\beta du,$$

$$t \geqq T,$$

where D is a positive constant. We have $x_1(t) = o(P^1_{2n-2m-2}(t))$ as $t \to \infty$.

Defining

$$v(t) = \sup_{s \geqq t} \frac{|x_1(s)|}{P^1_{2n-2m-2}(s)}, \quad t \geqq T_0$$

and arguing as in the proof of the case a) we can show that $x_1(t) = O(P^1_{2n-2m-3}(t))$ as $t \to \infty$. Then continuing as in the corresponding part of the proof of Lemma 4 we conclude that $(x_1(t), \ldots, x_n(t))$ satisfies (18). This completes the proof of the lemma.

The main result of this paper is the following theorem which describes the behavior of all solutions of (A).

Theorem 1. *Suppose that either* (A) *is sublinear and*

(34) $$\int^\infty P_k^n(\tau_1{}^*(t))\,\omega(t, cP^1_{2n-k-1}(\tau_1(t)))\,dt < \infty \quad \text{for all} \quad c > 0\,, \quad k = 0, 1\,,$$

(35) $$\int^\infty Q_{k-2}(t)\,a_{n-1}(t)\,\left(\int_{\tau_n(t)}^\infty \omega(s, cP^1_{2n-k-1}(\tau_1(s)))\,R_k(\tau_1(s))\,P_1^n(s)\,ds\right)^\beta dt < \infty\,,$$
$$\text{for all} \quad c > 0\,, \quad 2 \leqq k \leqq 2n - 1\,,$$

or (A) *is superlinear and*

(36) $$\int^\infty P_1^n(t)\,\omega(t, cP^1_{2n-1}(\tau_1(t)))\,dt < \infty \quad \text{for all} \quad c > 0\,,$$

(37) $$\int^\infty Q_{k-2}(t)\,a_{n-1}(t)\,\left(\int_{\tau_n(t)}^\infty \omega(s, cP^1_{2n-k}(\tau_1(s)))\,P_1^n(s)\,ds\right)^\beta dt < \infty$$
$$\text{for all} \quad c > 0\,, \quad 2 \geqq k \leqq 2n - 1\,.$$

If $(x_1(t), \ldots, x_n(t))$ is a solution of (A) then exactly one of the following cases occurs:

(I) $\limsup_{t\to\infty} \dfrac{|x_i(t)|}{P^i_{2n-2i+1}(t)} = \infty\,, \quad i = 1, \ldots, n\,,$

(II) there exists a nonzero number α_0 such that

$$\lim_{t\to\infty} \frac{x_n(t)}{P_1^n(t)} = \alpha_0\,, \quad \limsup_{t\to\infty} \frac{|x_i(t)|}{P^i_{2n-2i+1}(t)} < \infty\,, \quad i = 1, \ldots, n - 1\,;$$

(III) there exists a nonzero number α_1 such that

$$\lim_{t\to\infty} x_n(t) = \alpha_1\,, \quad \lim_{t\to\infty} \frac{x_i(t)}{P^i_{2n-2i}(t)} = g(\alpha_1)\,, \quad i = 1, \ldots, n - 1\,;$$

(IV) there exist an integer k, $2 \leqq k \leqq 2n - 1$ and a nonzero α_k such that

$$\lim_{t\to\infty} \frac{x_i(t)}{P^i_{2n-2i-k+1}(t)} = \alpha_k\,, \quad \lim_{t\to\infty} x_j(t) = 0\,, \quad 1 \leqq i \leqq n - \left[\frac{k}{2}\right] < j \leqq n\,;$$

(V) $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, \ldots, n$.

Proof. Let $(x_1(t), \ldots, x_n(t))$ be a solution of (A) on $[t_0; \infty)$ and let $T \geqq t_0$ be such that $\min(\tau_1(t), \tau_n(t)) \geqq t_0$ for $t \geqq T_0$. We shall show:

$$\text{If} \quad \limsup_{t\to\infty} \frac{|x_1(t)|}{P^1_{2n-1}(t)} = \infty \quad \text{then} \quad \limsup_{t\to\infty} \frac{|x_j(t)|}{P^j_{2n-2j+1}(t)} = \infty \quad \text{for} \quad j = 2, \ldots, n\,.$$

Suppose the contrary. Then there exists an integer j, $2 < j < n - 1$ such that

$$\limsup_{t\to\infty} \frac{|x_j(t)|}{P^j_{2n-2j+1}(t)} < \infty\,,$$

which means there exist a positive constant L and a $t_1 \geqq T$ such that $|x_j(t)| \leqq \leqq LP^j_{2n-2j+1}(t)$ for $t \geqq t_1$. From (1) (with $i = 1$, $k = j$, $s = t_1$) we have

$$|x_1(t)| \leqq \sum_{i=1}^{j-1} |x_i(t_1)|\,P^1_{2i-2}(t) + \sum_{i=1}^{j-1} |p_i(t_1)\,x_i'(t_1)|\,P^1_{2i-1}(t) + LP^1_{2n-1}(t)\,, \quad t \geqq T_1\,,$$

which implies, by Lemma 1, that

$$\limsup_{t\to\infty} \frac{|x_1(t)|}{P^1_{2n-1}(t)} < \infty\,,$$

a contradiction. Now, let $j = n$, i.e.

$$\limsup_{t\to\infty} \frac{|x_n(t)|}{P^n_1(t)} < \infty\,.$$

Then there exist a positive constant L_1 and a $t_1 \geqq T$ such that $|x_n(t)| \leqq L_1 P^n_1(t)$, $|x_n(\tau_n(t))| \leqq L_1 P^n_1(\tau_n(t))$ for $t \geqq T_1$. Using this fact in (2) (with $i = 1$, $s = t_1$) we get

$$|x_1(t)| \leqq \sum_{i=1}^{n-1} |x_i(t_1)|\, P^1_{2i-2}(t) + \sum_{i=1}^{n-1} |p_i(t_1)\, x'_i(t_1)|\, P^1_{2i-1}(t) + KL^\beta_1\, P^1_{2n-1}(t)\,, \quad t \geqq t_1\,.$$

From this relation we have

$$\limsup_{t\to\infty} \frac{|x_1(t)|}{P^1_{2n-1}(t)} < \infty$$

again, a contradiction. Hence Case (I) occurs.

Suppose now

$$\limsup_{t\to\infty} \frac{|x_1(t)|}{P^1_{2n-1}(t)} < \infty\,,$$

i.e. there exist a positive constant M and a $T_1 \geqq T$ such that $|x_1(\tau_1(t))| \leqq$ $\leqq MP^1_{2n-1}(\tau_1(t))$ for $t \geqq T_1$.

According to (34) with $k = 0$ or (36) provided (A) is sublinear or superlinear, respectively, we have $f(t, x_1(\tau_1(t))) \in L^1[T_1; \infty)$, and so the last equation of (A) yields

$$p_n(t)\, x'_n(t) = \alpha_0 - \int_t^\infty f(s, x_1(\tau_1(s)))\, ds\,, \quad t \geqq T_1 \tag{38}$$

where $\alpha_0 = p_n(T_1)\, x'_n(T_1) + \int_T^\infty f(s, x_1(\tau_1(s)))\, ds$.

So, we have $\lim_{t\to\infty} p_n(t)\, x'_n(t) = \alpha_0$. Using this fact one can easily see that $\lim_{t\to\infty} (x_n(t)/P^n_1(t)) = \alpha_0$ and so there exist a positive constant c and a $T_2 \geqq T_1$ such that $|x_n(\tau_n(t))| \leqq cP^n_1(\tau_n(t))$ for $t \geqq T_2$. Now, from (2) (with $s = T_2$) we have

$$|x_i(t)| \leqq \sum_{j=1}^{n-i} |x_j(T_2)|\, P^i_{2j-2}(t) + \sum_{j=1}^{n-i} |p_j(T_2)\, x'_j(T_2)|\, P^i_{2j-1}(t) +$$
$$+ Kc^\beta\, P^i_{2n-2i+1}(t)\,, \quad t \geqq T_2\,, \quad 1 \leqq i \leqq n-1\,.$$

This implies, by Lemma 1, that Case (II) occurs.

Let $\alpha_0 = 0$ and let either (34) with $k = 1$ or (36) hold provided A is sublinear or superlinear, respectively. We show that Case (III) occurs. By virtue of Lemma 3 the components of a solution of (A) have the following properties: $x_i(t) =$ $= O(P^i_{2n-2i}(t))$ as $t \to \infty$, $i = 1, \ldots, n$. Hence $x_1(t) = O(P^1_{2n-2}(t))$ as $t \to \infty$.

Consequently, it is clear that

$$\left|\int_T^\infty \frac{1}{p_n(s)} \int_s^\infty f(u, x_1(\tau_1(u)))\, du\, ds\right| < \infty .$$

Now (38) (with $\alpha_0 = 0$) yields

(39) $$x_n(t) = \alpha_1 + \int_t^\infty \frac{1}{p_n(s)} \int_s^\infty f(u, x_1(\tau_1(u)))\, du\, ds , \quad t \geqq T - \text{sufficiently large} ,$$

where

$$\alpha_1 = x_n(T) - \int_T^\infty \frac{1}{p_n(s)} \int_s^\infty f(u, x_1(\tau_1(u)))\, du\, ds .$$

We see that $\lim_{t\to\infty} x_n(t) = \alpha_1$ and $\lim_{t\to\infty} g(x_n(\tau_n(t)) = g(\alpha_1)$. Then the $(n-1)$st equation of (A) yields

$$x_{n-1}(t) = x_{n-1}(T) + p_{n-1}(T)\, x'_{n-1}(T) \int_T^t \frac{ds}{p_{n-1}(s)} +$$
$$+ \int_T^t \frac{1}{p_{n-1}(s)} \int_T^s a_{n-1}(u)\, g(x_n(\tau_n(u)))\, du\, ds , \quad t \geqq T,$$

which implies that $\lim_{t\to\infty} (x_{n-1}(t)/P_2^{n-1}(t)) = g(\alpha_1)$. Suppose that $\lim_{t\to\infty} (x_i(t)/P^i_{2n-2i}(t)) = g(\alpha_1)$ holds for some integer i, $1 < i \leqq n-1$. Then $\lim_{t\to\infty} (x_{i-1}(t)/P^{i-1}_{2n-2i-2}(t)) = g(\alpha_1)$, which follows from the $(i-1)$st equation of (A). If $\alpha_1 \neq 0$, a solution of (A) belongs to Case (III). It remains to examine the case when $\alpha_1 = 0$. We prove this part by mathematical induction. We shall show that if $\alpha_k = 0$ for some integer k, $1 \leqq k < 2n - 1$, i.e. the components of a solution of (A) satisfy

(40) $$x_i(t) = o(P^i_{2n-2i-k+1}(t)), \quad x_j(t) = o(1) \quad \text{as} \quad t \to \infty , \quad 1 \leqq i \leqq n - \left[\frac{k}{2}\right] < j \leqq n ,$$

and either (35) (with $k+1$) or (37) (with $k+1$) holds, then the components of the solution of (A) belong to Case (IV) with $\alpha_{k+1} \neq 0$.

Let $\alpha_k = 0$ and let (35) or (37) (with $k+1$) hold. By Lemma 4 and Lemma 5 we know that the components $x_i(t)$ of the solutions of (A) satisfy (40). Let us consider the following two cases.

a) Let k be even, i.e. $k = 2m$. It is a matter of an easy computation to derive from (19)–(22) the equations

(41) $$x_i(t) = \int_t^\infty Q_{2n-2i-1}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^\infty \frac{1}{p_n(v)} \int_v^\infty f(s, x_1(\tau_1(s)))\, ds\, dv\right) du ,$$
$$n - m + 1 \leqq i n - 1 ,$$

(42)
$$p_{n-m}(t)\, x'_{n-m}(t) = - \int_t^\infty Q_{2m-2}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^\infty \frac{1}{p_n(v)} \int_v^\infty f(s, x_1(\tau_1(s)))\, ds\, dv\right) du ,$$
$$t \geqq T .$$

Because $x_1(t) = O(P^1_{2n-2m-2}(t))$ as $t \to \infty$, there exist a positive constant c and a $T_1 \geqq T$ such that $|x_1(\tau_1(t))| \leqq cP^1_{2n-2m-2}(\tau_1(t))$, $t \geqq T_1$. By Lemma 1 and the properties of the functions $f(t, u)$, $g(v)$ we have

$$\left|\int_{T_1}^{\infty} Q_{2n-2i-1}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\right| \leqq$$

$$\leqq M \int_{T_1}^{\infty} Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^{\infty} \omega(s, cP^1_{2n-2m-2}(\tau_1(s)))\, P^n_1(s)\, ds\right)^{\beta} du < \infty\,,$$

$n - m + 1 \leqq i < n - 1$, $M > 0$, and

$$\left|\int_{T_1}^{\infty} Q_{2m-2}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\right| \leqq$$

$$\leqq L \int_{T_1}^{\infty} Q_{2m-1}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^{\infty} \omega(s, cP^1_{2n-2m-2}(\tau_1(s)))\, P^n_1(s)\, ds\right)^{\beta} du < \infty\,, \quad L > 0\,.$$

Hence $\lim_{t\to\infty} x_i(t) = 0$, $n - m + 1 \leqq i \leqq n$ and

(43)

$$x_{n-m}(t) = \alpha_{2m+1} + \int_t^{\infty} Q_{2m-1}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\,, \quad t \geqq T\,,$$

where

$$\alpha_{2m+1} = x_{n-m}(T) + \int_T^{\infty} Q_{2m-1}(u, T)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du,$$

which follows from (19), (20), (41), (42). Since the integral on the right-hand side of (43) is convergent by the assumption (35) or (37) we see that $\lim_{t\to\infty} x_{n-m}(t) = \alpha_{2m+1}$. Using this fact we successively obtain $\lim_{t\to\infty} (x_i(t)/P^i_{2n-2m-2i}(t)) = \alpha_{2m+1}$, $1 \leqq i < n - m$, by the first $(n - m - 1)$ equations of (A).

b) Now let k be odd, i.e. $k = 2m + 1$. As above, we know from Lemma 4 and Lemma 5 that $x_1(t) = O(P^1_{2n-2m-3}(t))$ as $t \to \infty$. (35) or (37) implies that the integrals

$$\int^{\infty} Q_i(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\,, \quad 0 \leqq i \leqq 2m\,,$$

$$\int^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv$$

are convergent. Hence from (19), (20), (41), (43) with $\alpha_{2m+1} = 0$ we get that $\lim_{t\to\infty} x_i(t) = 0$, $n - m \leqq i \leqq n$.

Combining the $(n - m - 1)$st equation of (A) and (43) with $\alpha_{2m+1} = 0$ we find

$$p_{n-m-1}(t)\, x'_{n-m-1}(t) =$$

$$= \alpha_{2m+2} - \int_t^{\infty} Q_{2m}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\,, \quad t \geqq T\,,$$

where

$$\alpha_{2m+2} = p_{n-m-1}(T)\, x'_{n-m-1}(T) +$$

$$+ \int_T^\infty Q_{2m}(u, T)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^\infty \frac{1}{p_n(v)} \int_v^\infty f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\,,$$

which implies $\lim_{t\to\infty} p_{n-m-1}(t)\, x'_{n-m-1}(t) = \alpha_{2m+2}$ and consequently $\lim_{t\to\infty} (x_{n-m-1}(t)/P_1^{n-m-1}(t)) = \alpha_{2m+2}$. Using this fact we successively obtain that $\lim_{t\to\infty} (x_i(t)/P^i_{2n-2m-2i+1}) = \alpha_{2m+2}$, $1 \leqq i \leqq n - m - 1$ from the first $(n - m - 2)$ equations of (A). Hence Case (IV) can occur if $\alpha_k \neq 0$, $2 \leqq k \leqq 2n - 1$.

If $\alpha_{2n-1} = 0$ a solution of (A) belongs to Case (V). The proof of Theorem 1 is complete.

Corollary. *Let all assumptions of Theorem* 1 *be fulfilled and let* $g(u) = |u|^\beta \operatorname{sgn} u$. *Then* (I), (III), (VI), (V) *of Theorem* 1 *and*

(II′) $\displaystyle\lim_{t\to\infty} \frac{x_i(t)}{P^i_{2n-2i+1}(t)} = \alpha_0^\beta\,, \quad i = 1, \ldots, n-1$

hold.

Example. Consider the system

$$(t^{1/3}\, x_1'(t))' = \tfrac{8}{3} t^{1/4} (x_2(t^{1/2}))^{1/2}\,,$$
$$(t^{-2}\, x_2'(t))' = -\tfrac{8}{9} t^{-20/3} (x_1(t^3))^{1/2}\,.$$

As one can easily check, condition (34) with $k = 1$ is violated and the system has a solution $x_1(t) = t^2$, $x_2(t) = t^{1/3}$ which has the following properties:

$$\lim_{t\to\infty} \frac{x_1(t)}{P_3^1(t)} = 0\,, \quad \lim_{t\to\infty} \frac{x_2(t)}{P_1^2(t)} = 0 \quad \text{but} \quad \lim_{t\to\infty} \frac{x_1(t)}{P_2^1(t)} = \infty\,, \quad \lim_{t\to\infty} x_2(t) = \infty\,.$$

This example shows that the violation of the integral condition of Theorem 1 for some integer $k \in \{0, 1, \ldots, 2n - 1\}$ may give rise to solutions with different asymptotic nature.

On the basis of Theorem 1 we want to determine some properties of all nonoscillatory solutions of (A) for which the following sign assumptions are given:

$$(44) \qquad f(t, v)\, v < 0 \quad \text{for} \quad v \neq 0\,, \quad (t, v) \in [a; \infty) \times (-\infty, \infty)\,,$$
$$g(u)\, u > 0 \quad \text{for} \quad u \neq 0\,, \quad u \in (-\infty; \infty)\,, \quad \liminf_{u\to\infty} |g(u)| \neq 0\,.$$

We remark that under these assumptions a solution of (A) is oscillatory [nonoscillatory] if and only if it is weakly oscillatory [weakly nonoscillatory]. We will need a lemma.

Lemma 6. *Let* (44) *hold. Then for a nonoscillatory solution* $(x_1(t), \ldots, x_n(t))$ *of* (A) *we have*:

1) *there exists a* $t_0 \geqq a$ *such that* $x_1(t)\, x_i'(t) > 0$ *for* $t \geqq t_0$, $i = 1, \ldots, n$;
2) *there exist an integer* $k \in \{1, \ldots, n\}$ *and a* $t_0 \geqq a$ *such that for* $t \geqq t_0$ *we have* $x_1(t)\, x_i(t) > 0$, $i = 1, \ldots, k$; $x_1(t)\, x_i(t) < 0$, $i = k + 1, \ldots, n$.

Theorem 2. *Assume that* (44) *and the hypotheses of Theorem* 1 *are satisfied. If* $(x_1(t), \ldots, x_n(t))$ *is a nonoscillatory solution of* (A) *then exactly one of Cases* (II)–(IV) *of Thoerem* 1 *occurs.*

Proof. Let $(x_1(t), \ldots, x_n(t))$ be a nonoscillatory solution of (A) such that $x_i(t) \neq 0$, $i = 1, \ldots, n$ on $[t_0; \infty)$. Take a $T \geqq t_0$ such that $\min(\tau_1(t), \tau_n(t), t) \geqq t_0$ for $t \geqq T$. Without loss of generality we may assume that $x_1(t) > 0$ for $t \geqq t_0$. Lemma 6 implies that $x_1(t)$ is increasing on $[T; \infty)$ and hence $\liminf_{t\to\infty} x_1(t) > 0$. Case (V) of Theorem 1 is excluded.

Now, from the n-th equation of (A) we see that $p_n(t)\, x_n'(t)$ is decreasing and

$$0 < p_n(t)\, x_n'(t) \leqq p_n(T)\, x_n'(T)\,, \quad t \geqq T. \tag{45}$$

Dividing (45) by $p_n(t)$ and then integrating from T to $t \geqq T$ we obtain

$$|x_n(t)| \leqq |x_n(T)| + |p_n(T)\, x_n'(T)|\, P_1^n(t, T) \leqq c P_1^n(t)\,, \quad t \geqq T_1 \geqq T\,,$$

c a positive constant. Substituting this inequality in (1) (with $i = 1$, $s = T_1$) we find

$$|x_1(t)| \leqq \sum_{i=1}^{n-1} |x_i(T_1)|\, P_{2i-2}^1(t) + \sum_{i=1}^{n-1} |p_i(T_1\, x_i'(T_1)|\, P_{2i-1}^1(t) + Kc^\beta\, P_{2n-1}^1(t)\,, \; t \geqq T_1\,.$$

This shows that $|x_1(t)|/P_{2n-1}^1(t)$ is bounded from above by a positive constant by Lemma 1. Hence Case (I) of Theorem 1 is excluded. This completes the proof.

Now, we turn to an investigation of the behavior of oscillatory solutions of (A). No sign conditions are introduced but the following conditions on $\tau_1(t)$ are needed.

We use the notation

$$h^*(t) = \sup_{a \leqq s \leqq t} (\max(\tau_1(s), s)\,, \quad h_*(t) = \inf_{s \geqq t} (\min \tau_1(s), s))\,.$$

We say that condition (G^*) [or (G_*)] is satisfied if there exists a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to \infty$ as $n \to \infty$ and $h^*(t_n) = t_n$ [$h_*(t_n) = t_n$] for $n = 1, 2, \ldots$.

Theorem 3. *Assume that* (A) *is sublinear and* (G^*) *is satisfied. If* (34), (35) *hold then every oscillatory solution* $(x_1(t), \ldots, x_n(t))$ *of* (A) *has the property* $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, \ldots, n$.

Proof. Let $(x_1(t), \ldots, x_n(t))$ be an oscillatory solution of (A) defined on $[t_0; \infty)$. Choose a $T \geqq t_0$ such that $h_*(T) \geqq t_0$. Since the solution is oscillatory by hypotheses, the Cases (II)–(IV) of Theorem 1 can never occur, so the solution must satisfy either (I) or (V). Suppose that (I) is true. We can choose a $T_1 \geqq T$ such that $P_1^n(t) \geqq 1$ for

$t \geqq T_1$. From the n-th equation of (A) we get

$$|x_n(t)| \leqq |x_n(T_1)| + |p_n(T_1)\, x_n'(T_1)|\, P_1^n(t) + \int_{T_1}^t \frac{1}{p_n(s)} \int_{T_1}^s \omega(u, |x_1(\tau_1(u))|)\, \mathrm{d}u\, \mathrm{d}s \leqq$$
$$\leqq P_1^n(t) \left(|x_n(T_1)| + |p_n(T_1)\, x_n'(T_1)| + \int_{T_1}^t \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s\right)$$

or

$$(46) \qquad |x_n(t)|^\beta \leqq (P_1^n(t))^\beta \left(c^\beta + \beta c^{\beta-1} \int_{T_1}^t \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s\right), \quad t \geqq T_1$$

where $c = |x_n(T_1)| + |p_n(T_1)\, x_n'(T_1)|$.

Substituting (46) in (2) (with $i = 1$, $s = T_2$) for $T_2 \geqq T_1$ such that $\tau_n(t) \geqq T_1$ for $t \geqq T_2$ we find

$$|x_1(t)| \leqq \sum_{i=1}^{n-1} |x_i(T_2)|\, P_{2i-2}^1(t) + \sum_{i=1}^{n-1} |p_i(T_2)\, x_i'(T_2)|\, P_{2i-1}^1(t) +$$
$$+ Kc^\beta P_{2n-1}^1(t) + K\beta c^{\beta-1} P_{2n-1}^1(t) \int_{T_2}^t \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s\,, \quad t \geqq T_2\,.$$

By Lemma 1 there exists positive constant L such that

$$(47) \qquad \frac{|x_1(t)|}{P_{2n-1}^1(t)} \leqq L + M \int_{T_2}^t \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s\,, \quad t \geqq T_2\,, \quad M = K\beta c^{\beta-1}\,.$$

Let us put

$$u(t) = \sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P_{2n-1}^1(s)}$$

and choose T_3, T_4, T_5 such that $T_2 < T_3 < T_4 < T_5$, $T_0 = h_*(T_3) \geqq T_2$, $|x_1(T_0)| > P_{2n-1}^1(T_0)$,

$$\sup_{T_0 \leqq s \leqq t} \frac{|x_1(s)|}{P_{2n-1}^1(s)} = \sup_{T_4 \leqq s \leqq t} \frac{|x_1(s)|}{P_{2n-1}^1(s)}\,,$$

$$\int_{T_4}^\infty \omega(s, P_{2n-1}^1(\tau_1(s)))\, \mathrm{d}s \leqq \frac{1}{4M}\,,$$

$$L + M \int_{T_2}^{T_4} \omega(s, |x_1(\tau_1(s))|)\, \mathrm{d}s \leqq \frac{1}{2} \frac{|x_1(T_5)|}{P_{2n-1}^1(T_5)}\,.$$

Using the sublinearity of (A) and the fact that $u(t)$ is nondecreasing we derive from (47)

$$u(t) \leqq \tfrac{1}{2} u(t) + M \int_{T_4}^t u(\tau_1(s))\, \omega(s, P_{2n-1}^1(\tau_1(s)))\, \mathrm{d}s \leqq$$
$$\leqq \tfrac{1}{2} u(t) + u(h^*(t))\, M \int_{T_4}^t \omega(s, P_{2n-1}^1(\tau_1(s)))\, \mathrm{d}s\,,$$

which implies $u(t)/u(h^*(t)) \leqq \frac{1}{2}$, $t \geqq T_5$.

Because of (G^*) this is a contradiction and so the solution $(x_1(t), \ldots, x_n(t))$ belongs to the case (V) of Theorem 1.

Theorem 4. *Assume that* (A) *is superlinear,* $\beta = 1$ *and* (G_*) *is satisfied. If* (36), (37) *hold then every oscillatory solution* $(x_1(t), \ldots, x_n(t))$ *of* (A) *has the property from Case* (I) *of Theorem* 1.

Proof. Let $(x_1(t), \dots, x_n(t))$ be an oscillatory solution of (A) defined on $[t_0; \infty)$. Similarly as in the proof of Theorem 3 we can show that it must belong either to Case (I) or to Case (V) of Theorem 1. Let Case (V) occur .We can choose a $T_1 \geqq T$ such that $|x_1(\tau_1(t))| \leqq 1$, $P_i^1(\tau_1(t)) \geqq 1$ for $t \geqq T_1$. Then (36) implies

$$|\int_{T_1}^{\infty} f(s, x_1(\tau_1(s)))\, ds| \leqq \int_{T_1}^{\infty} P_1^n(s)\, \omega(s, 1)\, ds \leqq \int_{T_1}^{\infty} P_1^n(s)\, \omega(s, P_{2n-1}^1(\tau_1(s)))\, ds < \infty ;$$

$$\left|\int_{T_1}^{\infty} \frac{1}{p_n(s)} \int_s^{\infty} f(u, x_1(\tau_1(u)))\, du\, ds\right| \leqq \int_{T_1}^{\infty} P_1^n(s)\, \omega(s, P_{2n-1}^1(\tau_1(s)))\, ds < \infty .$$

Using the fact that $x_n(t)$, $x_n'(t)$ are oscillatory we obtain from the last equation of (A)

$$p_n(t)\, x_n'(t) = - \int_t^{\infty} f(s, x_1(\tau_1(s)))\, ds$$

because

$$p_n(T_1)\, x_n'(T_1) + \int_{T_1}^{\infty} f(s, x_1(\tau_1(s)))\, ds = 0 ,$$

$t \geqq T_1$, and consequently

$$x_n(t) = \int_t^{\infty} \frac{1}{p_n(s)} \int_s^{\infty} f(u, x_1(\tau_1(u)))\, du\, ds$$

(48) because

$$x_n(T_1) - \int_{T_1}^{\infty} \frac{1}{p_n(s)} \int_s^{\infty} f(u, x_1(\tau_1(u)))\, du\, ds = 0 ,$$

$t \geqq T_1$. Again, (37) (with $k = 2, 3, \dots, 2n - 1$) implies that

$$\left|\int_{T_1}^{\infty} Q_{k-2}(u, T_1)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du\right| \leqq$$

$$\leqq K \int_{T_1}^{\infty} Q_{k-2}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^{\infty} \omega(s, P_{2n-k}^1(\tau_1(s)))\, P_1^n(s)\, ds\, du < \infty .\right.$$

Analogously as above, taking into account that the components $x_i(t)$, $x_i'(t)$, $i = 1, \dots, n - 1$ are oscillatory we get

$$(49) \qquad x_i(t) = \int_t^{\infty} \frac{1}{p_i(s)} \int_s^{\infty} a_i(u)\, x_{i+1}(u)\, du\, ds , \quad i = 1, \dots, n - 2 ,$$

$$(50) \qquad x_{n-1}(t) = \int_t^{\infty} \frac{1}{p_{n-1}(s)} \int_s^{\infty} a_{n-1}(u)\, g(x_n(\tau_n(u)))\, du\, ds , \quad t \geqq T_1 .$$

Combining (48)–(50) we have

$$x_1(t) = \int_t^{\infty} Q_{2n-3}(u, t)\, a_{n-1}(u)\, g\left(\int_{\tau_n(u)}^{\infty} \frac{1}{p_n(v)} \int_v^{\infty} f(s, x_1(\tau_1(s)))\, ds\, dv\right) du$$

or

$$(51) \quad |x_1(t)| \leqq K \int_t^{\infty} Q_{2n-3}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^{\infty} \omega(s, |x_1(\tau_1(s))|)\, P_1^n(s)\, ds\right) du , \quad t \geqq T_1 .$$

Now we define $v(t) = \sup_{s \geqq t} |x_1(s)|$, $t \geqq T_2$ and choose T_2, T_3 such that $T_1 < T_2 < T_3$, $h_*(T_3) \geqq T_2$, $|x_1(t)| \leqq 1$ for $t \geqq T_2$. Using the superlinearity of (A) and the properties of $v(t)$ we find from (51)

$$v(t) \leqq v(h_*(t)) \int_{t_1}^{\infty} Q_{2n-3}(u)\, a_{n-1}(u) \left(\int_{\tau_n(u)}^{\infty} \omega(s, 1)\, P_1^n(s)\, \mathrm{d}s\right) \mathrm{d}u\,. \tag{52}$$

This is a contradiction because the right-hand side of (52) tends to zero as $t \to \infty$ while the left-hand side equals to 1 along a sequence diverging to infinity by (G_*). It follows that Case (V) of Theorem 1 is the only possibility.

Theorems 1–4 extend the results of Kitamura, Kusano [1]. If (A) is equivalent to a differential equation of order $2n$ with deviating arguments the theorems yield the results proved in [4].

References

[1] *Y. Kitamura, T. Kusano:* Asymptotic properties of solutions of two-dimensional differential systems with deviating argument, Hiroshima Math. J. *8* (1978), 305–326.

[2] *Y. Kitamura, T. Kusano:* On the oscillation of a class of nonlinear differential systems with deviating argument, J. Math. Anal. Appl. *66* (1978), 20–36.

[3] *Y. Kitamura, T. Kusano:* Oscillation and a class of nonlinear differential systems with general deviating arguments, Nonlinear Analysis, Theory, Methods and Appl. Vol. *2*, No. 5 (1978), 337–351.

[4] *Y. Kitamura, T. Kusano, M. Naito:* Asymptotic properties of solutions of n-th order differential equations with deviating argument, Proc. Japan Acad., Vol. *54*, Ser. A, No. 1 (1978), 13–16.

[5] *P. Marušiak:* On the oscillation of nonlinear differential systems with retarded arguments, Math. Slovaca *34*, No. 1 (1984), 73–88.

[6] *M. Švec:* Behaviour of nonoscillatory solutions of some nonlinear differential equations, Acta Math. Univ. Comen. *XXXIX* (1980), 115–129.

Author's address: 041 54 Košice, Jesenná 5, Czechoslovakia (Prírodovedecká fakulta UPJŠ).