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ITERATIVE PROCESSES AND OPEN MAPPING THEOREMS

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1. INTRODUCTION

The different kinds of open mapping and closed graph theorems play central role in the implicit function theorem, existence of solution of nonlinear systems, optimization and so on ([1]–[4] and their references). These theorems represent abstract forms of approximation processes in some sense, so in the heart of their proofs are iterative and induction methods. The most familiar iterative method, which is used in some situations, is the Banach iteration. Dmitruk, Miljutin and Osmolovskij [1] give a very nice collection of applications of a type of open mapping theorems and they emphasize that the iteration used by them is different from the Banach's one. They call their induction process *Ljusternik iteration*. Obtaining a theorem of the closed graph type Pták [3, 4] has found a simple but very applicable theorem about families of sets in complete metric spaces which gives the abstract background of many results containing iterative processes in existence proofs. We shall call this theorem or the iterative process represented by it *Pták iteration*. The aim of the present paper is to show that the iterative and induction process of Pták is stronger than the Ljusternik iteration in the sense that open mapping theorems of [1] can be proved by a suitable form of Pták iteration.

In the section 2 we deal with a slight modification of Pták's theorem and so called *uniform open mapping theorems* will be deduced from it. The main part of the paper is the section 3. Here we give such a form of Pták's theorem which uses the notion of the system introduced by Miljutin [1]. This theorem yields open mapping theorems which have non-uniform character. Along the way known theorems are proved and new assertions are obtained. The methods of proofs are simple and elementary and we shall use some ideas of Pták [3, 4].

V. Pták kindly informed me of his paper [5] after the reading of this article and I discovered that our Theorem 3.1 can be found in [5] in a little different form. Recently a systematic treatment of applications of the Nondiscrete Induction Principle has been published [6].

We are grateful to professor Pták for his helpful suggestions after reading of the first version of our paper.

2. NOTATIONS

Now we introduce some notations. $\boldsymbol{R}, \boldsymbol{R}_+, \boldsymbol{R}_+^0$ denotes, respectively, the real, nonnegative real, positive real numbers. If X is a metric space with the metric d, A is a non-void subset of X and $r > 0$, the $B(A, r)$ stands for the open ball around A with radius r i.e. $B(A, r) = \{x \in X: d(x, A) < r\}$. By $\mathscr{P}(X)$ we denote the set of all subsets of X. If Y is another metric space, then a relation P from X to Y is a subset of $X \times Y$, P is closed relation if it is closed subset of the product space $X \times Y$. We write $y \in Px$ and $x \in P^{-1}y$ iff $(x, y) \in P$. Similarly for $A \subseteq X$ and $B \subseteq Y$ we write

$$P(A) = \bigcup\{Pa: a \in A\} \quad \text{and} \quad P^{-1}(B) = \bigcup\{P^{-1}b: b \in B\} .$$

By $R(P)$ we denote the range of the relation or function. Let $\{A_n\}$ be a sequence of subsets of X, the limit inferior of the sequence is defined by $\liminf_{n \to +\infty} A_n = \{x \in X: \exists x_n \in A_n, x_n \to x\}$. If F is a function from X to $\mathscr{P}(Y)$ then the subset $\{y \in Y: \exists x_n \to \bar{x}, y_n \in F(x_n), y_n \to y\}$ is called the *limit superior of the function* F at $\bar{x}$. The composition of function will be denoted by $\circ$, and f^n denotes the n-th iterate of the function, f^0 is the identity mapping by convention.

3. UNIFORM CASE

In this section we deal with a slight generalization of the induction theorem of Pták [3, 4], which yields a uniform open-mapping theorem. Among the corollaries it can be found the closed graph theorem of Pták [3]. The first theorem is a discrete form of the induction principle and it can be considered as a lemma for the second theorem of the section, but it seems to be interesting in itself.

Theorem 3.1. *Let X be a complete metric space. Suppose that the sequence of nonnegative numbers $\{\tau_n\}, \{\varrho_n\}$ and the mapping $Z: \{\tau_n\} \to \mathscr{P}(X)$ satisfy the assumptions*

(1) $$\tau_n \to 0 , \quad \sum_{n=0}^{\infty} \varrho_n < +\infty ,$$

and

(2) $$Z(\tau_n) \subseteq B(Z(\tau_{n+1}), \varrho_n) \quad \textit{for all} \quad n .$$

Then the inclusion

(3) $$Z(\tau_0) \subseteq B(\liminf_{n \to \infty} Z(\tau_n), \sum_{n=0}^{\infty} \varrho_n)$$

holds.

Proof. Let $x_0 \in Z(\tau_0)$. Since $Z(\tau_0) \subseteq B(Z(\tau_1), \varrho_0)$, there is a point x_1 such that $x_1 \in Z(\tau_1)$ and $d(x_0, x_1) < \varrho_0$. Similarly by $Z(\tau_1) \subseteq B(Z(\tau_2), \varrho_1)$ we have got an x_2 such that $x_2 \in Z(\tau_2)$ and $d(x_1, x_2) < \varrho_1$. Continuing this process we have a sequence $\{x_n\}$ such that

$$x_n \in Z(\tau_n) \quad \text{and} \quad d(x_{n-1}, x_n) < \varrho_{n-1}$$

for $n = 1, 2, \ldots$. That gives at once

$$d(x_n, x_{n+m}) \leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+m}) < \sum_{j=n}^{m+n} \varrho_j$$

thus by (1) the sequence $\{x_n\}$ is fundamental and by completeness of X converges to a limit x_∞ and $d(x_0, x) < \sum_{j=0}^{\infty} \varrho_j$. The inclusion $x_\infty \in \liminf_{n\to\infty} Z(\tau_n)$ is evident by (1) and the definition of the limit inferior, hence (3) follows at once. ∎

Theorem 3.2. *Let X be a complete metric space. Let t_0 be a positive number and α and γ be mappings from $(0, t_0)$ to R_+. Suppose that the inverse mapping of α α^{-1} exists and β is a mapping from $(0, t_0)$ to $R(\alpha)$. If a mapping Z: $R(\alpha) \cup \cup R(\beta) \to \mathscr{P}(X)$ and the mappings α, β, γ satisfy the assumptions*

(1) $$\lim (\alpha^{-1} \circ \beta)^n (t) = 0\,, \quad \alpha(t) \to 0 \textit{ if } t \to 0\,,$$

(2) $$\sigma(t) \doteq \sum_{n=0}^{\infty} \gamma_0(\alpha^{-1} \circ \beta)^n (t) < +\infty\,,$$

and

(3) $$Z(\alpha(t)) \subseteq B(Z(\beta(t), \gamma(t))$$

for all $t \in (0, t_0)$, then the inclusion

(4) $$Z(\alpha(t)) \subseteq B(\limsup_{h\to+0} Z(h), \sigma(t))$$

holds for all $t \in (0, t_0)$.

The existence of α^{-1} is not an essential assumption, since an arbitrary right inverse of α: $(0, t_0) \to R(\alpha)$ (which always exists from $R(\alpha)$ to $(0, t_0)$ would serve the purpose.

Proof. Let $t \in (0, t_0)$ and $x_0 \in Z(\alpha(t))$. Taking $(\alpha^{-1} \circ \beta)^n (t)$ in (3) instead of t we have the inclusion

$$Z \circ \alpha((\alpha^{-1} \circ \beta)^n (t)) \subseteq B(Z \circ \alpha((\alpha^{-1} \circ \beta)^{n+1}(t)), \gamma \circ (\alpha^{-1} \circ \beta)^n (t))$$

for all n. From this and by (1) and (2) the sequences

$$\tau_n = (\alpha^{-1} \circ \beta)^n (t)\,, \quad \varrho_n = \gamma \circ (\alpha^{-1} \circ \beta)^n (t)$$

and the mapping $Z \circ \alpha$ satisfy the assumptions of Theorem 2.1, thus the inclusion

$$Z(\alpha(t)) \subseteq B(\liminf_{n\to+\infty} Z \circ \alpha(\tau_n), \sigma(t))$$

holds. Since by (1) $\liminf_{u\to+\infty} Z(\tau_n) \subseteq \limsup_{h\to0} Z(h)$, the inclusion (4) is proved. ∎

Now we assert a uniform open mapping theorem.

Theorem 3.3. *Let t_0 be a positive number and α and γ be mappings from $(0, t_0)$ to R_+. Suppose the inverse mapping of α α^{-1} exists and β is a mapping from $(0, t_0)$ to $R(\alpha)$. Let P be a close relation in the product $X \times Y$ of metric spaces X and Y, where X is supposed complete.*

If the relation P and the mappings α, β, γ satisfy the assumptions

(1) $$\lim (\alpha^{-1} \circ \beta)^n (t) = 0\,, \quad \alpha(t) \to 0 \text{ if } t \to 0\,,$$

(2) $$\sigma(t) \doteq \sum_{n=0}^{\infty} \gamma \circ (\alpha^{-1} \circ \beta)^n (t) < +\infty$$

and

(3) $$B(Px, \alpha(t)) \subseteq B(P(B(x, \gamma(t)), \beta(t))$$

for all $t \in (0, t_0)$*, then the inclusion*

(4) $$B(Px, \alpha(t)) \subseteq P(B(x, \sigma(t)))$$

holds for all $t \in (0, t_0)$.

The remark made after the Theorem 3.2 concerning the existence of the inverse of α is valid here, too.

Proof. Let $t_1 \in (0, t_0)$ and $y \in B(Px, \alpha(t_1))$. Since $B(y, \alpha(t_1)) \cap Px \neq \emptyset$ is equivalent to $x \in P^{-1}(B(y, \alpha(t_1)),\ y \in B(Px, \alpha(t_1))$ if and only if $x \in P^{-1}(B(y, \alpha(t_1)))$.

Define now the mapping $Z\colon R(\alpha) \cup R(\beta) \to \mathscr{P}(X)$ by the following way

(3.1) $$Z(t) = P^{-1}(B(y, t)) \quad \text{for all} \quad t \in R(\alpha) \cup R(\beta)\,.$$

According to this definition $y \in B(Px, \alpha(t_1))$ is equivalent to

(3.2) $$x \in Z(\alpha(t_1))$$

If $x \in Z(\alpha(t)) = P^{-1}(B(y, \alpha(t))$ $(t \in (0, t_0))$ then $Px \cap B(y, \alpha(t)) \neq \emptyset$ thus $y \in B(Px, \alpha(t))$. Hence using (3) we have $y \in B(P(B(x, \gamma(t)), \beta(t))$, which implies

$$B(y, \beta(t)) \cap P(B(x, \gamma(t)) \neq \emptyset \quad \text{i.e.} \quad P^{-1}(B(y, \beta(t)) \cap B(x, \gamma(t)) \neq \emptyset\,,$$

so we get finally $x \in B(P^{-1}(B(y, \beta(t)), \gamma(t)) = B(Z(\beta(t)), \gamma(t))$. Hence we have

(3.3) $$Z(\alpha(t)) \subseteq B(Z(\beta(t)), \gamma(t)) \quad \text{for all} \quad t \in (0, t_0)\,.$$

This inclusion and assumptions (1) and (2) give that Z, α, β, γ satisfy the conditions of Theorem 3.2, and so the inclusion

(3.4) $$Z(\alpha(t)) \subseteq B(\limsup_{h \to +0} Z(h), \sigma(t))$$

holds.

We now show that $\limsup_{h \to +0} Z(h) \subseteq P^{-1}y$. (Actually equality holds but it is unnecessary to us.) Indeed, if $u \in \limsup_{h \to +0} Z(h)$, then there are sequences $\{h_n\}$, $\{u_n\}$ such that $h_n \to +0$, $u_n \in Z(h_n)$ and $u_n \to u$. By $u_n \in Z(h_n) = P^{-1}(B(y, h_n))$ we have $Pu_n \cap B(y, h_n) \neq \emptyset$, thus there is a sequence $\{y_n\}$ such that $y_n \in Pu_n$ and $y_n \to y$. Since P is closed this gives $y \in Pu$ i.e. $u \in P^{-1}y$, which was asserted.

Using the previous observation and (3.4) we have

$$Z(\alpha(t)) \subseteq B(P^{-1}y, \sigma(t))\,.$$

Applying this inclusion for $t = t_1$ by (3.2) we get $x \in B(P^{-1}y, \sigma(t_1))$, which is equivalent to $y \in P(B(x, \sigma(t_1))$, hence the inclusion (4) holds for all $t_1 \in (0, t_0)$. ■

Take the mappings α, β, γ in the previous theorem $\beta(t) = \alpha(\varepsilon t)$ and $\gamma(t) = t$, where $0 < \varepsilon < 1$. Then we get the following

Corollary 3.4. *Let X be a complete metric space and Y be a metric space. Suppose that t_0 is a positive number, α is an invertible mapping from $(0, t_0)$ to R_+ and $\varepsilon \in (0, 1)$. If a closed relation P in $X \times Y$ satisfies the assumption*

$$(1) \qquad B(Px, \alpha(t)) \subseteq B(P(B(x, t)), \alpha(\varepsilon t))$$

for all $t \in (0, t_0)$, then the inclusion

$$(2) \qquad B(Px, \alpha(t)) \subseteq P(B(x, t/(1 - \varepsilon))$$

holds for all $t \in (0, t_0)$, provided that $\alpha(t) \to 0$ if $\alpha \to 0$.

Since $\mathrm{cl}\,(P(B(x, t)) = \bigcap_{\delta > 0} B(P(B(x, t)), \delta)$, the previous theorem is a generalization of the following one.

Corollary 3.5. (*Closed-graph theorem of Pták* [3]. *Let X be a complete metric space and Y be a metric space. Suppose that t_0 is a positive number and P is a closed relation in $X \times Y$. If there is an invertible mapping $\alpha: (0, t_0) \to R^+$ such that*

$$(1) \qquad B(Px, \alpha(t)) \subseteq \mathrm{cl}(P(B(x, t))$$

for all $t \in (0, t_0)$, then the inclusion

$$(2) \qquad B(Px, \alpha(t)) \subseteq P(B(x, t'))$$

holds for all $t, t' \in (0, t_0)$ and $t < t'$.

4. NON-UNIFORM CASE

First we introduce the notion of the total system in a metric space, following [1].

Definition 4.1. *Let X be a metric space with the metric d, and let $\mathscr{A}$ be a relation in the product $X \times R^0_+$. We shall say that $\mathscr{A}$ is a total system for X, if $(x, \varrho) \in \mathscr{A}$, $(x', \varrho') \in X \times R^0_+$ and $d(x, x') + \varrho' \leqq \varrho$ imply $(x', \varrho') \in \mathscr{A}$.*

Let S be a subset of X. The most important example of the total system is the family

$$\{(x, \varrho) \in X \times R^0_+ : B(x, \varrho) \subseteq S\},$$

which will be denoted by $T(S)$.

Now we can assert a non-uniform version of the iteration process of Pták.

Theorem 4.1. *Let X be a complete metric space and let $\mathscr{A}$ be a total system for X. Suppose that the sequences of nonnegative numbers $\{\tau_n\}$, $\{\varrho_n\}$ and the mapping $Z: \{\tau_n\} \to \mathscr{P}(X)$ satisfy the assumptions*

$$(1) \qquad \tau_n \to 0$$

$$(2) \qquad \varrho_n + \tau_{n+1} \leqq \tau_n \quad \textit{for} \quad n = 0, 1, 2, \ldots,$$

and

(3) $$Z(\tau_n) \cap \{x \in X: (x, \tau_n) \in \mathscr{A}\} \subseteq B(Z(\tau_{n+1}), \varrho_n)$$

for all n. Then the inclusion

(4) $$Z(\tau_0) \cap \{x \in X: (x, \tau_0) \in \mathscr{A}\} \subseteq B(\liminf Z(\tau_n) \cap \{x: (x, \tau_n) \in \mathscr{A}\}, \tau_0)$$

holds.

Proof. Let $x_0 \in Z(\tau_0)$ and $(x_0, \tau_0) \in \mathscr{A}$. Using (3) with $n = 0$, there exists a point $x_1 \in Z(\tau_1)$ such that $d(x_0, x_1) < \varrho_0$, thus by (2) we have $d(x_0, x_1) + \tau_1 \leqq \tau_0$ and so $(x_1, \tau_1) \in \mathscr{A}$. Similarly from $x_1 \in Z(\tau_1)$, $(x_1, \tau_1) \in \mathscr{A}$ and by (3) we have got a point $x_2 \in Z(\tau_2)$ with $d(x_1, x_2) < \varrho_1$, thus by (2) $d(x_1, x_2) + \tau_2 \leqq \tau_1$ so that $(x_2, \tau_2) \in \mathscr{A}$. Continuing this process we have a sequence $\{x_n\}$ such that

(4.1) $$x_n \in Z(\tau_n), \quad d(x_{n-1}, x_n) < \varrho_{n-1} \quad \text{and} \quad (x_n, \tau_n) \in \mathscr{A} \quad \text{for} \quad n = 1, 2, \dots .$$

This gives at once

$$d(x_n, x_{n+m}) \leqq d(x_n, x_{n+1}) + \dots + d(x_{n+m-1}, x_{n+m}) < \sum_{j=n}^{m+n} \varrho_j .$$

From (2) we get easily $\sum_{j=n}^{m+n} \varrho_j \leqq \tau_n - \tau_{n+m+1}$, thus the sequence $\{x_n\}$ is fundamental by (1), and so using the completeness of X it converges to a limit x_∞ and $d(x_0, x_\infty) < \tau_0$. The relation $x_\infty \in \liminf Z(\tau_n) \cap \{x \in X: (x, \tau_n) \in \mathscr{A}\}$ follows from (4.1), (1) and the definition of the limit inferior. ■

Theorem 4.2. *Let X be a complete metric space and $\mathscr{A}$ be a total system for X. Let t_0 be a positive number and α and γ be mappings from $(0, t_0)$ to R_+. Suppose that the inverse mapping of α α^{-1} exists and β is a mapping from $(0, t_0)$ to $R(\alpha)$. If a mapping $Z: R(\alpha) \cup R(\beta) \to \mathscr{P}(X)$ and the mappings α, β, γ satisfy the assumptions*

(1) $$(\alpha^{-1} \circ \beta)^n (t) \to 0, \quad \alpha(t) \to 0 \text{ if } t \to 0,$$

(2) $$\gamma \circ (\alpha^{-1} \circ \beta)^n (t) + (\alpha^{-1} \circ \beta)^{n+1}(t) \leqq (\alpha^{-1} \circ \beta)^n (t),$$

and

(3) $$Z(\alpha(t)) \cap \{x \in X: (x, t) \in \mathscr{A}\} \subseteq B(Z(\beta(t)), \gamma(t))$$

for all $t \in (0, t_0)$, then the inclusion

(4) $$Z(\alpha(t)) \cap \{x \in X: (x, t) \in \mathscr{A}\} \subseteq B(\limsup_{h \to +0} (Z(h) \cap \{x: (x, h) \in \mathscr{A}\}), t)$$

holds for all $t \in (0, t_0)$.

It would be sufficient to assume that α, β, γ are defined only on the set $\{t: \exists x \in X$ $(x, t) \in \mathscr{A}\}$. The invertability of α is not necessary, since α^{-1} can be choose a right inverse to α.

Proof. Let $(x_0, t) \in \mathscr{A}$, $t \in (0, t_0)$ and $x_0 \in Z(t)$. Taking $(\alpha^{-1} \circ \beta)^n (t)$ in (3) in the place of t we get

$$Z \circ \alpha((\alpha^{-1} \circ \beta)^n (t)) \cap \{x: (x, (\alpha^{-1} \circ \beta)^n (t) \in \mathscr{A}\} \subseteq$$
$$\subseteq B(Z \circ \alpha(\alpha^{-1} \circ \beta)^{n+1} (t)), \quad \gamma \circ (\alpha^{-1} \circ \beta)^n (t))),$$

for $n = 0, 1, 2, \ldots$. Hence the sequences

$$\tau_n = (\alpha^{-1} \circ \beta)^n (t) \quad \text{and} \quad \varrho_n = \gamma \circ (\alpha^{-1} \circ \beta)^n (t)$$

and the mapping $Z \circ \alpha: \{\tau_n\} \to \mathscr{P}(X)$ satisfy the assumptions of Theorem 4.1, so that we have the inclusion

$$Z(\alpha(t)) \cap \{x: (x, t) \in \mathscr{A}\} \subseteq B(\liminf_{n \to +\infty} (Z \circ \alpha(\tau_n) \cap \{x: (x, \tau_n) \in \mathscr{A}\}), t),$$

which gives (4) immediately. ■

Next we turn to investigate non-uniform open mapping theorems.

Theorem 4.3. *Let X and Y be metric spaces, $\mathscr{A}$ be a total system for X and suppose that X is complete. Let t_0 be a positive number and α and γ be mappings from $(0, t_0)$ to R_+. Suppose that the mapping α is invertible and β is a mapping from $(0, t_0)$ to $R(\alpha)$. If a closed relation P in $X \times Y$ and the mappings α, β, γ satisfy the assumptions*

(1) $$\lim_{n \to +\infty} (\alpha^{-1} \circ \beta)^n (t) \to 0, \quad \alpha(t) \to 0 \text{ if } t \to 0,$$

(2) $$\gamma \circ (\alpha^{-1} \circ \beta)^n (t) + (\alpha^{-1} \circ \beta)^{n+1}(t) \leqq (\alpha^{-1} \circ \beta)^n (t),$$

and

(3) $$B(Px, \alpha(t)) \subseteq B(P(B(x, \gamma(t)), \beta(t))$$

for all $(x, t) \in \mathscr{A}$, $t \in (0, t_0)$, then the inclusion

(4) $$B(Px, \alpha(t)) \subseteq P(B(x, t))$$

holds for all $(x, t) \in \mathscr{A}$, $t \in (0, t_0)$.

The remark made after Theorem 4.2 concerning the existence of α^{-1} is valid here, too.

Proof. Let $(\bar{x}, \bar{t}) \in \mathscr{A}$, $\bar{t} \in (0, t_0)$ and $y \in B(Px, \alpha(\bar{t}))$. It is easy to see, as at the beginning of the proof of Theorem 3.3, that these conditions coincide with

(4.2) $$x \in P^{-1}(B(y, \bar{t})), \quad \bar{t} \in (0, t_0) \quad \text{and} \quad (x, \bar{t}) \in \mathscr{A}.$$

Define now the mapping $Z: R(\alpha) \cup R(\beta) \to \mathscr{P}(X)$ by the following way

(4.3) $$Z(t) = P^{-1}(B(y, t)) \quad \text{for all} \quad t \in R(\alpha) \cup R(\beta).$$

If $x \in Z(\alpha(t)) = P^{-1}(B(y, \alpha(t)))$, $t \in (0, t_0)$ and $(x, t) \in \mathscr{A}$, then $y \in B(Px, \alpha(t))$ thus by (3) $y \in B(P(B(x, \gamma(t)), \beta(t))$, which gives as in proof of Theorem 3.3

$$x \in B(P^{-1}(B(y, \beta(t)), \gamma(t)) = B(Z(\beta(t)), \gamma(t)).$$

Hence we have

(4.4) $$Z(\alpha(t)) \cap \{x: (x, t) \in \mathscr{A}\} \subseteq B(Z(\beta(t)), \gamma(t))$$

for all $t \in (0, t_0)$.

By (4.4), (1) and (2), the assumptions of the Theorem 4.2 are satisfied, thus the inclusion

(4.5) $$Z(\alpha(t)) \cap \{x: (x, t) \in \mathscr{A}\} \subseteq B(\limsup_{h\to+0} (Z(h) \cap \{x: (x, h) \in \mathscr{A}\}), t)$$

holds for all $t \in (0, t_0)$.

We now show that $\limsup_{h\to+0} (Z(h) \cap \{x: (x, h) \in \mathscr{A}\}) \subseteq P^{-1}y$.

Really if u is an element of the left-hand side, then there are sequences $\{h_n\}$, $\{u_n\}$ such that $h_n \to +0$, $u_n \to u$, $u_n \in Z(h_n)$ and $(u_n, h_n) \in \mathscr{A}$. By $u_n \in Z(h_n) = P^{-1}(B(y, h_n))$ we get $Pu_n \cap B(y, h_n) \neq \emptyset$, thus there is a sequence $\{y_n\}$ such that $y_n \in Pu_n$ and $y_n \to y$. Since P is closed this gives $y \in Pu$ i.e. $u \in P^{-1}y$. Hence by (4.5) we have

$$Z(\alpha(t)) \cap \{x: (x, t) \in \mathscr{A}\} \subseteq B(P^{-1}y, t)$$

for all $t \in (0, t_0)$.

Applying this inclusion for $t = \bar{t}$ by (4.2) we get $x \in B(P^{-1}y, \bar{t})$, which is equivalent to $y \in P(B(x, \bar{t}))$, hence summarizing we have finally

$$B(Px, \alpha(\bar{t})) \subseteq P(B(x, \bar{t})) \quad \text{for all} \quad (x, \bar{t}) \in \mathscr{A}, \quad \bar{t} \in (0, t_0),$$

which was to be proved. ■

We mention two special cases of the previous theorem, the following corollary is a theorem of Dmitruk [1, 2].

Corollary 4.4. *Let Y be a metric space and let X be a complete metric space and $\mathscr{A}$ a total system for X. Let $0 \leqq b < a$. If a closed relation P in $X \times Y$ satisfies the assumption*

(1) $$B(Px, at) \subseteq B(P(Bx, (1 - b/a)\, t), bt$$

for all $(x, t) \in \mathscr{A}$ and $t \in (0, t_0)$, then the inclusion

(2) $$B(Px, at) \subseteq P(B(x, t))$$

holds for all $(x, t) \in \mathscr{A}$ and $t \in (0, t_0)$.

In the Theorem of Dmitruk P is a function from X to Y so the Corollary is a slight generalization of his theorem. It is easy to see that Dmitruk's Theorem is the best possible consequence of the previous theorem provided that the mapping α, β, γ are products with any constants. The linear form of Dmitruk's Theorem can be found in [7] even in a stronger form.

Proof. Take $\alpha(t) = at$, $\beta(t) = bt$ and $\gamma(t) = ((a - b)/a)\, t$ in the previous theorem. ■

Choose $\beta(t) = \alpha(\varepsilon t)$ in the Theorem 4.3. The condition (1) is satisfied if $0 \leqq \varepsilon < 1$, the condition (2) holds if $\gamma(\tau) \leqq (1 - \varepsilon)\, \tau$, thus we have the following Corollary.

Corollary 4.5. *Let Y be a metric space, X be a complete metric space and $\mathscr{A}$ be a total system for X. Let t_0 be a positive number and α be an invertible mapping from $(0, t_0)$ to R_+. If P is a closed relation in $X \times Y$ and $0 \leqq \varepsilon < 1$, and the* *inclusion*

$$B(Px, \alpha(t)) \subseteq B(P(B(x, (1-\varepsilon)t), \alpha(\varepsilon t)) \tag{1}$$

holds for all $t \in (0, t_0)$ *and* $(x, t) \in \mathscr{A}$, *then the inclusion*

$$B(Px, \alpha(t)) \subseteq P(B(x, t)) \tag{2}$$

holds for all $t \in (0, t_0)$, $(x, t) \in \mathscr{A}$, *provided that* $\alpha(t) \to 0$ *if* $t \to 0$.

Similarly as in the previous section this yields a closed graph theorem closely related to Theorem of Pták [3].

Corollary 4.6. *Let* Y *be a metric space,* X *a complete metric space and* $\mathscr{A}$ *a total system for* X. *Let* t_0 *be a positive number and* α *be an invertible mapping from* $(0, t_0)$ *to* R_+. *If* P *is a closed relation in* $X \times Y$ *and* $0 \leqq \varepsilon < 1$, *then the inclusion*

$$B(Px, \alpha(t)) \subseteq \mathrm{cl}(P(B(x, (1-\varepsilon)t)) \quad \text{for all} \quad t \in (0, t_0), (x, t) \in \mathscr{A} \tag{1}$$

implies the inclusion

$$B(Px, \alpha(t)) \subseteq P(B(x, t)) \quad \text{for all} \quad t \in (0, t_0), (x, t) \in \mathscr{A}. \tag{2}$$

Finally we remark that the open mapping theorems have also discrete forms according to iteration processes (Theorem 3.1,4.1).

References

[1] *A. V. Dmitruk, A. A. Miljutin* and *N. P. Osmolovski:* Theorem of Lusternik and Theory of Extremum, UMN *35* (1980), 11—46 (in Russian).

[2] *A. D. Ioffe* and *V. M. Tihomirov:* The Theory of Extremal Problems, Nauka, Moskow (1974) (in Russian).

[3] *V. Pták:* A Quantitative refinement of the Closed Graph Theorem, Czechoslovak Math. J. *24* (1974), 503—506.

[4] *V. Pták:* Nondiscrete Mathematical Induction and Iterative Existence Proofs, Linear Algebra and its Applications *13* (1976), 223—238.

[5] *V. Pták:* A Nonlinear Subtraction Theorem, Proc. R. Ir. Acad. Vol. *82A*, No. 1, 47—53 (1982).

[6] *F. A. Potra* and *V. Pták:* Nondiscrete induction and iterative processes, Boston etc., Pitman Advanced Publishing Program, 1984.

[7] *V. Pták:* Some metric aspects of the open mapping theorem, Math. Annalen *165* (1966), 95—104.

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