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ON THE IDEALS OF THE SEMIGROUP OF THE 1-SPHERE

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1. INTRODUCTION

With the single exception of its use in the title, the word ideal when unmodified will mean two-sided ideal. When we discuss one-sided ideals the appropriate adjective will be used. Furthermore, it will be assumed throughout that a prime ideal of a semigroup is a proper subset of that semigroup. By the semigroup of a topological space X we mean $S(X)$ the semigroup, under composition, of all continuous selfmaps of X. This paper developed as a consequence of our interest in the existence of prime ideals in $S(X)$. In the spaces we have studied thus far, we have really turned up very few. In [2] we showed that the semigroup of every local dendrite with finite branch number which is not an arc, does contain prime ideals. According to Theorem (3.3) and Corollary (3.4) of [2], one locates such ideals by choosing a subcontinuum K of such a space X with the property that the collection $\mathscr{K}$ of all subspaces of X homeomorphic to K is a filterbase and then takes $J(\mathscr{K})$ to be the collection of all functions in $S(X)$ which are not injective on any member of $\mathscr{K}$. For local dendrites with finite branch number, this produces only finitely many prime ideals. For example, $S(S^1)$ the semigroup of the 1-sphere has only one ideal of this type and it is obtained by taking K to be S^1 itself. Evidently, $\mathscr{K} = \{S^1\}$ in this case. Nevertheless, $S(S^1)$ has an enormous number of prime ideals and this is due to the well known fact that the mapping deg, where $\deg f$ is the degree of an element $f \in S(S^1)$, is a homomorphism from $S(S^1)$ onto the multiplicative semigroup of integers. All this underscores the fact that there are a lot of spaces X (including all Euclidean N-cells) for which it is impossible to define the degree of a function $f \in S(X)$ in such a manner that the resulting mapping from $S(X)$ to the integers enjoys the same properties as the mapping deg from $S(S^1)$ to the integers. As we mentioned before, the existence of such a map implies the existence of prime ideals and Theorem (3.10) of [2] describes a class of spaces which includes all N-cells whose semigroups have no prime ideals whatsoever.

Definition (1.1). An ideal J of $S(S^1)$ is said to be *basic* if $|\deg f| \neq 1$ for all $f \in J$ and $\deg f \neq 0$ for some $f \in J$.

Basic ideals are abundant and include most principal ideals. Evidently, a principal ideal J will be basic if and only if $|\deg f| \neq 0, 1$ where f is any generator of J. It is also easy to see that arbitrary unions of basic ideals are basic. In this note, we completely characterize the basic prime ideals of $S(S^1)$. For each of these ideals, we characterize the minimal generating sets and this permits us to determine precisely which of the basic prime ideals are principal as well. Finally, we show that the partially ordered family of all basic prime ideals of $S(S^1)$ is order isomorphic to the partially ordered family of all nonempty subsets of a countably infinite set so that, among other things, the cardinality of the family of basic prime ideals of $S(S^1)$ is the cardinality of the continuum.

As we mentioned previously, we began these investigations because of our interest in prime ideals. However, we needed certain results about principal one-sided and two-sided ideals and these results led naturally to some other questions which we have answered. The results obtained include a number of facts concerning Green's relations. They are all contained in Section 2 and as it turns out, this section is considerably more extensive than Section 3 which contains the results about the basic prime ideals.

2. PRINCIPAL ONE SIDED AND TWO SIDED IDEALS

We will regard the points of S^1 as complex numbers with the usual multiplication. The symbol $[n]$ will denote that function in $S(S^1)$ which is defined by $[n](z) = z^n$. We will denote by E the function from the reals R to S^1 which is defined by $E(x) = e^{2\pi x i}$ and E_I will denote the restriction of E to the closed unit interval $I = [0, 1]$. For a real number a, $\langle a \rangle$ will denote the constant function from I to R which maps everything into a. We want to prove a number of results in this section about principal one-sided and two-sided ideals but it is appropriate to first recall some facts about the degree of a function $f \in S(S^1)$. It is well known that for each $f \in S(S^1)$, there exists a continuous function f_* from I into the reals R such that the following diagram commutes:

(2.1)
$$\begin{array}{ccc} I & \xrightarrow{f_*} & R \\ {\scriptstyle E_I}\downarrow & & \downarrow{\scriptstyle E} \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Definition (2.2). Any such function f_* will be referred to as a *lift* of f and f will be referred to as the *support* of f_*.

The lift of a function is not unique. However, any two lifts f_{*1} and f_{*2} of f will differ by an integer. That is, $f_{*1} = f_{*2} + \langle n \rangle$ for some integer n. Since diagram (2.1) commutes, it follows that for any lift f_* of f, $f_*(1) - f_*(0)$ is an integer which does not depend upon the particular lift and, by definition, is $\deg f$. Evidently, not every continuous function α from I into R can be the lift of a function $f \in S(S^1)$ for we have just seen that a necessary condition is that $\alpha(1) - \alpha(0)$ be an integer. But it is quite easy to see that this condition is sufficient as well. The function f defined by $f(e^{2\pi x i}) = e^{2\pi\alpha(x)i}$ for $x \in I$ is a continuous selfmap of S^1 and α is a lift of f. Thus, a continuous function α mapping I into R is a lift of some function in $S(S^1)$ if and only if $\alpha(1) - \alpha(0)$ is an integer. We note that although each support has a countably infinite number of lifts, each lift has only one support. In our first theorem, we collect a number of facts about lifts and degrees which will be useful to us. Most of these are well known and those which perhaps may not be are easily verified so we omit the proofs.

Theorem (2.3). *The following statements about $f, g \in S(S^1)$ and any two lifts f_* and g_* are valid.*

(2.3.1) $\deg f \circ g = (\deg f)(\deg g)$.

(2.3.2) *A function α mapping I into R is a lift of f if and only if $\alpha = f_* + \langle n \rangle$ for some integer n.*

(2.3.3) *$f = g$ if and only if $f_* = g_* + \langle n \rangle$ for some integer n.*

(2.3.4) *If g maps I into I then $f_* \circ g_*$ is a lift for $f \circ g$.*

(2.3.5) *f is a homeomorphism from S^1 onto S^1 if and only if f_* is a homeomorphism from I onto some closed interval $[a, b]$ such that $|a - b| = 1$.*

(2.3.6) *If f is a homeomorphism from S^1 onto S^1 then some lift of f maps I homeomorphically onto I if and only if $f(1) = 1$.*

(2.3.7) *If* $\deg f \neq 0$, *then* $\operatorname{Ran} f = S^1$.

Throughout this paper, we will not hesitate to use the facts stated in Theorem (2.3) without explicit mention. Similarly, we will use many times throughout the paper the fact that Diagram (2.1) commutes and we will not always call specific attention to that either. Statement (2.3.4) tells us that in a few instances we can get a lift of $f \circ g$ by composing f_* with g_* but quite often this cannot be done since Ran g_* (the range of g_*) may well not be contained in I for any lift g_*. It is most certainly not if $|\deg g| > 1$ and it is not even contained in I in a lot of cases where $|\deg g| = 0$ or 1. If g is a homeomorphism then $|\deg g| = 1$ and we see from (2.3.6) that Ran $g_* \subseteq I$ for some lift g_* if and only if $g(1) = 1$. If Ran g is a proper subset of S^1, we can be assured that Ran $g_* \subseteq I$ for some lift g_* but it may not be otherwise. Of course, $\deg g = 0$ whenever Ran g is a proper subset of S^1. Nevertheless, there exist functions $g \in S(S^1)$ such that $\deg g = 0$, Ran $g_* \subseteq I$ for some lift g_* and g maps S^1 onto S^1. For such an example, let $g_*(x) = 4x - 4x^2$ and let g be the support of g_*.

For $f \in S(S^1)$, we let $L(f)$, $R(f)$ and $I(f)$ denote respectively the principal left, right and two-sided ideal generated by f. The symbols $\mathscr{L}$ and $\mathscr{R}$ and $\mathscr{J}$ will denote the familiar Green's relations. That is, $(f, g) \in \mathscr{L}$ if and only if $L(f) = L(g)$, $(f, g) \in \mathscr{R}$ if and only if $R(f) = R(g)$ and $(f, g) \in \mathscr{J}$ if and only if $I(f) = I(g)$.

Theorem (2.4). *Suppose $f, g \in S(S^1)$ and* $\deg f \neq 0$. *Then f and g are $\mathscr{L}$-equivalent* *if and only if $f = k \circ g$ for some homeomorphism k mapping S^1 onto S^1. If, in addition, either f or g is finite-to-one, then f and g are $\mathscr{R}$-equivalent if and only if $f = g \circ k$ for some homeomorphism k mapping S^1 onto S^1.*

Proof. If $f = k \circ g$ and k is a homeomorphism, it is immediate that f and g are $\mathscr{L}$-equivalent. Suppose, conversely that f and g are $\mathscr{L}$-equivalent. Then $f = k \circ g$ and $g = h \circ f$ for some $h, k \in S(S^1)$ and we have $f = (k \circ h) \circ f$. Since $\deg f \neq 0$, $\operatorname{Ran} f = S^1$ so that $k \circ h$ is the identity map. Similarly, $g = (h \circ k) \circ g$ and since $\deg g \neq 0$ by (2.3.1), $\operatorname{Ran} g = S^1$ and hence $h \circ k$ is also the identity map. Consequently, h and k are homeomorphisms.

It is immediate that f and g are $\mathscr{R}$-equivalent if $f = g \circ k$ for some homeomorphism k so suppose f and g are $\mathscr{R}$-equivalent. Then $f = g \circ k$ and $g = f \circ h$ for some $h, k \in S(S^1)$ and we have $f = f \circ (h \circ k)$. If either f or g is finite-to-one then both must be finite-to-one so, by hypothesis, f is finite-to-one. Furthermore, $\deg h \neq 0 \neq \deg k$ by (2.3.1) so that both h and k and hence $h \circ k$ as well, map S^1 onto S^1. It now follows from Lemma (3.12) of [3] that $h \circ k$ is injective and must therefore be a homeomorphism from S^1 onto S^1. Since the only subspace of S^1 homeomorphic to S^1 is S^1 itself, condition (2.12.5) of Theorem (2.12) of [1] is satisfied and it follows from (2.12.3) of that theorem that both h and k are homeomorphisms.

We wish to get a result for the $\mathscr{J}$-relation which is analogous to the results we have for the $\mathscr{L}$ and $\mathscr{R}$ relations. To help us prove the result about the $\mathscr{R}$ relation, we had at our disposal a set-theoretic lemma (Lemma (3.12) of [3]) which states that if f and g are two selfmaps of a set X such that f is finite-to-one, g is surjective and $f \circ g = f$, then g is bijective. Unfortunately, there is no analogous result for a finite-to-one function f whenever $f = h \circ f \circ g$ even when both h and g are surjective. To see this consider the following

Example (2.5). Let $\{A_n\}_{n \in Z}$ be a decomposition of the integers Z into mutually disjoint nonempty finite sets where Z also serves as the index set. Choose the sets A_n so that $\operatorname{card}(A_n) = \operatorname{card}(A_0)$ for $n < 0$, $\operatorname{card}(A_0) > \operatorname{card}(A_1 \cup A_2)$ and $\operatorname{card}(A_n) = \operatorname{card}(A_1)$ for $n > 1$ where card denotes cardinality. Let g be any function which maps A_n onto A_{n+1} for $n < 0$, A_0 onto $A_1 \cup A_2$ and A_n onto A_{n+2} for $n > 0$. Define $f(x) = n$ for $x \in A_n$ and define $h(n) = n - 1$ for $n < 1$, $h(1) = h(2) = 0$ and $h(n) = n - 2$ for $n > 2$. One easily verifies that $f = h \circ f \circ g$. Evidently, all three functions are surjective but neither h nor g is injective.

The previous example shows that requiring f to be finite-to-one is not enough to enable one to conclude that h and g must be injective whenever $f = h \circ f \circ g$ for two surjections h and g. We can, however, get a result which is sufficient for our purposes here. By an *N-to-one* function we mean a function f with the property that $f^{-1}(y)$ consists of exactly N elements for each $y \in \operatorname{Ran} f$.

Lemma (2.6). *Let X be a set, let f be an N-to-one surjection of X and let h and g be two surjections of X such that $f = h \circ f \circ g$. Then both h and g are bijections of X.*

Proof. First, we show that h is injective. Suppose $h(a) = h(b)$ for $a \neq b$ and let $A = (f \circ g)^{-1}(a)$ and $B = (f \circ g)^{-1}(b)$. Then $\operatorname{card}(A \cup B) \geqq 2N$. For any $x, y \in A \cup B$, we have

$$f(x) = h \circ f \circ g(x) = h \circ f \circ g(y) = f(y).$$

But this contradicts the fact that f is N-to-one and we conclude that h must be injective.

Let $\Delta(f) = \{f^{-1}(y): y \in \operatorname{Ran} f\}$ and let $A \in \Delta(f)$. For $A \in \Delta(f)$ and any two points $a, b \in g[A]$, there exist points $x, y \in A$ such that $g(x) = a$ and $g(y) = b$ and, of course, $f(x) = f(y)$ and we have

$$h(f(a)) = h \circ f \circ g(x) = f(x) = f(y) = h \circ f \circ g(y) = h(f(b)).$$

Since h is injective this means that $f(a) = f(b)$ and hence that $g[A] \subseteq B$ for some $B \in \Delta(f)$. Suppose $g[A] \neq B$. Since g is surjective, we must have $g[C] \subseteq B$ for some $C \in \Delta(f)$ such that $C \neq A$. Now, for $x, y \in A \cup C$, we get

$$f(x) = h \circ f \circ g(x) = h \circ f \circ g(y) = f(y)$$

which contradicts the fact that f is N-to-one. Thus, for each $A \in \Delta(f)$, there exists a $B \in \Delta(f)$ such that $g[A] = B$ and for any $C \in \Delta(f)$ distinct from A, $g[C] \neq g[A]$. Since g is surjective and each $A \in \Delta(f)$ has exactly N points, g must be a bijection.

Theorem (2.7). *Suppose $f, g \in S(S^1)$, $\deg f \neq 0$ and suppose further that f is N-to-one. Then f and g are $\mathscr{J}$-equivalent if and only if $f = h \circ g \circ k$ for two homeomorphisms h and k from S^1 onto S^1.*

Proof. Only the necessity needs verification so suppose f and g are $\mathscr{J}$-equivalent. Then $f = h \circ g \circ k$ and $g = s \circ f \circ t$ for $h, k, s, t \in S(S^1)$. It follows from (2.3.1) and (2.3.7) that all the functions here are surjective. Since $f = (h \circ s) \circ f \circ (t \circ k)$, Lemma (2.6) applies and we conclude that $h \circ s$ and $t \circ k$ are both bijections and hence homeomorphisms. Again we appeal to Theorem (2.12) of [1] to conclude that h, k, s, t are all homeomorphisms from S^1 onto S^1.

It is immediate from (2.3.1) that if $g \in R(f)$ then $\deg f \mid \deg g$. Our next results, among other thinks, characterizes those functions $f \in S(S^1)$ with nonzero degree for which $R(f)$ consists of all such functions g.

Theorem (2.8). *Let $f \in S(S^1)$ and suppose $\deg f = n \neq 0$. Then the following statements are equivalent.*

(2.8.1) $R(f) = \{g \in S(S^1): n \mid \deg g\}$.

(2.8.2) $[n] \in R(f)$.

(2.8.3) *f and $[n]$ are $\mathscr{R}$-equivalent.*

(2.8.4) *$f = [n] \circ k$ where k is a homeomorphism from S^1 onto S^1.*

Proof. It is immediate that (2.8.1) implies (2.8.2). We show next that (2.8.2) implies (2.8.3). We need only show that $f \in R([n])$. Let f_* be any lift of f and define $g_*(x) = (f_*(x))/n$. Since $f_*(1) - f_*(0) = n$, $g_*(1) - g_*(0) = 1$ so that g_* is a lift for its support $g \in S(S^1)$ and we use the fact that Diagram (2.1) commutes to get

$$[n] \circ g(e^{2\pi x i}) = [n]\,(e^{2\pi g_*(x) i}) = e^{2\pi f_*(x) i} = f(e^{2\pi x i})$$

for $x \in I$. Thus $f = [n] \circ g$ and f and $[n]$ are $\mathscr{R}$-equivalent.

It follows from Theorem (2.4) that (2.8.3) implies (2.8.4) so to complete the proof we need only show that (2.8.4) implies (2.8.1). Since $f = [n] \circ k$ for some homeomorphism k, f and $[n]$ are $\mathscr{R}$-equivalent and thus generate the same principal right ideal. The proof will therefore be complete when we show that $R([n]) = \{g \in S(S^1)\colon n \mid \deg g\}$. Let $\deg g = m$ and suppose first that $m \neq 0$. Then $m = nr$ for some nonzero integer r. The argument used in proving that (2.8.3) implies (2.8.4) allows us to conclude that $g = [m] \circ h$ for some $h \in S(S^1)$. Thus, $g = [n] \circ [r] \circ h$ and we see that, in this case, $g \in R([n])$. Now consider the case where $\deg g = 0$ and let g_* be any lift of g. Here, we must have $g_*(0) = g_*(1)$. As before, define $h_*(x) = (g_*(x))/n$. Then $h_*(0) = h_*(1)$ so that h_* is a lift for its support $h \in S(S^1)$. Just as before, one verifies that $g = [n] \circ h$ so that in this case also, $g \in R([n])$. This concludes the proof.

Corollary (2.9). *Suppose $f \in S(S^1)$ and $n \neq 0$. Then $f \in R([n])$ if and only if $n \mid \deg f$.*

Corollary (2.10). $R([n]) = I([n])$ *for each integer $n \neq 0$.*

Proof. This follows immediately from (2.3.1) and the previous corollary.

The previous results show that the principal right ideal generated by $[n]$ is quite extensive. The next several results show, among other things, that the situation is far different for principal left ideals.

Theorem (2.11). *Let $f \in S(S^1)$ and suppose $\deg f = n > 0$ and $f(1) = 1$. Then $f \in L([n])$ if and only if there exists a continuous function α from I into R such that $\alpha(0) = 0$, $\alpha(1) = 1$ and the function f_* defined by*

(2.11.1) $$f_*(x) = \alpha(nx - j) + j \quad \text{for} \quad j/n \leqq x \leqq (j+1)/n \quad \text{and} \quad j = 0, 1, 2, \ldots, n-1$$

is a lift for f. Moreover, if such a function α does exist then $f = g \circ [n]$ where g is the support for α.

Proof. Suppose f_* is a lift for f and let g be the support of α. Then for $x \in [j/n, (j+1)/n]$, $j = 0, 1, 2, \ldots, n-1$, we have

$$g \circ [n]\,(e^{2\pi x i}) = g(e^{2\pi n x i}) = g(e^{2\pi(nx - j) i}) = e^{2\pi(f_*(x) - j) i} = e^{2\pi f_*(x) i} = f(e^{2\pi x i}).$$

Thus, $f = g \circ [n]$ which is to say $f \in L([n])$. Conversely, suppose $f = g \circ [n]$. Then

$$1 = f(1) = g \circ [n]\,(1) = g(1)$$

and $\deg g = 1$ so there exists a lift g_* of g such that $g_*(0) = 0$ and $g_*(1) = 1$.

Since $f(1) = 1$ we can choose a lift f_* of f so that $f_*(0) = 0$. Let $A_j = [j/n, (j+1)/n]$, $j = 0, 1, 2, \ldots, n-1$ and first consider $x \in A_0$. We have

$$e^{2\pi f_*(x)i} = f(e^{2\pi xi}) = g \circ [n]\,(e^{2\pi xi}) = g(e^{2\pi nxi}) = e^{2\pi g_*(nx)i}\,.$$

This means that for each $x \in A_0$, $f_*(x)$ and $g_*(nx)$ must differ by an integer $M(x)$. Since M is a continuous function of x, it must be constant so that in fact, $f_*(x) = g_*(nx) + m$ for some integer m. Since $f_*(0) = g_*(0) = 0$, it follows that $m = 0$ and we have

$$(2.11.2) \qquad f_*(x) = g_*(nx) \quad \text{for} \quad x \in A_0\,.$$

Now, let $x \in A_1$ and we get

$$e^{2\pi f_*(x)i} = f(e^{2\pi xi}) = g \circ [n]\,(e^{2\pi xi}) = g(e^{2\pi nxi}) = g(e^{2\pi(nx-1)i}) = e^{2\pi g_*(nx-1)i}\,.$$

This means, as before, that $f_*(x)$ and $g_*(nx-1)$ must differ by an integer which is the same for all $x \in A_1$. That is, $f_*(x) = g_*(nx-1) + m$ for some m and all $x \in A_1$. Take $x = 1/n$. From (2.11.2) we get $f_*(1/n) = g_*(1) = 1$ and since $g_*(0) = 0$, it readily follows that $m = 1$ and we get

$$(2.11.3) \qquad f_*(x) = g_*(nx-1) + 1 \quad \text{for} \quad x \in A_1\,.$$

One continues by induction and gets

$$(2.11.4) \qquad f_*(x) = g_*(nx-j) + j \quad \text{for} \quad x \in A_j\,, \quad j = 0, 1, 2, \ldots, n-1\,.$$

This completes the proof.

We next consider the case where $\deg f < 0$

Theorem (2.12). *Let $f \in S(S^1)$ and suppose $\deg f = n < 0$ and $f(1) = 1$. Then $f \in L([n])$ if and only if there exists a continuous function α from I into R such that $\alpha(0) = 0$, $\alpha(1) = 1$ and the function f_* defined by*

$$(2.12.1) \qquad f_*(x) = \alpha(nx + j + 1) - n - j - 1 \quad \textit{for}$$
$$j/|n| \leqq x \leqq (j+1)/|n| \quad \textit{and} \quad j = 0, 1, 2, \ldots, |n| - 1$$

is a lift for f. Moreover, if such a function α does exist then $f = g \circ [n]$ where g is the support for α.

Proof. Suppose first that there exists a function α so that f_* defined as in (2.12.1) is a lift for f and let g be the support of α. For $j/|n| \leqq x \leqq (j+1)/|n|$, we have $0 \leqq nx + j + 1 \leqq 1$ and from this we get

$$g \circ [n]\,(e^{2\pi xi}) = g(e^{2\pi nxi}) = g(e^{2\pi(nx+j+1)i}) = e^{2\pi\alpha(nx+j+1)i} =$$
$$= e^{2\pi(f_*(x)+n+j+1)i} = e^{2\pi f_*(x)i} = f(e^{2\pi xi})\,.$$

Thus, $f = g \circ [n]$ and $f \in L([n])$.

Suppose, conversely, that $f \in L([n])$ and define $t(z) = 1/z$. Then $\deg t = -1$ and $[n] \circ t = [m]$ where $m = -n$. Thus, $f \circ t \in L([m])$ and by Theorem (2.11) there exists a continuous function α from I into R such that $\alpha(0) = 0$, $\alpha(1) = 1$ and the function $(f \circ t)_*$ defined by

$$(2.12.2) \quad (f \circ t)_*(x) = \alpha(mx - k) + k \quad \text{for} \quad k/m \leqq x \leqq (k+1)/m \quad \text{and} \quad k = 0, 1, 2, \ldots, m-1 .$$

Moreover, $f \circ t = g \circ [m]$ where g is the support of α. Define $t_*(x) = 1 - x$ for $x \in I$. Then t_* is a lift for t and $(f \circ t)_* \circ t_*$ is a lift for $f = (f \circ t) \circ t$ according to (2.3.4). After a little computation, it follows from (2.12.2) by letting $k = m - j - 1 = -n - j - 1$ that

$$(2.12.3) \qquad (f \circ t)_* \circ t_*(x) = \alpha(nx + j + 1) - n - j - 1$$
$$\text{for} \quad j/|n| \leqq x \leqq (j+1)/|n| \quad \text{and} \quad j = 0, 1, 2, \ldots, |n| - 1 .$$

Since $(f \circ t)_* \circ t_*$ is a lift for f and

$$f = (f \circ t) \circ t = (g \circ [m]) \circ t = g \circ [n] ,$$

the proof is complete.

Corollary (2.13). *Let* $f \in S(S^1)$. *Suppose* $\deg f = n \neq 0$, $f(1) = 1$, $f \in L([n])$ *and let* f_* *be any lift of* f. *Then* $f_*(j/|n|)$ *is an integer for* $j = 0, 1, 2, \ldots, |n|$.

Proof. It follows from Theorems (2.11) and (2.12) that there exists a lift f_{*1} of f such that $f_{*1}(j/|n|)$ is an integer for $j = 0, 1, 2, \ldots, |n|$. Since any two lifts differ by an integer, the corollary follows.

Some Remarks. The latter corollary accentuates the fact that $R([n]) = I([n])$ contains many functions which do not belong to $L([n])$. Simply let f_* be any continuous function from I into R such that $f_*(0) = 0$, $f_*(1) = n$ and $f_*(j/|n|)$ is not an integer for at least one integer j between 0 and $|n|$ and let f be the support of f_*. Then $f \in R([n])$ by Corollary (2.9) but $f \notin L([n])$ in view of Corollary (2.13).

Now let $\alpha(x) = 7x - 6x^2$ for $x \in I$, let n be a positive integer, define f_* as in (2.11.1) and let f be the support of f_*. Then deg $= n$ and $f \in L([n])$ by Theorem (2.11). However, $[n] \notin L(f)$. To see this, note that $f = g \circ [n]$ where g is the support of α by Theorem (2.11). Now suppose $[n] \in L(f)$. Then f and $[n]$ are $\mathscr{L}$-equivalent and thus $f = k \circ [n]$ for some homeomorphism k by Theorem (2.4). This means $g \circ [n] = k \circ [n]$ which, in turn, implies $g = k$ since $[n]$ is surjective. But g is not a homeomorphism by (2.3.5) and we have a contradiction.

In our next several results, we characterize those functions of degree $n \neq 0$ which fix 1 whose principal left ideals contain the function $[n]$.

Theorem (2.14). *Let* $f \in S(S^1)$ *and suppose* $\deg f = n > 0$ *and* $f(1) = 1$. *Then the following statements are equivalent.*

(2.14.1) $[n] \in L(f)$.

(2.14.2) $f = k \circ [n]$ *for some homeomorphism mapping* S^1 *onto* S^1.

(2.14.3) f *and* $[n]$ *are* $\mathscr{L}$*-equivalent.*

(2.14.4) *There exists an increasing homeomorphism* α *from* I *onto* I *such that the function* f_* *defined by*

$f_*(x) = \alpha(nx - j) + j$ *for* $j/n \leqq x \leqq (j + 1)/n$ *and*

$j = 0, 1, 2, \ldots, n - 1$

is a lift for f. *Moreover* $f = k \circ [n]$ *where* k *is the support of* α.

Proof. We show first that (2.14.1) implies (2.14.4) so suppose $[n] = g \circ f$ for some $g \in S(S^1)$. Then

$$1 = [n](1) = g(f(1)) = g(1)$$

and since $\deg g = 1$ there exists a lift g_* of g such that $g_*(0) = 0$ and $g_*(1) = 1$. Since $f(1) = 1$, we can choose a lift f_* of f so that $f_*(0) = 0$. Now suppose $f_*(a)$ is an integer for some $a \in I$. We then have

$$e^{2\pi nai} = [n](e^{2\pi ai}) = g \circ f(e^{2\pi ai})$$
$$g(e^{2\pi f_*(a)i}) = g(1) = 1\,.$$

Then na must be an integer and we have shown that

(2.14.5) If $f_*(a)$ is an integer for $a \in I$, then $a = j/n$ for some integer j such that $0 \leqq j \leqq n$.

Since $f_*(0) = 0$ and $\deg f = n$ we must have $f_*(1) = n$ so that $[0, n] \subseteq \operatorname{Ran} f_*$. It follows readily from this and (2.14.5) that

(2.14.6) $$f_*(j/n) = j \quad \text{for} \quad j = 0, 1, 2, \ldots, n\,.$$

Now let $A_j = [j/n, (j + 1)/n]$, $j = 0, 1, 2, \ldots, n - 1$. Since $[n] \circ E_I$ is injective on A_j° (the interior of A_j), and $[n] = g \circ f$, it follows that f_* must also be injective on A_j° and therefore on A_j as well. From this and (2.14.6), we conclude that

(2.14.7) f_* maps A_j homeomorphically onto $[j, j + 1]$ for $j = 0, 1, 2, \ldots, n - 1$.

This implies further that f_* maps I homeomorphically onto $[0, n]$. Now let us consider A_0. Then $f_*[A_0] = I$ and for $x \in A_0$, we have

$$e^{2\pi nxi} = [n](e^{2\pi xi}) = g \circ f(e^{2\pi xi}) = g(e^{2\pi f_*(x)i}) = e^{2\pi g_*(f_*(x))i}\,.$$

This means that for each $x \in A_0$, nx and $g_*(f_*(x))$ must differ by an integer $M(x)$. As in the proof of Theorem (2.11), M is a continuous function of x so that $g_*(f_*(x)) = nx + m$ for some integer m and all $x \in A_0$. Take $x = 0$ and get $0 = g_*(f_*(0)) = m$. Thus, $g_*(f_*(x)) = nx$ for $x \in A_0$. This tells us two things. Since $f_*[A_0] = I$, it tells us that g_* is injective and hence

(2.14.8) $$g_* \text{ is a homeomorphism from } I \text{ onto } I\,.$$

It further tells us that

(2.14.9) $$f_*(x) = g_*^{-1}(nx) \quad \text{for} \quad x \in A_0\,.$$

Next, consider A_1. The function f_* maps A_1 homeomorphically onto $[1, 2]$ by

(2.14.7) and $f_*(1/n) = 1$ and $f_*(2/n) = 2$ by (2.14.6). Let $x \in A_1$ and we have

$$e^{2\pi nxi} = [n](e^{2\pi xi}) = g \circ f(e^{2\pi xi}) = g(e^{2\pi f_*(x)i}) = g(e^{2\pi(f_*(x)-1)i}) = e^{2\pi g_*(f_*(x)-1)i}.$$

This means, as before, that $g_*(f_*(x) - 1) = nx + m$ for some integer m and all $x \in A_1$. Take $x = 1/n$ and get $0 = g_*(f_*(1/n) - 1) = 1 + m$. Thus $g_*(f_*(x) - 1) = nx - 1$ and we have

$$(2.14.10) \qquad f_*(x) = g_*^{-1}(nx - 1) + 1 \quad \text{for} \quad x \in A_1 .$$

One continues by induction and gets

$$(2.14.11) \quad f_*(x) = g_*(nx - j) + j \quad \text{for} \quad x \in A_j \quad \text{and} \quad j = 0, 1, 2, \ldots, n - 1 .$$

Take $\alpha = g_*^{-1}$ to get the first portion of (2.15.4). Now, $g \circ f = [n]$ and by (2.3.5) and (2.14.8), g is a homeomorphism from S^1 onto S^1 so that $f = g^{-1} \circ [n]$. Since α is a lift for g^{-1} the second portion of (2.14.4) follows and we have shown that (2.14.1) implies (2.14.4).

If (2.14.4) holds then k is a homeomorphism since α is a homeomorphism from I onto I so that (2.15.4) implies (2.15.2). Evidently, (2.14.2) implies (2.14.3) and (2.14.3) implies (2.14.1) so that the theorem is proved.

One derives the next result from Theorem (2.14) in much the same manner as we derived Theorem (2.12) from Theorem (2.12) so for that reason, we omit the proof.

Theorem (2.15). *Let $f \in S(S^1)$ and suppose $\deg f = n < 0$ and $f(1) = 1$. Then the following statements are equivalent.*

(2.15.1) $[n] \in L(f)$.

(2.15.2) $f = k \circ [n]$ *for some homeomorphism k mapping S^1 onto S^1.*

(2.15.3) *f and $[n]$ are $\mathscr{L}$-equivalent.*

(2.15.4) *There exists an increasing homeomorphism α from I onto I such that the function f_* defined by*
$f_*(x) = \alpha(nx + j + 1) - n - j - 1$ *for* $j/|n| \leq x \leq (j + 1)/|n|$ *and*
$j = 0, 1, 2, \ldots, |n| - 1$
is a lift for f. Moreover, $f = k \circ [n]$ where k is the support of α.

Corollary (2.16). *Let $f \in S(S^1)$ with $f(1) = 1$ and $\deg f = n \neq 0$. Let $[n] \in L(f)$ and let f_* be any lift of f. Then f_* is a homeomorphism from I into R such that $f_*(j/|n|)$ is an integer for $j = 0, 1, 2, \ldots, |n|$.*

Proof. This follows from Theorems (2.14) and (2.15) and the fact that any two lifts differ by an integer.

Some Remarks. The function f_* in (2.14.4) looks a great deal like the function f_* in (2.11.1) but in (2.14.4), f_* is formed under considerably more stringent conditions than in (2.11.1). Both are formed from a function α but in (2.14.4) α must be a homeomorphism from I onto I while in (2.11.1) the only requirements are that it be continuous, $\alpha(0) = 0$ and $\alpha(1) = 1$. Analogous remarks hold for (2.15.4) and (2.12.1).

It is now apparent from Theorems (2.11), (2.12), (2.14) and (2.15) that in order to get all $f \in S(S^1)$ such that $f(1) = 1$, $\deg f = n \neq 0$, $f \in L([n])$ but f and $[n]$ are not $\mathscr{L}$-equivalent, choose continuous functions α from I to R which are not injective but $\alpha(0) = 0$ and $\alpha(1) = 1$. Then form f_* as in Theorem (2.11) if $n > 0$ and as in Theorem (2.12) if $n < 0$ and take f to be the support of f_*. This is what we did in the example discussed in our remarks following the proof of Corollary (2.13).

We conclude this section with a result concerning principal two-sided ideals which we will need in the next section when we investigatate the prime ideals.

Corollary (2.17). *Let* $\deg f = n \neq 0$. *Then the following statements are equivalent.*

(2.17.1) $[n] \in I(f)$.

(2.17.2) $[n] = h \circ f \circ k$ *where* h *and* k *are homeomorphisms from* S^1 *onto* S^1.

(2.17.3) $[n]$ *and* f *are* $\mathscr{J}$*-equivalent.*

Proof. It follows immediately from Theorem (2.7) that (2.15.2) and (2.15.3) are equivalent and it is immediate that (2.15.3) implies (2.15.1). Since $f \in R([n]) = I([n])$ by Corollaries (2.4) and (2.10), we see that (2.15.1) implies (2.15.3) and the proof is complete.

3. THE BASIC PRIME IDEALS

We recall our assumption that a prime ideal of a semigroup is a proper subset of that semigroup.

Definition (3.11). Let $J_0 = \{f \in S(S^1): \deg f = 0\}$.

Lemma (3.2). J_0 *is a prime ideal of* $S(S^1)$ *and* $J_0 \subseteq J$ *for each basic prime ideal* J *of* $S(S^1)$.

Proof. It follows immediately from (2.3.1) that J_0 is a prime ideal. Let J be any basic prime ideal of $S(S^1)$, let $f \in J_0$ and choose $g \in J$ such that $\deg g = n \neq 0$. It follows from Corollary (2.9) that $g = [n] \circ h$ for some $h \in S(S^1)$ and it follows from (2.3.1) that $\deg h = 1$. Consequently, $h \notin J$ and so $[n] \in J$. But $f = [n] \circ k$ for some $k \in S(S^1)$ by Corollary (2.9) and the lemma is proved.

Definition (3.3). Let P be any nonempty collection of prime numbers. We let

$$J(P) = \{f \in S(S^1): p \text{ divides } \deg f \text{ for some } p \in P\}.$$

Theorem (3.4). *For each nonempty collection* P *of prime numbers,* $J(P)$ *is a basic prime ideal of* $S(S^1)$. *Conversely, if* J *is a basic prime ideal of* $S(S^1)$, *then* $J = J(P)$ *for some nonempty collection* P *of prime numbers.*

Proof. It follows readily from (2.3.1) that $J(P)$ is a basic prime ideal of $S(S^1)$.

Conversely, let J be a basic prime ideal of $S(S^1)$ and let

$$P = \{p\colon p \text{ is prime and } [p] \in J\}\ .$$

We show that $P \neq \emptyset$. Let f be a function in J such that $\deg f = n \neq 0$. By Corollary (2.9), $f = [n] \circ g$ for some $g \in S(S^1)$. Since $\deg g = 1$, $g \notin J$ so we must have $[n] \in J$. If $n < 0$, define $t(z) = 1/z$. Then $\deg t = -1$ and we have $[m] = [n] \circ t \in J$ where $m = -n > 0$. Consequently, we lose no generality if we assume that $n > 0$. We then have

$$[n] = [p_1] \circ [p_2] \circ \ldots \circ [p_M]$$

where the p_i are the prime factors of n and it follows that $[p_i] \in J$ for one of the primes p_i. This tells us two things. It tells first of all that $p_i \in P$ so that $P \neq \emptyset$. Secondly, it tells us that $f \in J(P)$ since $p_i \mid \deg f$. It now follows from this, Lemma (3.2) and the fact $J(P)$ is a basic prime ideal that $J \subseteq J(P)$.

Now let $f \in J(P)$. If $\deg f = 0$, then $f \in J$ by Lemma (3.2) so consider the case where $\deg f = n \neq 0$. Again, we invoke Corollary (2.9) and conclude that $f = [n] \circ g$ for some $g \in S(S^1)$. Again, $g \notin J(P)$ since $\deg g = 1$ so we have $[n] \in J(P)$. Thus $n = pm$ where $p \in P$ and $[p] \in J$. Since $f = [p] \circ [m] \circ g$ it follows that $f \in J$. Therefore, $J = J(P)$ and the proof is complete.

Theorem (3.5). *Let J be a basic prime ideal of $S(S^1)$ and let P be the (evidently unique) set of primes for which $J = J(P)$. A subset G of J is a minimal set of generators for J if and only if*

$$G = \{h_p \circ [p] \circ k_p\colon p \in P\}$$

where h_p and k_p are homeomorphisms from S^1 onto S^1 for each $p \in P$.

Proof. The ideal generated by a subset H will be denoted by $\langle H \rangle$. We show first that $\langle G \rangle = J$. Certainly, $\langle G \rangle \subseteq J$. Take any $f \in J$ and let $\deg f = n$. Then $p \mid n$ for some $p \in P$ and $f = [p] \circ g$ for some $g \in S(S^1)$ by Corollary (2.9). Thus, we have

$$f = h_p^{-1} \circ h_p \circ [p] \circ k_p \circ k_p^{-1} \circ g \in \langle G \rangle\ .$$

As for the minimality of G, suppose $H \subseteq G$ and $\langle H \rangle = J$ and choose any $h_p \circ [p] \circ k_p \in G$. Then $h_p \circ [p] \circ k_p = f \circ t \circ g$ for some $f, g \in S(S^1)$ and $t \in H$. That is,

$$h_p \circ [p] \circ k_p = f \circ h_q \circ [q] \circ k_q \circ g$$

for some prime $q \in P$ where $h_q \circ [q] \circ k_q \in H$. It readily follows that $p = q$ and hence $h_p \circ [p] \circ k_p \in H$. Thus, $H = G$.

Now we show that if $G \subseteq J$ is a minimal set of generators of J, then

$$G = \{h_p \circ [p] \circ k_p\colon p \in P\}$$

for appropriate homeomorphisms h_p, k_p. Let $g \in G$ be given and let $\deg g = n$. Then $p \mid n$ for some $p \in P$ where $[p] \in J$. Then $[p] = f_1 \circ g_1 \circ t_1$ for some $g_1 \in G$ and since $g = [p] \circ f_2$ by Corollary (2.9) we have $g = f_1 \circ g_1 \circ t_1 \circ f_2$. But this means $g = g_1$ since G is minimal and so

$$[p] = f_1 \circ g \circ t_1 \in I(g).$$

Thus, there are homeomorphisms h_p and k_p such that $g = h_p \circ [p] \circ k_p$ by Corollary (2.17). We have shown thus far that for each $g \in G$, there exists a prime $p \in P$ and homeomorphisms h_p and k_p such that $g = h_p \circ [p] \circ k_p$. On the other hand let any prime $q \in P$ be given. Then $[q] \in J$ by definition of P and thus $[q] = f \circ g \circ t$ for some $g \in G$. But $g = h_p \circ [p] \circ k_p$ and we have $[q] = f \circ h_p \circ [p] \circ k_p \circ t$. It readily follows that $q = p$ so that for each $q \in P$, there exist homeomorphisms h_q and k_q such that $h_q \circ [q] \circ k_q \in G$. This concludes the proof.

The following corollary is an immediate consequence of the previous theorem.

Corollary (3.6). *The principal basic prime ideals of $S(S^1)$ are precisely the ideals of the form $J(\{p\})$ where p is a prime number. Moreover a function $g \in S(S^1)$ is a generator for $J(\{p\})$ if and only if $g = h \circ [p] \circ k$ where h and k are homeomorphisms from S^1 onto S^1.*

Now, let $\mathscr{BP}$ denote the collection of all basic prime ideals of $S(S^1)$ and partially order $\mathscr{BP}$ in the usual manner. Next, let X be any countably infinite set, denote by $\mathscr{NS}(X)$ the collection of nonempty subsets of X and partially order $\mathscr{NS}(X)$ in the usual manner.

Theorem (3.7). *$\mathscr{BP}$ is order isomorphic to $\mathscr{NS}(X)$.*

Proof. Let Π denote the collection of all prime numbers. For $J_1, J_2 \in \mathscr{BP}$, there exist, in view of Theorem (3.4), $P_1, P_2 \in \mathscr{NS}(\Pi)$ such that $J_1 = J(P_1)$ and $J_2 = J(P_2)$. One easily verifies that $J_1 \subseteq J_2$ if and only if $P_1 \subseteq P_2$ and it follows that the map which sends J to P (where $J = J(P)$) is an order isomorphism from $\mathscr{BP}$ onto $\mathscr{NS}(\Pi)$. This concludes the proof since $\mathscr{NS}(\Pi)$ is order isomorphic to $\mathscr{NS}(X)$ for any countably infinite set X.

It is easy to see from Theorem (3.7) that $\mathscr{BP}$ is a complete upper semilattice and that not only is it not a complete lower semilattice, it fails quite badly at being even a lower semilattice. The intersection of two prime ideals in any semigroup will always be an ideal but it will rarely be prime. In fact, it will be prime if and only if one of the ideals is contained in the other. So if $J(P)$ and $J(Q)$ are any two basic prime ideals of $S(S^1)$ with the property that neither is contained in the other, then their intersection, although it will be a basic ideal, will fail to be prime. Nevertheless $J(P) \wedge J(Q)$ does exist in some cases. Specifically, one can show with little effort that $J(P) \wedge J(Q)$ exists if and only if $P \cap Q \neq \emptyset$. When $P \cap Q \neq \emptyset$, $J(P) \wedge J(Q) = J(P \cap Q)$ and this will be properly contained in $J(P) \cap J(Q)$ when neither of these ideals is contained in the other. One easily verifies, however, that the process of taking least upper bounds does coincide with the process of taking unions. That is

$$\bigvee J(P_\alpha) = J(\bigcup P_\alpha) = \bigcup J(P_\alpha)$$

where α ranges over any given index set.

The minimal elements of $\mathscr{BP}$ are, of course just the basic prime ideals. The greatest element is $J(\Pi)$ and this consists of all $f \in S(S^1)$ such that $|\deg f| \neq 1$. In any semigroup with identity, the union of any collection of prime ideals will always be a prime ideal so if a semigroup has prime ideals at all, it will have a largest prime ideal. It si reasonable to ask if $J(\Pi)$ is the largest prime ideal of $S(S^1)$ and the answer turns out to be no. Let L consist of all functions in $S(S^1)$ which are not injective on S^1 or, in other words, are not homeomorphisms from S^1 onto S^1. We noted in the introduction that L is a prime ideal and, in fact, it is quite easy to show this. Certainly, $J(\Pi) \subseteq L$ and, moreover, the inclusion is proper. For example, define $f_*(x) = 7x - 6x^2$ for $x \in I$ and let f be the support of f_*. Evidently, $\deg f = f_*(1) - f_*(0) = 1$ but f is not a homeomorphism in view of (2.3.5) so that $f \in L - J(\Pi)$. Actually, L is not only the largest prime ideal of $S(S^1)$, it is, in fact, the largest ideal of $S(S^1)$. This is an immediate consequence of Theorem (3.3) of [1]. Of course, L is not a basic ideal.

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