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SEQUENTIAL CONVERGENCES ON LATTICE ORDERED GROUPS

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Sequential convergences on groups were investigated by J. Novák in [11], cf. also the surveys of R. Frič and V. Koutník [4, 5]. The notion of sequential convergence on an abelian lattice ordered group was introduced in [8]; the non-abelian case was dealt with in [9]. Several particular cases of convergences on lattice ordered groups were studied by C. J. Everett and S. Ulam [3] and by F. Papangelou [12]. The relations between the system of all convergences on a lattice ordered group G and higher degrees of distributivity of G were investigated by J. Jakubík in [10].

Let G be a lattice ordered group. The system of all sequential convergences on G will be denoted by Conv G (for definitions, cf. Section 1 below). This system is partially ordered by the set inclusion. In the present paper the order properties of Conv G will be investigated. We establish that Conv G is a complete lower semilattice and every closed interval of Conv G is a complete Brouwerian lattice. The equivalence of the following four conditions will be shown:

(1) Conv G has a greatest element;
(2) Conv G is an upward-directed set;
(3) Conv G is a lattice;
(4) Conv G is a complete lattice.

The atoms of Conv G are described constructively in the case when G is abelian.

Some of these results were announced at the Conference on Convergence in 1984 (cf. [7]).

1. PRELIMINARIES

For notation and terminology we refer to G. Birkhoff [2] and L. Fuchs [6]. Throughout the paper, G denotes a lattice ordered group and G^+ denotes the positive cone of G.

Let N be the set of all positive integers. The set of all sequences in G will be denoted by G^N. The set G^N is a lattice ordered group with respect to the induced operation and order of G. The constant sequence $(g, g, g, \ldots)$ is denoted by const (g). If $S \in G^N$ then $S(n)$ denotes the n-th term of the sequence S.

1.1. Definition. A subset Y of $(G^N)^+$ is said to be *G-normal*, if $\text{const}(g) + S - \text{const}(g) \in Y$ whenever $g \in G$ and $S \in Y$.

1.2. Definition. A G-normal convex subsemigroup P of $(G^N)^+$ will be called *a convergence on* G if the following conditions are satisfied:

(I) If S is a sequence belonging to P, then each subsequence of S belongs to P as well.

(II) Let $S \in (G^N)^+$. If each subsequence of S has a subsequence belonging to P, then S belongs to P.

(III) Let $g \in G$. Then $\text{const}(g) \in P$ if and only if $g = 0$.

The system of all convergences on G will be denoted by Conv G.

1.3. Remark. Let $P \in \text{Conv } G$. Further, let $S \in G^N$ and $g \in G$. We denote by T the sequence with $T(n) = |S(n) - g|$ for each $n \in N$. We put $S \to_P g$ if and only if $T \in P$.

Let $P \in \text{Conv } G$. It is easy to verify that the following assertions are valid (for detailed proofs cf. [9]):

(i) if $S \to_P g$ then $T \to_P g$ for each subsequence T of S;

(ii) if $S \in G^N$ and if for each subsequence S_1 of S there exists a subsequence S_2 of S_1 such that $S_2 \to_P g$, then $S \to_P g$;

(iii) $\text{const}(g) \to_P g$ whenever $g \in G$;

(iv) if $S \to_P g_1$ and $S \to_P g_2$ then $g_1 = g_2$;

(v) if $S_1 \to_P g_1$ and $S_2 \to_P g_2$ then $(S_1 - S_2) \to_P (g_1 - g_2)$, $(S_1 \wedge S_2) \to_P (g_1 \wedge g_2)$ and $(S_1 \vee S_2) \to_P (g_1 \vee g_2)$;

(vi) if $S_1 \to_P g$, $S_2 \to_P g$ and if $S \in G^N$ with $S_1(n) \leqq S(n) \leqq S_2(n)$ for each $n \in N$, then $S \to_P g$.

In view of the above properties each convergence on G gives a convergence group in the sense of [4, 5, 11].

Conversely, let us have a partial function $\to$ from G^N into G fulfilling (i)–(vi); if we put $P = \{S \in (G^N)^+ : S \to 0\}$ then $P \in \text{Conv } G$ and the partial functions $\to$ and $\to_P$ coincide.

1.4. Remark. Let $P, Q \in \text{Conv } G$. Then $P \subseteq Q$ if and only if $S \to_P g$ implies $S \to_Q g$ whenever $S \in G^N$ and $g \in G$. Therefore there is a one-to-one order preserving mapping from Conv G into the set $\{\to_P : P \in \text{Conv } G\}$, both naturally ordered.

2. SEMILATTICE Conv G

Again, let G be a lattice ordered group. Let the set Conv G be partially ordered by inclusion. In this section we are concerned with the properties of the partially ordered set Conv G. We denote by I a non-empty system of indices.

2.1. Lemma. *Let* $P_i \in \text{Conv } G$ *for each* $i \in I$. *Then the infimum of* $\{P_i : i \in I\}$ *in* Conv G *exists. Namely,* $\inf \{P_i : i \in I\} = \bigcap_{i \in I} P_i$.

Proof. Immediate; it suffices to verify that $\bigcap_{i\in I} P_i \in \operatorname{Conv} G$ (by Definition 1.2).

The following construction will help us to solve the question about suprema in Conv G.

Let Y be a non-empty subset of $(G^N)^+$. We denote

$\delta Y = \{S \in (G^N)^+ :$ there exists $R \in Y$ such that S is a subsequence of $R\}$;

$\langle Y\rangle = \{S \in (G^N)^+ :$ there exist $R_1, R_2, \ldots, R_k \in Y$ and $g_1, g_2, \ldots, g_k \in G$ such that $S(n) = g_1 + R_1(n) - g_1 + g_2 + R_2(n) - g_2 + \ldots + g_k + R_k(n) - g_k$ for each $n \in N\}$;

$[Y] = \{S \in (G^N)^+ :$ there exists $R \in Y$ such that $S(n) \leqq R(n)$ for each $n \in N\}$;

$Y^* = \{S \in (G^N)^+ :$ for each subsequence S_1 of S there exists a subsequence S_2 of S_1 such that $S_2 \in Y\}$;

$\bar{Y} = [\langle\delta Y\rangle]^*$.

2.2. Theorem. *Let Y be a non-empty subset of $(G^N)^+$. If $[\langle\delta Y\rangle]$ does not contain* const (g) *for any $g \in G$, $g \neq 0$, then $\bar{Y}$ is the smallest element of* Conv G *containing Y. In the opposite case there exists no $P \in$* Conv G *containing Y.*

This assertion was established for the abelian case in [7] (Theorem 2). In the non-abelian case only slight modifications in the proof are needed (for details cf. [9], Theorem 1.18).

2.3. Lemma. *Let $P_i \in$* Conv G *for each $i \in I$. If there is $P_{up} \in$* Conv G *such that $P_i \subseteq P_{up}$ for each $i \in I$ then there exists* $\sup\{P_i : i \in I\}$ *in* Conv G. *Namely* $\sup\{P_i : i \in I\} = \langle\bigcup_{i\in I} P_i\rangle^*$.

Proof. The system $\mathscr{P} = \{P \in \operatorname{Conv} G : P \subseteq P_{up}\}$ has a greatest element. By Lemma 2.1, $\mathscr{P}$ is a complete lattice. Therefore there exists a supremum P_{sup} of the system $\{P_i : i \in I\}$ in $\mathscr{P}$. Clearly, P_{sup} is the supremum of $\{P_i : i \in I\}$ in Conv G. Denote $Y = \bigcup_{i\in I} P_i$. It is easy to see that $Y \subseteq P_{up}$ and thus also $[\langle\delta Y\rangle] \subseteq P_{up}$. Since P_{up} has no constant sequence except const (0), $[\langle\delta Y\rangle]$ cannot have it, either. By Theorem 2.2, $P_{sup} = \bar{Y}$. In order to complete the proof it suffices (because of $\delta Y = Y$ and $[\langle Y\rangle] \supseteq \langle Y\rangle$) to prove that $[\langle Y\rangle] \subseteq \langle Y\rangle$. Then $P_{sup} = \bar{Y} = [\langle\delta Y\rangle]^* = \langle Y\rangle^* = \langle\bigcup_{i\in I} P_i\rangle^*$. So, let $S \in [\langle\delta Y\rangle]$, i.e. $S \in (G^N)^+$ and there is $T \in \langle Y\rangle$ such that $S(n) \leqq T(n)$ for each $n \in N$. There exist $k \in N$, $T_j \in Y$, $g_j \in G$ for each $j \in \{1, 2, \ldots, k\}$ such that $T = \sum_{j=1}^{k} (\operatorname{const}(g_j) + T_j - \operatorname{const}(g_j))$.

For the moment fix $n \in N$.

We have $0 \leqq S(n) \leqq \sum_{j=1}^{k} (g_j + T_j(n) - g_j)$. Because of the Riesz property of a lattice ordered group (see for example [6]) there are $S_1(n), S_2(n), \ldots, S_k(n)$ in G such that $0 \leqq S_j(n) \leqq g_j + T_j(n) - g_j$ for each $j \in \{1, 2, \ldots, k\}$, and $S(n) = \sum_{j=1}^{k} S_j(n)$.

In this way we get sequences $S_j \in (G^N)^+$, $j \in \{1, 2, \ldots, k\}$ with $S_j \leqq \operatorname{const}(g_j) + T_j - \operatorname{const}(g_j)$ and $S = \sum_{j=1}^{k} S_j$. Now, $T_j \in Y$ implies $S_j \in Y$ for each $j \in \{1, 2, \ldots, k\}$ and thus $S \in \langle Y\rangle$.

2.4. Lemma. *Let $\{P_i : i \in I\}$ be a chain in* Conv G. *Then* $(\bigcup_{i\in I} P_i)^* \in$ Conv G.

Proof. Straightforward.

A partially ordered set K is said to be a *complete lower semilattice* if each non-empty subset of K has an infimum in K.

2.5. Theorem. (a) Conv G *is a complete lower semilattice.*

(b) *Every chain of* Conv G *is bounded.*

(c) *Every closed interval of* Conv G *is a complete Brouwerian lattice.*

Proof. (a) is a corollary of Lemma 2.1.

(b) is a corollary of Lemma 2.4.

(c): An arbitrary closed interval of Conv G has a greatest element and by Lemma 2.1, it contains infima of all of its non-empty subsets. Therefore it is a complete lattice. In view of [2], it suffices to prove that the infinite meet-distributive law holds for this complete lattice. We will do it. Let $Q_1, Q_2 \in \text{Conv } G$ and $Q_1 \subseteq Q_2$. Consider the closed interval of Conv G from Q_1 to Q_2. Let $P \in \text{Conv } G$ such that $Q_1 \subseteq P \subseteq Q_2$. Let I be a non-empty system of indices and let $P_i \in \text{Conv } G$ and $Q_1 \subseteq P_i \subseteq Q_2$ for each $i \in I$. According to Lemmas 2.1 and 2.3, it suffices to verify that $P \cap \langle \bigcup_{i \in I} P_i \rangle^* = \langle \bigcup_{i \in I} (P \cap P_i) \rangle^*$.

"$\subseteq$": Let $S \in P \cap \langle \bigcup_{i \in I} P_i \rangle^*$ and let T be a subsequence of S. Then there is a subsequence R of T belonging to $\langle \bigcup_{i \in I} P_i \rangle$. Therefore there are $S_1, S_2, \ldots, S_k \in \in \bigcup_{i \in I} P_i$ and $g_1, g_2, \ldots, g_k \in G$ such that $R = \sum_{j=1}^{k} (\text{const}(g_j) + S_j - \text{const}(g_j))$. Since $g_j + S_j(n) - g_j \leqq R(n)$ for each $n \in N$, $j \in \{1, 2, \ldots, k\}$ and $R \in P$, thus $S_j \in P$ for each $j \in \{1, 2, \ldots, k\}$. Because of $S_j \in \bigcup_{i \in I} P_i$ we have $\{S_1, S_2, \ldots, S_k\} \subseteq \subseteq \bigcup_{i \in I} (P \cap P_i)$ and $R \in \langle \bigcup_{i \in I} P \cap P_i) \rangle$. Therefore $S \in \langle \bigcup_{i \in I} (P \cap P_i) \rangle^*$. The converse inequality ("$\supseteq$") is obvious.

Let G and Conv G be as above. Then the discrete convergence on G defined by $d(G) = \{S \in (G^N)^+ : S(n) = 0 \text{ for all but finitely many } n \in N\}$ is the smallest element of Conv G. On the other hand, Conv G need not have a greatest element (cf. [7, 9]). J. Jakubík has shown in [10] that if G is a completely distributive archimedean lattice ordered group, then Conv G has a greatest element.

2.6. Theorem. *The following conditions are equivalent*:

(1) Conv G *has a greatest element.*

(2) Conv G *is an upward-directed set.*

(3) Conv G *is a lattice.*

(4) Conv G *is a complete lattice.*

Proof. (1) implies (2), trivially. (2) and Lemmas 2.1, 2.3 imply (3). (3) implies (4): Suppose that Conv G is a lattice but not a complete one. In view of Lemmas 2.1 and 2.3 there are $P_i \in \text{Conv } G$, $i \in I$, such that $\langle \bigcup_{i \in I} P_i \rangle^* \notin \text{Conv } G$. By Theorem 2.2 (with $Y = \bigcup_{i \in I} P_i$) there exists $g \in G$, $g \neq 0$, such that $\text{const}(g) \in \langle \bigcup_{i \in I} P_i \rangle$. Hence there are sequences $S_1, S_2, \ldots, S_k$ in $\bigcup_{i \in I} P_i$ and elements $g_1, g_2, \ldots, g_k$ in G such that $g = \sum_{j=1}^{k} (g_j + S_j(n) - g_j)$ for each $n \in N$. For each $j \in \{1, 2, \ldots, k\}$ there is

$i(j) \in I$ with $S_j \in P_{i(j)}$. However, it follows that a finite subset of Conv G, e.g., $\{P_{i(1)}, P_{i(2)}, \ldots, P_{i(k)}\}$ has no upper bound in Conv G; this contradicts (3). Finally, (4) implies (1) trivially.

3. ATOMS OF Conv G

In this section we assume that G is an abelian lattice ordered group. First we recall some notions. Let S be a sequence in G. Then S is said to be *orthogonal* if $S(n) > 0$ for each $n \in N$ and $S(i) \wedge S(j) = 0$ for each $i, j \in N$, $i \neq j$.

An element $b \in G$ is basic if $b > 0$ and the interval from zero to b is a chain.

The notion of an atom of the partially ordered set Conv G has the usual meaning. Thus, a convergence $P \in$ Conv G is an atom in Conv G if for each $Q \in$ Conv G, $Q \subseteq P$ implies $Q = d(G)$ or $Q = P$ and $P \neq d(G)$.

3.1. Lemma. *Let G have no basic element and let $S \in G^N$ with $S(n) > 0$ for each $n \in N$. Then there exist T and S_0 in G^N such that*
T is orthogonal,
S_0 is a subsequence of S and
$T(n) \leqq S_0(n)$ for each $n \in N$.

Proof. Since $S(1)$ is not basic, there exist $a_1, b_1 \in G$ such that $0 < a_1 < S(1)$, $0 < b_1 < S(1)$ and $a_1 \wedge b_1 = 0$.

Denote $N_1 = \{n \in N: S(n) \wedge a_1 > 0\}$.

If N_1 is a finite set, then put $T(1) = a_1$ and denote by S' the sequence that arises from S by deleting the terms $S(n)$ for which $S(n) \wedge a_1 > 0$.

If N_1 is infinite, then put $T(1) = b_1$ and denote by S' the sequence that arises from $S \wedge \text{const}(a_1)$ by deleting the first term and all zero members.

In both of these cases our choice yields $T(1)$ and S' such that $0 < T(1) < S(1)$, $S'(n) > 0$ for each $n \in N$, $S' \wedge \text{const}(T(1)) = \text{const}(0)$ and a subsequence S_1 of S such that $S'(n) \leqq S_1(n)$ for each $n \in N$. Since $S'(1)$ is not basic either, we can repeat the same procedure as we did for S, now for S'. In this way we obtain $T(2) \in G$, $S'' \in G^N$ and a subsequence S_2 of S such that $0 < T(2) < S'(1)$, $S'' \wedge \text{const}(T(2)) = \text{const}(0)$ and $0 < S''(n) \leqq S_2(n)$ for each $n \in N$. Moreover, $T(1) \wedge T(2) = 0$.

We proceed by induction. The sequence T obtained is orthogonal and there is a subsequence S_0 of S with $T(n) \leqq S_0(n)$ for each $n \in N$.

3.2. Lemma. *If P is an atom of* Conv G *then there is no orthogonal sequence in P.*

Proof. Assume to the contrary that T is an orthogonal sequence and $T \in P$. Denote
$T_1(n) = T(2n)$ and
$T_2(n) = T(2n + 1)$ for each $n \in N$ and put
$P_1 = \overline{\{T_1\}}$ and

$P_2 = \overline{\{T_2\}}$. By [8] (Theorem 7.3 and Corollary 7.6) we have $P_1 \in \operatorname{Conv} G$, $P_2 \in \operatorname{Conv} G$ and $P_1 \neq P_2$. On the other hand, $P_1 \subseteq P$ and $P_2 \subseteq P$ (cf. Theorem 2.2). Since P is an atom in Conv G and neither P_1 nor P_2 is a discrete convergence, $P_1 = P = P_2$ is valid, which is a contradiction.

From 3.1 and 3.2 we obtain

3.3. Theorem. *If G has no basic element then* Conv G *has no atom.*

From now on throughout this section, let B denote the set of all basic elements of G. For a subset H of G we denote $H^{\perp} = \{g \in G: |g| \wedge |h| = 0 \text{ for all } h \in H\}$.

3.4. Lemma. *Let P be an atom in* Conv G *and let $S \in P$ with $S(n) > 0$ for each $n \in N$. Then $\{S(n): n \in N\} \cap B^{\perp}$ is a finite set.*

Proof. On the contrary, suppose that there exists a subsequence S' of S such that $S'(n) \in B^{\perp}$ for each $n \in N$. Since $B^{\perp}$ is a convex l-subgroup of G and thus an l-group without basic elements, we can apply Lemma 3.1 for $B^{\perp}$ and S'. Therefore there are T' and S'_0 in G^N such that

T' is orthogonal,

S'_0 is a subsequence of S' and

$T'(n) \leqq S'_0(n)$ for each $n \in N$.

Clearly, $T' \in P$, which contradicts Lemma 3.2.

3.5. Lemma. *Let H be a linearly ordered convex l-subgroup of G and let S be a decreasing sequence in H with* $\inf\{S(n): n \in N\} = 0$. *Then $\overline{\{S\}}$ is an atom in* Conv G.

Proof. Denote $P_a = \overline{\{S\}}$. By applying the results of [8], $P_a \in \operatorname{Conv} G$. We shall show that P_a is an atom in Conv G. Let $P \in \operatorname{Conv} G$ and let $P \subseteq P_a$. It is easy to verify that $P \cap H^N$ and $P_a \cap H^N$ are elements of Conv H. According to [8] (Theorem 3.9), Conv H has at most two elements including $d(H)$. Since $S \in P_a \cap H^N$ and thus $P_a \cap H^N \neq d(H)$, either $P \cap H^N = d(H)$ or $P \cap H^N = P_a \cap H^N$ follows. Because $P = (P \cap H^N)^*$ and $P_a = (P_a \cap H^N)^*$, we have $P = (P \cap H^N)^* = (d(H))^* = d(G)$ or $P = (P \cap H^N)^* = (P_a \cap H^N)^* = P_a$.

3.6. Theorem. *Let $P \in \operatorname{Conv} G$. Then the following conditions are equivalent:*

(i) *P is an atom in* Conv G;

(ii) *there exists a linearly ordered convex l-subgroup H of G which contains a decreasing sequence S such that* $\inf\{S(n): n \in N\} = 0$ *and $P = \overline{\{S\}}$.*

Proof. Let P be an atom in Conv G and let $T \in P$, $T \notin d(G)$. Assume that the set $\{n \in N: T(n) \wedge b > 0\}$ is finite for each $b \in B$. By Lemma 3.4, the set $\{n \in N:$ there is $b \in B$ such that $T(n) \wedge b > 0\}$ is infinite. Then there is a subsequence S_T of T and a one-to-one sequence $S_B \in B^N$ such that $S_T(n) \wedge S_B(n) > 0$ for each $n \in N$. If we denote $T_0(n) = S_T(n) \wedge S_B(n)$ for each $n \in N$, then $T_0 \in P$ and T_0 is orthogonal, which contradicts Lemma 3.2. So, our assumption above was not right and there is $b_0 \in B$ such that the set $\{n \in N: T(n) \wedge b_0 > 0\}$ is infinite. Let S_0 be the sequence that arises from T by deleting all terms $T(n)$ for which $T(n) \wedge b_0 = 0$. Clearly, $S_0 \in P$.

Denote $H = \{b_0\}^{\perp\perp}$. Then H is a linearly ordered convex l-subgroup of G (cf. [1], Proposition 3.2.3 and Corollary 7.2.5) and $S_0(n) \in H$ for each $n \in N$. It is easy to see (cf. [8], Lemma 3.3) that there exists a decreasing subsequence S of S_0; therefore $S \in P$ and $\overline{\{S\}} \subseteq P$. Since P is an atom in Conv G and $S \notin d(G)$, we have $\overline{\{S\}} = P$. By [8] (Lemma 3.2) we conclude $\inf\{S(n): n \in N\} = 0$. Lemma 3.5 completes the proof.

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