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REPRESENTATION OF MULTILINEAR OPERATORS ON $\mathsf{X}C_0(T_i)$

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INTRODUCTION

Let T_i, $i = 1, \ldots, d$, be locally compact Hausdorff spaces, and let $\mathsf{X}C_0(T_i)$ denote the Cartesian product $C_0(T_1) \times \ldots \times C_0(T_d)$, where $C_0(T_i)$, $i = 1, \ldots, d$, is the Banach space of all scalar $= K$-valued continuous functions on T_i tending to zero at infinity with the sup norm. In this paper we prove the Riesz (also the Bartle-Dunford-Schwartz) Representation Theorem type results for bounded d-linear operators U: $\mathsf{X}C_0(T_i) \to Y$ – a Banach space.

In the papers [12], [13] and [14] we already started developing an extension of the Lebesgue type integration to integration with respect to set functions of several variables – polymeasures. The bounded d-linear operators are represented, via this integration, either by separately countably additive Y-valued Baire d-polymeasures, see Theorems 2, 9 and 11, or by weak*-separately countably additive $Y = Z^*$, or Y^{**}-valued Baire d-polymeasures, see Theorems 4 and 5, respectively.

The representation theorems are easily derived from a deep result of A. Pelczyński from [32]. Not so easy was it to prove the Lebesgue bounded convergence result of Theorem 3, and the double limit characterization of Y-valuedness of the representing d-polymeasure given by Theorem 9.

The special case $d = 2$ was investigated in the papers [25]–[29], [35], [36] and [22]. The case of the Banach spaces of vector valued continuous functions $C_0(T_i, X_i)$ will be treated in [18]. We will freely use the notation from [12], [13] and [14], particularly the abbreviated notation.

1. OPERATOR VALUED BAIRE AND BOREL POLYMEASURES

Let T be a locally compact Hausdorff topological space. In accordance with our notation in [3], by $\mathscr{B}_0 = \delta(\mathscr{C}_0)$ we denote the δ-ring of all relatively compact Baire subsets of T. Similarly $\mathscr{B} = \delta(\mathscr{C})$ will denote the δ-ring of all relatively compact Borel subsets if T. The symbols $\sigma(\mathscr{B}_0)$ and $\sigma(\mathscr{B})$ stand for the σ-rings of Baire and Borel subsets of T, respectively.

We denote by $K(T)$ the linear space of all scalar valued continuous functions on T with compact support. Q will denote the set of all X valued continuous functions on T which are of the form $f = \sum_{j=1}^{r} \varphi_j x_j$, where $\varphi_j \in K(T)$ and $x_j \in X$, $j = 1, \ldots, r$. According to Proposition 1 in § 19 in [2] Q is dense in $C_0(T, X)$.

Let $m: \mathscr{B}_0 \to L(X, Y)$ be an operator valued Baire measure countably additive in the strong operator topology. By Theorem 1 in [3], $\hat{m}(E) = \sup\{|\int_E f \, dm|, f \in Q, \|f\|_E \leqq 1\}$ for each set $E \in \sigma(\mathscr{B}_0)$. Nonetheless, the proof given there needs a correction, since it may happen that $\|f\|_E > 1$ for the function f in that proof. We have $C_i \subset E_i \subset U_i$, $i = 1, \ldots, r$ at the top of page 16 in [3]. Since C_i, $i = 1, \ldots, r$ are pairwise disjoint compacts, there are pairwise disjoint open sets $U'_i \in \mathscr{B}_0$, $i = 1, \ldots, r$ such that $C_i \subset U'_i \subset U_i$ for each i. By virtue of Theorem B in § 51 in [20] there are functions $\varphi'_i \in K(T)$, $0 \leqq \varphi'_i \leqq 1$, $i = 1, \ldots, r$ such that $\varphi'_i(t) = 1$ for $t \in C_i$, and $\varphi'_i(t) = 0$ for $t \in T - U'_i$. Now $f' = \sum_{i=1}^{r} \varphi'_i x_i \in Q$ is such that $\|f'\|_E \leqq 1$ and $|\sum_{i=1}^{r} m(E_i) x_i| \leqq |\int_E f' \, dm| + \varepsilon$. This inequality implies that the proved equality holds also if $\hat{m}(E) = +\infty$.

Further let us note that if $m': \sigma(\mathscr{B}_0) \to L(X, Y)$ is countably additive in the strong operator topology and $m = m': \mathscr{B}_0 \to L(X, Y)$, then $\hat{m}'(E) = \hat{m}(E)$ for each $E \in \sigma(\mathscr{B}_0)$ by Theorem 14 in [4]. It is easy to verify that the above mentioned facts remain valid if $m: \mathscr{B} \to L(X, Y)$ and $m': \sigma(\mathscr{B}) \to L(X, Y)$ are additive Borel measures regular in the strong operator topology, hence also countably additive in this topology.

Now let T_i, $i = 1, \ldots, d$ be locally compact Hausdorff spaces with Baire (Borel) δ-ring $\mathscr{B}_{0,i}(\mathscr{B}_i)$, $i = 1, \ldots, d$. Let further $X_1, \ldots, X_d$ and Y be Banach spaces over the same scalar field. By $L^{(d)}(X_i; Y) = L^{(d)}(X_1, \ldots, X_d; Y)$ we denote the Banach space of all bounded d-linear operators $V: X_1 \times \ldots \times X_d \to Y$. There is a natural isometric isomorphism between the spaces $L^{(d)}(X_i; Y)$ and $L(X_1 \otimes^\wedge \ldots \otimes^\wedge X_d, Y)$, where $X_1 \otimes^\wedge \ldots \otimes^\wedge X_d$ is the completed projective tensor product, given by the equality $V(x_1, \ldots, x_d) = \dot{V}(x_1 \otimes^\wedge \ldots \otimes^\wedge x_d)$. We say that $V \in L^{(d)}(X_i; Y)$ is weakly compact, unconditionally converging, compact, etc., if $\dot{V}$ has the corresponding property, see [31].

Let Q_i, $i = 1, \ldots, d$ be the analog of Q for T_i and X_i, and let $\Gamma: \mathsf{X}\mathscr{B}_{0,i} \to L^{(d)}(X_i; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology, see [12]. From Theorem 8 in [5] and Theorem 2 in [13] we immediately obtain that $(f_i) \in I_1(\Gamma)$ if $(f_i) \in \mathsf{X}Q_i$. We now prove a generalization of Theorem 1 from [3], and Theorem 6 from [6].

Theorem 1. *Let $\Gamma: \mathsf{X}\mathscr{B}_{0,i} \to L^{(d)}(X_i; Y)$ be an operator valued Baire d-polymeasure separately countably additive in the strong operator topology. Then*

$$\hat{\Gamma}(A_i) = \sup\{|\int_{(A_i)} (f_i) \, d\Gamma|; \ (f_i) \in \mathsf{X}Q_i, \ \|f_i\|_{A_i} \leqq 1, \ i = 1, \ldots, d\}$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$, and

$$\hat{\Gamma}[(g_i), (A_i)] = \sup\{|\int_{(A_i)} (f_i) \, d\Gamma|; \ (f_i) \in \mathsf{X}Q_i, \text{ and } |f_i| \leqq |g_i|, \ i = 1, \ldots, d\}$$

for each $\mathscr{B}_{0,i}$*-measurable* $g_i\colon T_i \to X_i$ *(or* $g_i\colon T_i \to [0, +\infty]$*),* $i = 1, \ldots, d$*, and each* $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$*. By Theorem 4 in [13] the same equalities hold if* $\mathsf{X}\mathscr{B}_{0,i}$ *is replaced by* $\mathsf{X}\sigma(\mathscr{B}_{0,i})$*. These assertions remain valid if* $\mathscr{B}_{0,i}$ *is replaced by* $\mathscr{B}_i$*,* $i = 1, \ldots, d$*, and* Γ *is separately additive and regular in the strong operator topology.*

Proof. Let $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$ and let $\varepsilon > 0$. By Definition 3 in [12] $\hat{\Gamma}(A_i) = \sup \{|\int_{(A_i)} (g_i)\, d\Gamma|; (g_i) \in \mathsf{X}S(\mathscr{B}_{0,i}, X_i), \|g_i\|_{A_i} \leqq 1, i = 1, \ldots, d\}$, where $S(\mathscr{B}_{0,i}, X_i)$ denotes the linear space of all $\mathscr{B}_{0,i}$-simple X_i valued functions on T_i. Take $(g_i) \in \mathsf{X}S(\mathscr{B}_{0,i}, X_i)$ with $\|g_i\|_{A_i} \leqq 1$ for each $i = 1, \ldots, d$.

For $E_1 \in \mathscr{B}_{0,1}$ and $x_1 \in X_1$ put $m_1(E_1)\, x_1 = \int_{(E_1, A_2, \ldots, A_d)} (x_1 \cdot \chi_{E_1}, g_2, \ldots, g_d)\, d\Gamma$. Then $m_i\colon \mathscr{B}_{0,1} \to L(X_1, Y)$ is countably additive in the strong operator topology, and $\int_{(A_i)} (g_i)\, d\Gamma = \int_{A_1} g_1\, dm_1$. According to the proof of Theorem 1 in [3], see also the beginning of our proof above, there is an $f_1 \in Q_1$ with $\|f_1\|_{A_1}$ such that $|\int_{A_1} g_1\, dm_1| \leqq |\int_{A_1} f_1\, dm_1| + \varepsilon/d$. It is easy to verify that $\int_{A_1} f_1\, dm_1 = \int_{(A_i)} (f_1, g_2, \ldots, g_d)\, d\Gamma$.

For $E_2 \in \mathscr{B}_{0,2}$ and $x_2 \in X_2$ put $m_2(E_2)\, x_2 = \int_{(A_1, E_2, A_3, \ldots, A_d)} (f_1, x_2 \cdot \chi_{E_2}, g_3, \ldots, g_d)\, . d\Gamma$. Then there is again an $f_2 \in Q_2$ with $\|f_2\|_{A_2} \leqq 1$ such that $|\int_{A_2} g_2\, dm_2| \leqq |\int_{A_2} f_2\, dm_2| + \varepsilon/d$. Continuing in this way we obtain a d-tuple $(f_i) \in \mathsf{X}Q_i$ such that $\|f_i\|_{A_i} \leqq 1$ for each $i = 1, \ldots, d$, and

$$|\int_{(A_i)} (g_i)\, d\Gamma| \leqq |\int_{(A_i)} (f_i)\, d\Gamma| + \varepsilon\,.$$

From this inequality the equation with the semivariation $\hat{\Gamma}(A_i)$ is evident for both cases $\hat{\Gamma}(A_i) < +\infty$ and $\hat{\Gamma}(A_i) = \infty$. Since $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$ was arbitrary, the first assertion of the theorem is proved. The other assertions may be proved similarly. As we mentioned above, Theorem 1 from [3] is valid if $\mathscr{B}_0$ is replaced by $\mathscr{B}$, hence the last assertion of the theorem is evident. The theorem is proved.

2. REPRESENTATION THEOREMS

In accordance with [32] let $B^{(\Omega)}(T_i)$, $i = 1, \ldots, d$ denote the Banach space of all bounded scalar valued Baire measurable functions on T_i with the sup-norm. As this notation suggests, $B^{(\Omega)}(T_i)$ is the smallest class of bounded functions on T_i which contains $K(T_i)$ and is closed with respect to the pointwise convergence of bounded sequences of functions, see § 51 in [20] and Theorem 15 in [17]. In accordance with Definition on page 381 in [32] a sequence $f_{i,n} \in B^{(\Omega)}(T_i)$, $n = 1, 2, \ldots$, $i \in \{1, \ldots, d\}$ fixed, is said to be *ω*-convergent to a function* $f_i \in B^{(\Omega)}(T_i)$ provided $\sup_n \|f_{i,n}\|_{T_i} < +\infty$ and $\lim_{n\to\infty} f_{i,n}(t_i) = f_i(t_i)$ for any $t_i \in T_i$.

Our representation theorems are derived from the following basic result of A. Pełczyński, see Theorem 2 in [32], which obviously holds also for locally compact Hausdorff spaces T_i, $i = 1, \ldots, d$.

Theorem of A. Pelczyński. *Let* $U\colon \mathsf{X}C_0(T_i) \to Y$ *be a bounded d-linear operator and let us suppose that one of the following conditions is satisfied:*

(A) *no subspace of* Y *is isomorphic to the space* c_0;

(B) U *is weakly compact.*

Then there is a unique d-linear bounded operator $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y$ *such that*

1) U^{**} *is an extension of* U, *i.e.*, $U^{**}(f_i) = U(f_i)$ *for* $(f_i) \in \mathsf{X}C_0(T_i)$, *and*

2) *if* $g_{i,n}$, $n = 1, 2, \ldots, d$, $i = 1, \ldots, d$, *are* ω^**-convergent to* g_i *sequences of elements of* $B^{(\Omega)}(T_i)$, *then*

$$\lim_{n\to\infty} U^{**}(g_{i,n}) = U^{**}(g_i)\,.$$

Moreover, in the case (B) *the operator* U^{**} *is weakly compact.*

As a consequence we easily obtain

Theorem 2. *Let* $U\colon \mathsf{X}C_0(T_i) \to Y$ *be a bounded d-linear operator and suppose either* $c_0 \not\subset Y$, *or* U *is weakly compact. For* $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$ *put* $\gamma(A_i) = U^{**}(\chi_{A_i})$. *Then* $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y$ *is a separately countably additive vector Baire d-polymeasure. Further* $(g_i) \in I_1(\gamma) = I(\gamma)$, *and*

$$U^{**}(g_i) = \int_{(T_i)} (g_i)\,\mathrm{d}\gamma$$

for each $(g_i) \in \mathsf{X}B^{(\Omega)}(T_i)$, *in particular*

$$U(f_i) = \int_{(T_i)} (f_i)\,\mathrm{d}\gamma$$

for each $(f_i) \in \mathsf{X}C_0(T_i)$. *At the same time*

$$|U| = |U^{**}| = \|\gamma\|\,(T_i) = \sup_{|y^*|\leq 1} \|y^*\gamma\|\,(T_i)\,.$$

Moreover, the range of γ *is relatively weakly compact if and only if* U *is weakly compact.*

Proof. The separate countably additivity of γ is an easy consequence of assertion 2) of Theorem of A. Pelczyński.

Now let $(g_i) \in \mathsf{X}B^{(\Omega)}(T_i) = \mathsf{X}\bar{S}(\sigma(\mathscr{B}_{0,i}), K)$, and for each $i = 1, \ldots, d$ take a sequence $g_{i,n} \in S(\sigma(\mathscr{B}_{0,i}), K)$, $n = 1, 2, \ldots$ such that $\|g_{i,n} - g_i\|_{T_i} \to 0$. According to the Nikodym uniform boundedness theorem for polymeasures, see [12], we have $\|\gamma\|\,(T_i) < +\infty$. Hence by Theorem 1 and Definition 1 in [13], and assertion 2) of Theorem of A. Pelczyński we obtain

$$\int_{(T_i)} (g_i)\,\mathrm{d}\gamma = \lim_{n\to\infty} \int_{(T_i)} (g_{i,n})\,\mathrm{d}\gamma = \lim_{n\to\infty} U^{**}(g_{i,n}) = U^{**}(g_i)\,.$$

By Corollary of Theorem 5 in [14] we conclude that $I(\gamma) = I_1(\gamma)$.

The equality with norms follows from Theorem 1.

If U is weakly compact, then $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y$ is weakly compact by Theorem of A. Pelczyński, hence the range of γ is relatively weakly compact. Conversely, if the range of γ is relatively weakly compact, then using Krein-Šmuljan Theorem, see Theorem II.2.11 in [1], similarly as in the proof of Theorem VI.1.1 in [1] we obtain that $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y$ is weakly compact. The theorem is proved.

From Theorem 2 and from the elementary properties of the integral with respect to a polymeasure, see [13] and [14], we immediately obtain

Corollary. *There is an isometric isomorphism between the Banach space of all bounded d-linear functionals* $L^{(d)}(C_0(T_i); K)$ *and the Banach space of all separately countably additive d-polymeasures* $\mathrm{pm}\,(\mathsf{X}\sigma(\mathscr{B}_{0,i}), K)$ *with the norm* $\gamma \to \|\gamma\|\,(T_i)$, *given by the equalities*

$$V(f_i) = \int_{(T_i)} (f_i)\,\mathrm{d}\gamma\,, \quad (f_i) \in \mathsf{X}C_0(T_i)\,, \quad \text{and} \quad |V| = \|\gamma\|\,(T_i)\,.$$

If $U\colon \mathsf{X}C_0(T_i) \to Y$ is a bounded d-linear operator and either $c_0 \not\subset Y$ or U is weakly compact, then assertion 2) of Theorem of A. Pelczyński implies via Theorem 2 a Lebesgue Bounded Convergence Theorem type result for the integral with respect to the representing d-polymeasure γ of U. We prove in Theorem 3 below that for the integral of d-tuples of scalar valued functions with respect to arbitrary separately countably additive vector d-polymeasure this Lebesgue Bounded Convergence Theorem holds. Hence for any bounded d-linear operator $U\colon \mathsf{X}C_0(T_i) \to Y$ which can be represented by a separately countably additive $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y$, the assertions of Theorem of A. Pelcyński hold.

First we introduce a useful notion.

Definition 1. Let $T_i \neq \emptyset$, $i = 1, \ldots, d$ be arbitrary sets, let $\mathscr{S}_i \subset 2^{T_i}$ be σ-rings, and let $\gamma\colon \mathsf{X}\mathscr{S}_i \to Y$ be separately additive. Let further $g_i, g_{i,n}\colon T_i \to K$, $n = 1, 2, \ldots$ be $\mathscr{S}_i$-measurable for each $i = 1, \ldots, d$. We say that the d-tuples $(g_{i,n})$, $n = 1, 2, \ldots$, *$\mathsf{X}\omega^*$-converge to the d-tuple (g_i) γ-almost everywhere* if there are sets $N_i \in \mathscr{S}_i$, $i = 1, \ldots, d$ such that $\bar{\gamma}(\ldots, T_{i-1}, N_i, T_{i+1}, \ldots) = 0$ and the sequence $g_{i,n} \cdot \chi_{T_i - N_i}$ $n = 1, 2, \ldots$ ω^*-converges to the function $g_i \cdot \chi_{T_i - N_i}$ for each $i = 1, \ldots, d$.

Theorem 3. *Let $T_i \neq \emptyset$, $i = 1, \ldots, d$ be arbitrary sets, let $\mathscr{S}_i \subset 2^{T_i}$, $i = 1, \ldots, d$ be σ-rings and let $\gamma\colon \mathsf{X}\mathscr{S}_i \to Y$ be a separately countably additive vector d-polymeasure. Let further $g_i, g_{i,n}\colon T_i \to K$, $n = 1, 2, \ldots$ be bounded $\mathscr{S}_i$-measurable functions for each $i = 1, \ldots, d$, and let the sequence of d-tuples $(g_{i,n})$, $n = 1, 2, \ldots$, $\mathsf{X}\omega^*$-converge to the d-tuple (g_i) γ-almost everywhere. Then (g_i), $(g_{i,n}) \in I_1(\gamma) = I(\gamma)$, and*

$$(1) \qquad \lim_{n_1, \ldots, n_d \to \infty} \int_{(A_i)} (g_{i,n_i})\,\mathrm{d}\gamma = \int_{(A_i)} (g_i)\,\mathrm{d}\gamma$$

for each $(A_i) \in \mathsf{X}\mathscr{S}_i$. If in each of $(d-1)$ coordinates either γ is uniformly countably additive or the sequence $g_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to the function g_i, then the limit in (1) is uniform with respect to $(A_i) \in \mathsf{X}\mathscr{S}_i$.

Proof. First note that $\|\gamma\|(T_i) < +\infty$ by the Nikodým uniform boundedness theorem for polymeasures, see (N) in [12]. Since (g_i), $(g_{i,n}) \in \mathsf{X}\bar{S}(\mathscr{S}_i, K)$, we have (g_i), $(g_{i,n}) \in I_1(\gamma)$ by Theorem 2 in [13]. Further, $I_1(\gamma) = I(\gamma)$ by Corollary of Theorem 5 in [14]. According to Theorems 2 and 3 in [12],

$$\lim_{n_1,\ldots,n_d\to\infty} \|\gamma\|(A_{i,n_i}) = \lim_{n_1,\ldots,n_d\to\infty} \bar{\gamma}(A_{i,n_i}) = 0 \tag{2}$$

whenever $A_{i,n} \in \mathscr{S}_i$, $n = 1, 2, \ldots$ and $A_{i,n} \to \emptyset$ for each $i = 1, \ldots, d$. Without loss of generality we may suppose that $(g_{i,n})$, $n = 1, \ldots$ is $\mathsf{X}\omega^*$-convergent to (g_i) everywhere. However, then by the definition of ω^*-convergence there is a constant $C > 0$ such that $|g_{i,n}(t_i)| \leqq C$ for each $i = 1, \ldots, d$, each $n = 1, 2, \ldots$, and each $t_i \in T_i$.

If now in each of $(d - 1)$ coordinates either γ is uniformly countably additive of the sequence $g_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to the function g_i, then from the proof of Theorem 7 in [13] it is easy to see that the limit in (1) is uniform with respect to $(A_i) \in \mathsf{X}\mathscr{S}_i$.

For a general γ we prove (1) by induction with respect to the dimension d. For $d = 1$ the theorem is already proved, since then γ is a uniform polymeasure. Suppose the theorem is proved for dimensions $1, \ldots, (d - 1)$.

Let $(A_i) \in \mathsf{X}\mathscr{S}_i$. For each $i = 1, \ldots, d$ take a countably generated σ-ring $\mathscr{S}'_i \subset \mathscr{S}_i$ such that χ_{A_i}, $g_{i,n}$, $n = 1, 2, \ldots$ are $\mathscr{S}'_i$-measurable. Let γ' be the restriction $\gamma' = \gamma\colon \mathsf{X}(G_i \cap \mathscr{S}'_i) \to Y$, where $G_i = \bigcup_{n=1}^{\infty} \{t_i \in T_i,\ g_{i,n}(t_i) \neq 0\} \in \mathscr{S}'_i$. Since (g_i), $(g_{i,n_i}) \in \mathsf{X}\bar{S}((G_i \cap \mathscr{S}'_i), K) \subset I_1(\gamma') \cap I(\gamma)$ for any $n_1, \ldots, n_d = 1, 2, \ldots$, obviously $\int_{(E_i)} (g_{i,n_i})\,\mathrm{d}\gamma' = \int_{(E_i)} (g_{i,n_i})\,\mathrm{d}\gamma$ and $\int_{(E_i)} (g_i)\,\mathrm{d}\gamma' = \int_{(E_i)} (g_i)\,\mathrm{d}\gamma$ for each $n_1, \ldots, n_d = 1, 2, \ldots$ and each $(E_i) \in \mathsf{X}\mathscr{S}'_i$, in particular for $(E_i) = (A_i)$. Hence it is enough to prove (1) when γ is replaced by γ'.

According to Theorem 11 in [12] there is a control d-polymeasure, say $\lambda_1 \times \ldots \times \lambda_d\colon \mathsf{X}(G_i \cap \mathscr{S}'_i) \to [0, +\infty)$, for the vector d-polymeasure γ'. Obviously

$$\begin{aligned}
&\int_{(A_i)} (g_{i,n_i} - g_i + g_i)\,\mathrm{d}\gamma' - \int_{(A_i)} (g_i)\,\mathrm{d}\gamma' = \int_{(A_i)} (g_{i,n_i} - g_i)\,\mathrm{d}\gamma' + \\
&+ \int_{(A_i)} (g_1, (g_{2,n_2} - g_2), \ldots, (g_{d,n_d} - g_d))\,\mathrm{d}\gamma' + \ldots\ldots \\
&\ldots + \int_{(A_i)} ((g_{1,n_1} - g_1), (g_{2,n_2} - g_2), \ldots, (g_{d-1,n_{d-1}} - g_{d-1}), g_d)\,\mathrm{d}\gamma' + \\
&+ \int_{(A_i)} (g_1, g_2, (g_{3,n_3} - g_3), \ldots, (g_{d,n_d} - g_d))\,\mathrm{d}\gamma' + \ldots \\
&\ldots + \int_{(A_i)} (g_1, \ldots, g_{d-1}, (g_{d,n_d} - g_d))\,\mathrm{d}\gamma' + \ldots + \int_{(A_i)} ((g_{1,n_1} - g_1), g_2, \ldots, g_d)\,\mathrm{d}\gamma'
\end{aligned}$$

for any $n_1, \ldots, n_d = 1, 2, \ldots$. Clearly the set functions:

$$\begin{aligned}
&(E_2, \ldots, E_d) \to \int_{(A_1,E_2,\ldots,E_d)} (g_1, \chi_{E_2}, \ldots, \chi_{E_d})\,\mathrm{d}\gamma', (E_2, \ldots, E_d) \in \\
&\in \mathscr{S}'_2 \times \ldots \times \mathscr{S}'_d, \ldots, (E_1, \ldots, E_{d-1}) \to \int_{(E_1,\ldots,E_{d-1},A_d)} (\chi_{E_1}, \ldots, \chi_{E_{d-1}}, g_d)\,\mathrm{d}\gamma', \\
&(E_1, \ldots, E_{d-1}) \in \mathscr{S}'_d \times \ldots \times \mathscr{S}'_{d-1}, (E_3, \ldots, E_d) \to \\
&\to \int_{(A_1,A_2,E_3,\ldots,E_d)} (g_1, g_2, \chi_{E_3}, \ldots, \chi_{E_d})\,\mathrm{d}\gamma', (E_3, \ldots, E_d) \in \mathscr{S}'_3 \times \ldots \times \mathscr{S}'_d, \ldots, E_d \to \\
&\to \int_{(A_1,\ldots,A_{d-1},E_d)} (g_1, \ldots, g_{d-1}, \chi_{E_d})\,\mathrm{d}\gamma', E_d \in \mathscr{S}'_d, \ldots, E_1 \to \\
&\to \int_{(E_1,A_2,\ldots,A_d)} (\chi_{E_1}, g_2, \ldots, g_d)\,\mathrm{d}\gamma', E_1 \in \mathscr{S}'_1
\end{aligned}$$

are $(d-1)$-, ..., $(d-1)$-, $(d-2)$-, ..., 1-, ..., 1-polymeasures, respectively. It is easy to see that the integrals of $((g_{2,n_2}-g_2), \ldots, (g_{d,n_d}-g_d)), \ldots, ((g_{1,n_1}-g_1), \ldots$ $\ldots, (g_{d-1,n_{d-1}}-g_{d-1})), ((g_{3,n_3}-g_3), \ldots, (g_{d,n_d}-g_d)), \ldots, (g_{d,n_d}-g_d), \ldots$ $\ldots, (g_{1,n_1}-g_1)$ with respect to them are equal to the corresponding integrals with respect to γ' written above. Now, let $\varepsilon > 0$. Then by the induction hypothesis there is a positive integer n_0 such that

$$(3) \qquad \left|\int_{(A_i)} (g_{i,n_i})\, \mathrm{d}\gamma' - \int_{(A_i)} (g_i)\, \mathrm{d}\gamma'\right| \leqq \left|\int_{(A_i)} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma'\right| + \varepsilon/2$$

whenever $n_1, \ldots, n_d \geqq n_0$.

According to the Egoroff-Lusin theorem, see Section 1.4 in [5], for each $i = 1, \ldots, d$ there are sets N_i, $G_{i,k} \in G_i \cap \mathscr{S}'_i$, $k = 1, 2, \ldots$ such that $\lambda_i(N_i) = 0$, $G_{i,k} \nearrow G_i - N_i$, and on each $G_{i,k}$, $k = 1, 2, \ldots$ the sequence $g_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to the function g_i. Evidently

$$(4) \qquad \int_{(A_i)} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma' = \int_{(A_i - N_i)} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma' =$$
$$\int_{((A_i - N_i - G_{i,k}) \cup G_{i,k})} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma' = \int_{(A_i - N_i - G_{i,k})} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma' +$$
$$+ \int_{(G_{1,k}, (A_2 - N_2 - G_{2,k}), \ldots, (A_d - N_d - G_{d,k}))} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma' + \ldots$$
$$\ldots + \int_{(G_{i,k})} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma' .$$

By (2) there is an integer k_0 such that

$$\left|\int_{(A_i - N_i - G_{i,k_0})} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma'\right| \leqq (2C)^d \|\gamma'\| (A_i - N_i - G_{i,k_0}) < \varepsilon/4 .$$

In the second, third, ..., $(2^d - 1)$-summand $=$ the last summand on the right hand of (4) we have uniform convergence in at least one coordinate. Hence there is an $n'_0 > n_0$ such that

$$\left|\int_{(A_i)} (g_{i,n_i} - g_i)\, \mathrm{d}\gamma'\right| \leqq \varepsilon/4 + \varepsilon/4 \quad \text{for} \quad n_1, \ldots, n_d \geqq n'_0 .$$

Thus

$$\left|\int_{(A_i)} (g_{i,n_i})\, \mathrm{d}\gamma' - \int_{(A_i)} (g_i)\, \mathrm{d}\gamma'\right| \leqq \varepsilon$$

for $n_1, \ldots, n_d \geqq n'_0$. Since $\varepsilon > 0$ was arbitrary, (1) is proved for the d-tuple (A_i). Since $(A_i) \in \mathsf{X}\mathscr{S}_i$ was arbitrary, the theorem is proved.

Let us note that Theorem 1 in [8], i.e., the Diagonal Convergence Theorem, is a generalization of Proposition 1 in [32].

Theorem 4. *Let Z be a Banach space. Then there is an isometric isomorphism between the Banach space $L^{(d)}(C_0(T_i); Z^*)$ of all bounded d-linear operators $U: \mathsf{X}C_0(T_i) \to Z^*$ and the Banach space of all separately weak*-countably additive vector d-polymeasures $\gamma: \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Z^*$, equipped with the norm $\gamma \to \|\gamma\| (T_i)$. This isometric isomorphism is given by the equations*

$$U(f_i) = \int_{(T_i)} (f_i)\, \mathrm{d}\gamma ,$$

and

$$|U| = \|\gamma\| (T_i) = \sup_{|z| \leqq 1} \|\gamma(\cdot)\, z\| (T_i) .$$

Proof. Let $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Z^*$ be a separately weak*-countably additive vector d-polymeasure. Then $\|\gamma\|(T_i) = \sup_{|z|\leq 1} \|\gamma(\cdot)\,z\|(T_i) < +\infty$ by Nikodým's uniform boundedness theorem for polymeasures, see (N) in [12], and by the uniform boundedness principle. Since $\mathsf{X}C_0(T_i) \subset \mathsf{X}\bar{S}(\mathscr{B}_{0,i}, K)$, we have $\mathsf{X}C_0(T_i) \subset I_1(\gamma(\cdot)\,z)$ for each $z \in Z$ by Theorem 2 in [13]. Since $\gamma(A_i)(\cdot)\colon Z \to K$ is a linear mapping for each $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$, the mapping $U(f_i)(\cdot)\colon Z \to K$ defined by the equality

$$U(f_i)\,z = \int_{(T_i)} (f_i)\,\mathrm{d}(\gamma(\cdot)\,z)$$

is also linear by the elementary properties of the integral, for each $(f_i) \in \mathsf{X}C_0(T_i)$. Clearly $U(\cdot)\,z\colon \mathsf{X}C_0(T_i) \to K$ is d-linear for each $z \in Z$. By elementary properties of the integral, see assertion 6) of Theorem 3 in [13], we obtain the inequalities

$$|U(f_i)\,z| \leqq \prod_{i=1}^{d} \|f_i\|_{T_i}\, \|\gamma(\cdot)\,z\|(T_i) \leqq \prod_{i=1}^{d} \|f_i\|_{T_i}\, \|\gamma\|(T_i)\,, \quad |z| < +\infty$$

for each $z \in Z$, hence $U(f_i) \in Z^*$ for each $(f_i) \in \mathsf{X}C_0(T_i)$. Now using Theorem 1 we obtain the equalities

$$|U| = \sup_{|z|\leq 1} |U(\cdot)\,z| = \sup_{|z|\leq 1} \|\gamma(\cdot)\,z\|(T_i) = \|\gamma\|(T_i)\,.$$

Conversely, let $U\colon \mathsf{X}C_0(T_i) \to Z^*$ be a bounded d-linear operator. By Corollary of Theorem 2 for each $z \in Z$ there is a unique separately countably additive scalar d-polymeasure $\gamma_z\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to K$ such that

$$U(f_i)\,z = \int_{(T_i)} (f_i)\,\mathrm{d}\gamma_z\,, \quad (f_i) \in \mathsf{X}C_0(T_i)\,,$$

and

$$\|\gamma_z\|(T_i) = |U|\,.\,|z| \leqq |U|\,.\,|z| < +\infty\,.$$

Since $B^{(\Omega)}(T_i)$ for each $i = 1, \ldots, d$ is the smallest class of functions $g_i\colon T_i \to K$ which is closed under the ω^*-convergence of sequences and which contains $C_0(T_i)$, by transfinite induction, using assertion 2) of Theorem of A. Pelczyński and the uniform boundedness principle, we obtain a $\mathsf{X}\omega^*$-weak*-continuous extension $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Z^*$. By Corollary of Theorem 2 this extension is of the form

$$U^{**}(g_i)\,z = \int_{(T_i)} (g_i)\,\mathrm{d}\gamma_z\,, \quad (g_i) \in \mathsf{X}B^{(\Omega)}(T_i)$$

for each $z \in Z$. Taking $(g_i) = (\chi_{A_i})$, $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$, we obtain that $U^{**}(\chi_{A_i})\,z = \gamma_z(A_i)$ for each $z \in Z$. Hence $\gamma(A_i) = U^{**}(\chi_{A_i}) \in Z^*$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$. The equality with the norms of U and γ was established in the first part of the proof. Hence the theorem is proved.

We immediately obtain

Corollary. *Let Z be a Banach space. Then every bounded d-linear operator $U\colon \mathsf{X}C_0(T_i) \to Z^*$ has a unique d-linear $\mathsf{X}\omega^*$-weak*-continuous extension $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Z^*$ given by the equality*

$$U^{**}(g_i) = \int_{(T_i)} (g_i)\,\mathrm{d}\gamma\,, \quad (g_i) \in \mathsf{X}B^{(\Omega)}(T_i)\,,$$

where $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Z^*$ is its representing d-polymeasure. Moreover, $|U^{**}| = \|\gamma\|(T_i) = |U|$.

Identifying Y with its canonical image in Y^{**}, from the preceding theorem we easily obtain

Theorem 5. *There is an isometric isomorphism between the Banach space of all bounded d-linear operators* $L^{(d)}(C_0(T_i); Y)$ *and the Banach space of all separately weak*-countably additive d-polymeasures* $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y^{**}$, *with the norm* $\gamma \to \|\gamma\|(T_i)$. *This isometric isomorphism is given by the equations*

$$U(f_i) = \int_{(T_i)} (f_i)\,\mathrm{d}\gamma\,, \quad (f_i) \in \mathsf{X}C_0(T_i)\,,$$

$$y^*\,U(f_i) = \int_{(T_i)} (f_i)\,\mathrm{d}(\gamma(\cdot)\,y^*)\,, \quad (f_i) \in \mathsf{X}C_0(T_i)\,, \quad y^* \in Y^*\,, \quad \textit{and}$$

$$|U| = \|\gamma\|(T_i) = \sup_{|y^*| \leq 1} \|\gamma(\cdot)\,y^*\|(T_i)\,.$$

From Corollary of Theorem 4 we obtain another

Corollary. *Every bounded d-linear operator* $U\colon \mathsf{X}C_0(T_i) \to Y$ *has a unique* $\mathsf{X}\omega^*$-*weak*-continuous extension* $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y^{**}$ *given by the equality*

$$U^{**}(g_i) = \int_{(T_i)} (g_i)\,\mathrm{d}\gamma\,, \quad (g_i) \in \mathsf{X}B^{(\Omega)}(T_i)\,,$$

where $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y^{**}$ *is the representing d-polymeasure of U. Moreover,* $|U^{**}| = \|\gamma\|(T_i) = |U|$.

Keeping the identification of Y with its canonical image in Y^{**}, we now prove

Theorem 6. *Let* $U\colon \mathsf{X}C_0(T_i) \to Y$ *be a bounded d-linear operator with the representing d-polymeasure* $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y^{**}$ *and extension* $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y^{**}$. *Then the following conditions are equivalent*:

a) γ *is Y-valued,*

b) γ *is Y-valued and is separately countably additive in the norm topology of Y,*

c) γ *is separately countably additive in the norm topology of* Y^{**},

d) $U^{**}\text{-}\ \mathsf{X}B^{(\Omega)}(T_i) \to Y^{**}$ *is* $\mathsf{X}\omega^*$-*norm-continuous, and*

e) $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y$, *and it is* $\mathsf{X}\omega^*$-*norm-continuous.*

Proof. a) $\Rightarrow$ b) by Theorem 5 and the Orlicz-Pettis theorem, since on $Y \subset Y^{**}$ the weak* and the weak topology coincide.

Evidently b) $\Rightarrow$ c).

c) $\Rightarrow$ d) by Theorem 3.

Trivially d) $\Rightarrow$ e).

e) $\Rightarrow$ a) since $\gamma(A_i) = U^{**}(\chi_{A_i})$. The theorem is proved.

Another generalization of the l-dimensional case is given in

Theorem 7. *For a bounded d-linear operator* $U\colon \mathsf{X}C_0(T_i) \to Y$ *with the representing d-polymeasure* $\gamma\colon \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y^{**}$ *and the extension* $U^{**}\colon \mathsf{X}B^{(\Omega)}(T_i) \to Y^{**}$ *the following conditions are equivalent*:

a) *U is (weakly) compact,*

b) *U^{**} is Y valued and (weakly) compact, and*

c) *γ is Y valued and its range is (weakly) compact.*

Proof. a) $\Rightarrow$ b) by Theorem of A. Pelczyński, using assertion e) of Theorem 6 and the fact that the $X\omega^*$-sequential closure of $XC_0(T_i)$ is $XB^{(\Omega)}(T_i)$.

Evidently b) $\Rightarrow$ c).

The implication c) $\Rightarrow$ a) in the case of a weakly compact range was already proved in Theorem 2. In the case of a compact range it can be proved similarly as Theorem VI.7.7 in [19].

Let $\gamma\colon X\sigma(\mathscr{B}_{0,i}) \to Y$ be separately countably additive. We say that γ is uniformly countably additive in the coordinate i if the vector measures $\gamma(\ldots, A_{i-1}, \cdot, A_{i+1}, \ldots)\colon \sigma(\mathscr{B}_{0,i}) \to Y, \ldots, A_{i-1} \in \sigma(\mathscr{B}_{0,i-1}), A_{i+1} \in \sigma(\mathscr{B}_{0,i+1}), \ldots$ are uniformly countably additive.

Theorem 8. *Let $\gamma\colon X\sigma(\mathscr{B}_{0,i}) \to Y$ be separately countably additive and let $U(f_i) = \int_{(T_i)} (f_i)\,d\gamma$, $(f_i) \in XC_0(T_i)$. Put $S_i = \{f_i \in C_{0,i}(T_i), \|f_i\|_{T_i} \leqq 1\}$, $i = 1, \ldots, d$. Then the following conditions are equivalent:*

a) $\lim\limits_{n\to\infty} \sup\limits_{\substack{f_j \in S_j \\ j \neq i}} |U(\ldots, f_{i-1}, f_{i,n}, f_{i+1}, \ldots)| = 0$ *whenever* $f_{i,n} \in S_i$, $n = 1, 2, \ldots$ *and* $f_{i,n_1} \cdot f_{i,n_2} = 0$ *for* $n_1 \neq n_2$, $n_1, n_2 = 1, 2, \ldots$;

b) *γ is uniformly countably additive in the coordinate i; and*

c) *for each $\varepsilon > 0$ there is a positive integer $N_{i,\varepsilon}$ such that $|U(\ldots, f_{i-1}, f_{i,n}, f_{i+1}, \ldots)| < \varepsilon$ for at least one $n \in \{1, \ldots, N_{i,\varepsilon}\}$ whenever $f_{i,n} \in S_i$, $n = 1, \ldots, N_{i,\varepsilon}$, $f_{i,n_1} \cdot f_{i,n_2} = 0$ for $n_1 \neq n_2$, $n_1, n_2 = 1, \ldots, N_{i,\varepsilon}$, and $f_j \in S_j$ for $j \neq i$.*

Proof. a)/$\Rightarrow$ b) by Lemma 1 in [30], which coincides with Lemma 8.3 on p. 267 in [33].

b) $\Rightarrow$ c), since a uniformly countably additive family of vector measures is uniformly absolutely continuous with respect to a finite non negative countably additive measure, see Theorem I.2.4 in [1] and Theorem 1 in [4].

Evidently c) $\Rightarrow$ a).

We now characterize those bounded d-linear operators $U\colon XC_0(T_i) \to Y$ whose representing d-polymeasure is Y valued in terms of U itself. For $d = 1$ the condition seems to be also new.

Theorem 9. *Let $U\colon XC_0(T_i) \to Y$ be a bounded d-linear operator and let $\gamma\colon X\sigma(\mathscr{B}_{0,i}) \to Y^{**}$ be its representing d-polymeasure. Then $\gamma\colon X\sigma(\mathscr{B}_{0,i}) \to Y$ if and only if $\lim\limits_{n\to\infty} U(\varphi_{i,n,k}) \in Y$ exists for any double sequence $\varphi_{i,n,k} \in XC_0(T_i)$ such that $0 \leqq \varphi_{i,n,k} \leqq 1$ for each $i = 1, \ldots, d$ and each $k, n = 1, 2, \ldots$, $\varphi_{i,n,k} \nearrow (\searrow)\, g_{i,n}$ as $k \to \infty$ for each $i = 1, \ldots, d$ and each $n = 1, 2, \ldots$, and $g_{i,n} \searrow (\nearrow)$ as $n \to \infty$ for each $i = 1, \ldots, d$.*

Proof. The necessity is a consequence of Theorem 3. First we show the sufficiency for the case $d = 1$. According to Theorem VI.7.3 in [19], $\gamma: \sigma(\mathscr{B}_0) \to Y$ ($T = T_1$, $\mathscr{B}_0 = \mathscr{B}_{0,1}$, etc.) if and only if the family of scalar measures $\{\gamma(\cdot)\, y^*: \sigma(\mathscr{B}_0) \to K,\ y^* \in Y^*,\ |y^*| \leqq 1\}$ is uniformly countably additive. By Grothendieck's result, see Lemma VI.2.13 in [1], this occurs if and only if $\gamma(O_j) \to 0$ whenever $O_j \in \sigma(\mathscr{B}_0)$, $j = 1, 2, \ldots$ is a sequence of pairwise disjoint open sets. Let O_j, $j = 1, 2, \ldots$ be such a sequence. Since $V_n = \bigcup_{j=n}^{\infty} O_j$, $n = 1, 2, \ldots$ are open F_σ sets, by Theorem B in § 50, [20], for each $n = 1, 2, \ldots$ there is a sequence $\varphi_{n,k} \in C_0(T)$, $0 \leqq \varphi_{n,k} \leqq 1$, $k = 1, 2, \ldots$ such that $\varphi_{n,k} \nearrow \chi_{V_n}$. By assumption $\lim_{n \to \infty} \gamma(V_n) = \lim_{n \to \infty} \mathsf{U}^{**}(\chi_{V_n}) \in Y$ exists, hence $\lim_{j \to \infty} \gamma(O_j) = \lim_{j \to \infty} \gamma(V_j - V_{j+1}) = \lim_{j \to \infty} \gamma(V_j) - \gamma(V_{j+1}) = 0$.

In the case $\nearrow$, $\searrow$ given in the bracket, let $C_j, j = 1, 2, \ldots$ be a sequence of pairwise disjoint compact G_δ sets. For $n = 1, 2, \ldots$ put $D_n = \bigcup_{j=1}^{n} C_j$. By Theorem B in § 50, [20], for each D_n, $n = 1, 2, \ldots$ take a sequence $\varphi_{n,k} \in C_0(T)$, $0 \leqq \varphi_{n,k} \leqq 1$, $k = 1, 2, \ldots$ such that $\varphi_{n,k} \searrow \chi_{D_n}$. By assumption $\lim_{n \to \infty} \gamma(D_n) = \lim_{n \to \infty} \mathsf{U}^{**}(\chi_{D_n}) \in Y$ exists, hence $\lim_{j \to \infty} \gamma(C_j) = 0$. Consequently, since C_j, $j = 1, 2, \ldots$ was an arbitrary sequence of pairwise disjoint compact G_δ sets, using the regularity of the scalar Baire measures $\gamma(\cdot)\, y^*: \sigma(\mathscr{B}_0) \to K$, $y^* \in Y^*$ we immediately obtain that $\lim_{j \to \infty} \gamma(O_j) = 0$ whenever $O_j \in \sigma(\mathscr{B}_0), j = 1, 2, \ldots$ is a sequence of pairwise disjoint open sets. Hence by Lemma VI.2.13 in [1] γ is Y valued. Hence for $d = 1$ the sufficiency is also proved.

Let $d > 1$ and let $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$. Then $(\chi_{A_i}) \in \mathsf{X}B^{(\Omega)}(T_i)$. Since $B^{(\Omega)}(T_i) = \bigcup_{\alpha < \Omega} B^{(\alpha)}(T_i)$, $i = 1, \ldots, d$, where $B^{(\alpha)}(T_i)$ stands for the α-th Baire class, using transfinite induction we immediately see that for each $i = 1, \ldots, d$ there is a countable family of functions $f_{i,n} \in C_0(T_i)$, $n = 1, 2, \ldots$ such that $\chi_{A_i} \in B(\{f_{i,n}\})$ = the smallest class of functions $f_i: T_i \to K$ which contains the family $\{f_{i,n}\}$ and which is closed under the formation of pointwise limits of sequences. Using the sequences $\{f_{i,n}\}$, $i = 1, \ldots, d$, similarly as in the proof of Theorem VI.7.6 in [19], concerning $B(\{f_{i,n}\})$ we may and will suppose that each T_i, $i = 1, \ldots, d$ is a σ-compact metric space. In particular, T_i, $i = 1, \ldots, d$ are separable metric spaces now. According to Lemma VI.8.4 in [19] each $C_0(T_i)$, $i = 1, \ldots, d$ is a separable Banach space. Let $h_{i,n}$, $n = 1, 2, \ldots$ be a countable dense set in $C_0(T_i)$, $i = 1, \ldots, d$.

By the case $d = 1$ proved above, for each $(f_2, \ldots, f_d) \in C_0(T_2) \times \ldots \times C_0(T_d)$ there is a unique countably additive vector measure $\gamma_{(f_2, \ldots, f_d)}: \sigma(\mathscr{B}_{0,1}) \to Y$ which represents the bounded linear operator $U_{(f_2, \ldots, f_d)}: C_0(T_1) \to Y$, $U_{(f_2, \ldots, f_d)}(f_1) = U(f_1, f_2, \ldots, f_d)$, $f_1 \in C_0(T_1)$. Let $\lambda_1: \sigma(\mathscr{B}_{0,1}) \to [0, 1]$ be a common countably additive $(0 \to 0)$ control measure for the countable family of countably additive vector measures $\gamma_{(h_{2,n_2}, \ldots, h_{d,n_d})}: \sigma(\mathscr{B}_{0,1}) \to Y$, $n_2, \ldots, n_d = 1, 2, \ldots$, see Lemma IV.10.5 in [19], or Corollary I.2.6 in [1]. Let $N_1 \in \sigma(\mathscr{B}_{0,1})$, and let $\lambda_1(N_1) = 0$.

We assert that $\gamma(N_1, E_2, \ldots, E_d) = 0$ for each $E_i \in \sigma(\mathscr{B}_{0,i})$, $i = 2, \ldots, d$. First we show that $\gamma_{(f_2,\ldots,f_d)}(N_1) = 0$ for each $(f_2, \ldots, f_d) \in C_0(T_2) \times \ldots \times C_0(T_d)$.

Let $(f_2, \ldots, f_d) \in C_0(T_2) \times \ldots \times C_0(T_d)$, and take subsequences $\{n_{i,k}\} \subset \{n\}$, $i = 2, \ldots, d$ such that $\|f_i - h_{i,n_{i,k}}\|_{T_i} \to 0$ for each $i = 2, \ldots, d$. Evidently $\|f_i\|_{T_i} \leqq \leqq \sup_k \|h_{i,n_{i,k}}\|_{T_i} = b_i < +\infty$ for each $i = 2, \ldots, d$. Put $b = \max_{2 \leqq i \leqq d} b_i$. Clearly $\gamma_{(f_2,\ldots,f_d)}(N_1) = \gamma_{(f_2,\ldots,f_d)}(N_1) - \gamma(h_{2,n_{2,k}}, \ldots, h_{d,n_{d,k}})(N_1)$, and

$$(f_2, \ldots, f_d) - (h_{2,n_{2,k}}, \ldots, h_{d,n_{d,k}}) =$$
$$= ((f_2 - h_{2,n_{2,k}}), f_3, \ldots, f_d) + \ldots + (h_{2,n_{2,k}}, \ldots, h_{d-1,n_{d-1,k}}, (f_d - h_{d,n_{d,k}}))$$

for each $n_{2,k}, \ldots, n_{d,k} = 1, 2, \ldots$. Since

$$|\gamma((f_2 - h_{2,n_{2,k}}), f_3, \ldots, f_d)(N_1)| \leqq \|\gamma_{(\ldots)}\|(T_1) = (\text{by Theorem 1}) =$$
$$= \sup_{\|f_1\|_{T_1} \leqq 1} |U(f_1, (f_2 - h_{2,n_{2,k}}), f_3, \ldots, f_d)| \leqq |U| \cdot \|f_2 - h_{2,n_{2,k}}\|_{T_2} b^{d-2},$$

. . . .

$$|\gamma(h_{2,n_{2,k}}, \ldots, h_{d-1,n_{d-1,k}}, (f_d - h_{d,n_{d,k}}))(N_1)| \leqq$$
$$\leqq \|\gamma_{(\ldots)}\|(T_1) \leqq |U| b^{d-2} \|f_d - h_{d,n_{d,k}}\|_{T_d}$$

for each $n_{2,k}, \ldots, n_{d,k} = 1, 2, \ldots$, we have $\gamma_{(f_2,\ldots,f_d)}(N_1) = 0$

Let $(E_2, \ldots, E_d) \in \sigma(\mathscr{B}_{0,2}) \times \ldots \times \sigma(\mathscr{B}_{0,d})$. According to Theorem 5 and its Corollary we have $\gamma(N_1, E_2, \ldots, E_d) = U^{**}(\chi_{N_1}, \chi_{E_2}, \ldots, \chi_{E_d})$. Since $U^{**}(\chi_{N_1}, f_2, \ldots \ldots, f_d) = U^{**}_{(f_2,\ldots,f_d)}(\chi_{N_1}) = \gamma_{(f_2,\ldots,f_d)}(N_1) = 0$ for each $(f_2, \ldots, f_d) \in C_0(T_2) \times \ldots \ldots \times C_0(T_d)$, we have $\gamma(N_1, E_2, \ldots, E_d)\, y^* = 0$ for each $y^* \in Y^*$ by Corollary of Theorem 5. Hence $\gamma(N_1, E_2, \ldots, E_d) = 0$, which we wanted to show.

By symmetry in coordinates there are countably additive measures $\lambda_i: \sigma(\mathscr{B}_{0,i}) \to \to [0, 1]$, $i = 2, \ldots, d$ with analogous properties as λ_1. Hence $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_d$ is a control d-polymeasure for the d-polymeasure $\gamma: \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y^{**}$.

By regularity of the Baire measures $\lambda_i: \sigma(\mathscr{B}_{0,i}) \to [0, 1]$, $i = 1, \ldots, d$, see Theorem G in § 52, [20], for each i there is a non decreasing sequence of compact G_δ subsets $C_{i,n} \subset T_i$, $n = 1, 2, \ldots$, and a non increasing sequence of open subsets $O_{i,n} \subset T_i$, $n = 1, 2, \ldots$ such that $C_{i,n} \subset A_i \subset O_{i,n}$ and $\lambda_i(O_{i,n} - C_{i,n}) < 1/n$ for each $n = = 1, 2, \ldots$. Hence $\lambda_i(\bigcap_{n=1}^{\infty} O_{i,n} - \bigcup_{n=1}^{\infty} C_{i,n}) = 0$ for each $i = 1, \ldots, d$. Now by Urysohn's Lemma for each $i = 1, \ldots, d$ and each $n = 1, 2, \ldots$ there is a non decreasing (non increasing) sequence $\varphi_{i,n,k} \in C_0(T_i)$, $0 \leqq \varphi_{i,n,k} \leqq 1$, $k = 1, 2, \ldots$ such that $\varphi_{i,n,k} \nearrow \nearrow \chi_{O_{i,n}} (\varphi_{i,n,k} \searrow \chi_{C_{i,n}})$. By assumption and Theorem 5,

$$\lim_{n\to\infty} \lim_{k\to\infty} U(\varphi_{i,n,k}) = \lim_{n\to\infty} \lim_{k\to\infty} \int_{(T_i)} (\varphi_{i,n,k})\, d\gamma = y \in Y$$

exists. Hence, using Corollary of Theorem 5 and the fact that $\lambda_1 \times \ldots \times \lambda_d$ is a control d-polymeasure for the d-polymeasure $\gamma: \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y^{**}$, we immediately obtain the equalities

$$y^*y = \lim_{n\to\infty} \lim_{k\to\infty} \int_{(T_i)} (\varphi_{i,n,k}) \, \mathrm{d}(\gamma(\cdot)\, y^*) = \gamma(\bigcap_{n=1}^{\infty} O_{i,n})\, y^* =$$

$$= \gamma(A_i)\, y^* \quad (= \gamma(\bigcup_{n=1}^{\infty} C_{i,n})\, y^*)$$

for each $y^* \in Y^*$. Thus $\gamma(A_i) = y \in Y$ by the Hahn-Banach theorem, which we wanted to show.

Since $(A_i) \in \mathsf{X}\sigma(\mathscr{B}_{0,i})$ was arbitrary, the theorem is proved.

Since our spaces T_i, $i = 1, \ldots, d$ are locally compact, the following "localization" of our integtal representation is of importance. Its proof is obvious.

Theorem 10. *For a bounded d-linear operator* $U: \mathsf{X}C_0(T_i) \to Y$ *the following conditions are equivalent:*

a) *the representing d-polymeasure* γ *of* U *is* Y *valued on* $\mathsf{X}\mathscr{B}_{0,i}$;

b) *the representing d-polymeasure* γ *of* U *is* Y *valued on* $\mathsf{X}\mathscr{B}_{0,i}$ *and separately countably additive on* $\mathsf{X}\mathscr{B}_{0,i}$;

c) *for any relatively compact open sets* $D_i \in \mathscr{B}_{0,i}$, $i = 1, \ldots, d$ *the restriction* $U_{(D_i)} = U: \mathsf{X}C_0(D_i) \to Y$ *is representable by a unique* Y *valued d-polymeasure* $\gamma_{(D_i)}: \mathsf{X}(D_i \cap \mathscr{B}_{0,i}) \to Y$.

If these conditions are fulfilled, then

$$U(f_i) = \int_{(T_i)} (f_i)\, \mathrm{d}\gamma\,, \quad (f_i) \in \mathsf{X}C_0(T_i)\,,$$

where $\gamma: \mathsf{X}\mathscr{B}_{0,i} \to Y$, $|U| = \|\gamma\|\,(T_i)$, *and* $\gamma_{(D_i)} = \gamma: \mathsf{X}(D_i \cap \mathscr{B}_{0,i}) \to Y$ *for any open* $D_i \in \mathscr{B}_{0,i}$, $i = 1, \ldots, d$.

The bilinear operator $U: c_0 \times c_0 \to c_0$ of pointwise multiplication $U(x, z) = (x(t), z(t)) \in C_0$ is a simple example of a separately compact operator, obviously bounded. However, its representing bimeasure is not Y valued on $2^N \times 2^N$, nonetheless, it is $Y = c_0$ valued on $\mathscr{B}_{0,1} \times \mathscr{B}_{0,2}$, where $\mathscr{B}_{0,1} = \mathscr{B}_{0,2}$ is the δ-ring of all finite subsets of N.

If Y contains no copy of l_∞ and T is a Stonean compact, then every bounded linear operator $U: C(T) \to Y$ is weakly compact by the important theorem of H. P. Rosenthal, see [34] and Theorem VI.2.10 in [1]. Now it is easy to check that the proof of Theorem of A. Pelczyński in [32], hence also the theorem itself remain valid if Y contains no isomorphic copy of l_∞ and each T_i, $i = 1, \ldots, d$ is a Stonean compact. Hence, similarly as Theorem 2, we have our concluding.

Theorem 11. *Let* Y *contain no copy of* l_∞, *in particular let* Y *be separable, and let* T_i, $i = 1, \ldots, d$ *be Stonean compacts. Then every bounded d-linear operator* $U: \mathsf{X}C(T_i) \to Y$ *has a unique representation in the form*

$$U(f_i) = \int_{(T_i)} (f_i)\, \mathrm{d}\gamma\,, \quad (f_i) \in \mathsf{X}C(T_i)\,,$$

where the representing d-polymeasure $\gamma: \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to Y$ *is separately countably* *additive. Moreover, U has a unique* $X\omega^*$*-norm-continuous extension* U^{**}: $XB^{(\Omega)}(T_i) \to Y$ *given by the equality*

$$U^{**}(g_i) = \int_{(T_i)} (g_i)\,\mathrm{d}\gamma\,, \quad (g_i) \in XB^{(\Omega)}(T_i)\,.$$

At the same time,

$$|U| = \|\gamma\|\,(T_i) = \sup_{|y^*| \leq 1} \|y^*\,\gamma(\cdot)\|\,(T_i) = |U^{**}|\,.$$

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