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ON UNBOUNDED POSITIVE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

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1. INTRODUCTION

This paper is concerned with the asymptotic behavior of positive solutions of the nonlinear ordinary differential equation

$$y^{(n)} + \sum_{i=1}^{N} q_i(t) f_i(y) = 0 \tag{1}$$

subject to the hypotheses

(2) (a) $n \geqq 2$;

(b) each q_i: $[0, \infty) \to \mathbb{R}$, $1 \leqq i \leqq N$, is continuous;

(c) each f_i: $[0, \infty) \to (0, \infty)$, $1 \leqq i \leqq N$, is continuous and nondecreasing.

Our attention will be focused on the case where each coefficient $q_i(t)$ in (1) is oscillating, that is, $q_i(t)$ changes its sign in any neighbourhood of infinity.

It is known that if, for some integer k, $0 \leqq k \leqq n - 1$, there is a constant $c > 0$ such that

$$\sum_{i=1}^{N} \int_0^\infty t^{n-k-1} |q_i(t)| f_i(ct^k) \, dt < \infty , \tag{3}$$

then equation (1) has a positive solution $y(t)$ which is asymptotic to the solution t^k of the corresponding unperturbed equation $y^{(n)} = 0$ in the sense that

$$\lim_{t\to\infty} \frac{y(t)}{t^k} = \text{const} > 0 ; \tag{4}$$

see Hale and Onuchic [1], Kitamura [2] and Švec [5].

In this paper we are interested in the situation in which equation (1) possesses a positive solution which is asymptotic to none of the solutions of $y^{(n)} = 0$; more precisely, we want to find criteria for the existence of a positive solution $y(t)$ of (1) with the property

$$\lim_{t\to\infty} \frac{y(t)}{t^k} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{y(t)}{t^{k-1}} = \infty \tag{5}$$

for some integer k, $1 \leqq k \leqq n - 1$. The desired existence criteria, given in Theorems 1 and 2 below, are formulated in terms of the positive part $(q_i)_+ (t)$ and the negative parts $(q_i)_-(t)$ of the coefficients $q_i(t)$:

(6) $\quad (q_i)_+(t) = \max\{q_i(t), 0\}\,, \quad (q_i)_-(t) = \max\{-q_i(t), 0\}\,, \quad 1 \leqq i \leqq N\,,$

and show that, in case k is such that $n \not\equiv k \pmod 2$ [resp. $n \equiv k \pmod 2$], there exists a solution $y(t)$ satisfying (5) provided the contribution of $(q_i)_+(t)$ is greater than that of $(q_i)_-(t)$ [resp. the contribution of $(q_i)_-(t)$ is greater than that of $(q_i)_+(t)$] in a suitable sense. Our results include part of the recent results of Kusano and Naito [3, 4] on the same problem for equation (1) in which all $q_i(t) > 0$ or all $q_i(t) < 0$ on $[0, \infty)$.

2. MAIN RESULTS

Our first result is the following

Theorem 1. (i) *Let k be an integer such that $1 \leqq k \leqq n - 1$ and $n \not\equiv k \pmod 2$. Then equation (1) has a positive solution $y(t)$ satisfying (5) if the following conditions are satisfied:*

(7) $\quad \sum_{i=1}^{N} \int_0^\infty t^{n-k-1}(q_i)_+(t) f_i(at^k)\,dt < \infty$ *for some* $a > 0\,,$

(8) $\quad \sum_{i=1}^{N} \int_0^\infty t^{n-k}(q_i)_-(t) f_i(bt^k)\,dt < \infty$ *for some* $b > 0\,,$

(9) $\quad \int_0^\infty t^{n-k}(q_{i_0})_+(t) f_{i_0}(ct^{k-1})\,dt = \infty$ *for some* $i_0\,,\ 1 \leqq i_0 \leqq N\,,$ *and all* $c > 0\,.$

(ii) *Let k be an integer such that $1 \leqq k \leqq n - 1$ and $n \equiv k \pmod 2$. Then equation (1) has a positive solution $y(t)$ satisfying (5) if the following conditions are satisfied:*

(10) $\quad \sum_{i=1}^{N} \int_0^\infty t^{n-k-1}(q_i)_-(t) f_i(at^k)\,dt < \infty$ *for some* $a > 0\,,$

(11) $\quad \sum_{i=1}^{N} \int_0^\infty t^{n-k}(q_i)_+(t) f_i(bt^k)\,dt < \infty$ *for some* $b > 0\,,$

(12) $\quad \int_0^\infty t^{n-k}(q_{i_0})_-(t) f_{i_0}(ct^{k-1})\,dt = \infty$ *for some* i_0, $1 \leqq i_0 \leqq N$, *and all* $c > 0\,.$

Proof. In either of the cases (i) and (ii) the desired solution of equation (1) will be obtained, via the Schauder-Tychonoff fixed point theorem, as a solution of the integral equation

(13)
$$y(t) = \frac{\alpha t^{k-1}}{(k-1)!} + (-1)^{n-k-1} \int_T^\infty \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r) f_i(y(r))\,dr\,ds\,,$$
$$t \geqq T\,,$$

for suitably chosen $\alpha > 0$ and $T > 0$.

(i) Let k, $1 \leqq k \leqq n - 1$, be such that $n \not\equiv k \pmod 2$. Let α, $0 < \alpha < \min\{a, b\}$, be fixed, where a and b are positive constants appearing in (7) and (8). Because of (7) and (8) there is a constant $T > 0$ such that

$$\sum_{i=1}^{N} \int_T^\infty t^{n-k-1}(q_i)_+ (t) f_i(\alpha(t+1)^k)\, \mathrm{d}t \leqq \alpha \tag{14}$$

and

$$\sum_{i=1}^{N} \int_T^\infty t^{n-k}(q_i)_- (t) f_i(\alpha(t+1)^k)\, \mathrm{d}t \leqq \tfrac{1}{2}\alpha\,. \tag{15}$$

Let $C[T, \infty)$ be the locally convex space of all continuous functions on $[T, \infty)$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$, and consider the closed convex subset of $C[T, \infty)$ defined by

$$Y = \left\{ y \in C[T, \infty) \colon \frac{\alpha t^{k-1}}{2(k-1)!} \leqq y(t) \leqq \frac{\alpha t^{k-1}}{(k-1)!} + \frac{\alpha t^k}{k!}\,, \quad t \geqq T \right\}. \tag{16}$$

Define the mapping $F\colon Y \to C[T, \infty)$ by

(17)
$$F\, y(t) = \frac{\alpha t^{k-1}}{(k-1)!} + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r) f_i(y(r))\, \mathrm{d}r\, \mathrm{d}s\,, \quad t \geqq T\,.$$

It can be shown that F is a continuous mapping which sends Y into a compact subset of Y.

(a) $F(Y) \subset Y$. Noting that $q_i(t) = (q_i)_+ (t) - (q_i)_- (t)$ and using (14), we have for $y \in Y$

$$\begin{aligned} Fy(t) &\leqq \frac{\alpha t^{k-1}}{(k-1)!} + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} (q_i)_+ (r) f_i(y(r))\, \mathrm{d}r\, \mathrm{d}s \leqq \\ &\leqq \frac{\alpha t^{k-1}}{(k-1)!} + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!}\, \mathrm{d}s \sum_{i=1}^{N} \int_T^\infty \frac{r^{n-k-1}}{(n-k-1)!} (q_i)_+ (r) f_i(\alpha(r+1)^k)\, \mathrm{d}r \leqq \\ &\leqq \frac{\alpha t^{k-1}}{(k-1)!} + \frac{\alpha t^k}{k!} \leqq \alpha(t+1)^k\,, \quad t \geqq T\,. \end{aligned} \tag{18}$$

On the other hand, $y \in Y$ implies

$$\begin{aligned} Fy(t) &\geqq \frac{\alpha t^{k-1}}{(k-1)!} - \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} (q_i)_- (r) f_i(y(r))\, \mathrm{d}r\, \mathrm{d}s = \\ &= \frac{\alpha t^{k-1}}{(k-1)!} - \int_T^t \frac{(t-\sigma)^{k-2}}{(k-2)!} \int_T^\sigma \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} (q_i)_- (r) f_i(y(r))\, \mathrm{d}r\, \mathrm{d}s \geqq \\ &\geqq \frac{\alpha t^{k-1}}{(k-1)!} - \int_T^t \frac{(t-\sigma)^{k-2}}{(k-2)!}\, \mathrm{d}\sigma \sum_{i=1}^{N} \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)_- (r) f_i(\alpha(r+1)^k)\, \mathrm{d}r\, \mathrm{d}s \end{aligned} \tag{19}$$

for $t \geqq T$. Since, by (15),

$$\sum_{i=1}^{N} \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)_- (r) f_i(\alpha(r+1)^k)\, \mathrm{d}r\, \mathrm{d}s = \tag{20}$$

$$= \sum_{i=1}^{N} \int_T^t \left(\int_T^r \frac{(r-s)^{n-k-1}}{(n-k-1)!} \, ds \right) (q_i)_- (r) f_i(\alpha(r+1)^k) \, dr +$$

$$+ \sum_{i=1}^{N} \int_t^\infty \left(\int_T^t \frac{(r-s)^{n-k-1}}{(n-k-1)!} \, ds \right) (q_i)_- (r) f_i(\alpha(r+1)^k) \, dr \leqq$$

$$\leqq \sum_{i=1}^{N} \int_T^\infty \frac{(r-T)^{n-k}}{(n-k)!} (q_i)_- (r) f_i(\alpha(r+1)^k) \, dr \leqq \frac{\alpha}{2}, \quad t \geqq T,$$

we see from (19) that for $y \in Y$

$$Fy(t) \geqq \frac{\alpha t^{k-1}}{(k-1)!} - \frac{\alpha t^{k-1}}{2(k-1)!} = \frac{\alpha t^{k-1}}{2(k-1)!}, \quad t \geqq T. \tag{21}$$

In deriving (21) from (19) we have assumed that $k \geqq 2$. It is simpler to verify that (21) also holds for $k = 1$. It follows that $y \in Y$ implies $Fy \in Y$, that is, $F(Y) \subset Y$.

(b) F is continuous. Let $\{y_\nu\}$ be a sequence of elements of Y converging to $y \in Y$ in the $C[T, \infty)$ topology. We then have

$$|Fy_\nu(t) - Fy(t)| \leqq \frac{t^k}{k!} \int_T^\infty r^{n-k-1} \sum_{i=1}^{N} |q_i(r)| \, |f_i(y_\nu(r)) - f_i(y(r))| \, dr$$

for $t \geqq T$. The integrand on the right hand side of the above tends to zero pointwise on $[T, \infty)$ as $\nu \to \infty$ and is bounded by

$$2r^{n-k-1} \sum_{i=1}^{N} [(q_i)_+ (r) + (q_i)_- (r)] \, f_i(\alpha(r+1)^k)$$

which is integrable on $[T, \infty)$ by (7) and (8), and so the Lebesgue dominated convergence theorem shows that $Fy_\nu(t) \to Fy(t)$ as $\nu \to \infty$ uniformly on each compact subinterval of $[T, \infty)$. Therefore, $Fy_\nu \to Fy$ as $\nu \to \infty$ in $C[T, \infty)$, which implies the continuity of F.

(c) $F(Y)$ is relatively compact. This follows from the observation that, for $y \in Y$, $(Fy)'$ is given by

$$(Fy)'(t) = \int_t^\infty \frac{(r-t)^{n-2}}{(n-2)!} \sum_{i=1}^{N} q_i(r) f_i(y(r)) \, dr, \quad k = 1,$$

$$(Fy)'(t) = \frac{\alpha t^{k-2}}{(k-2)!} + \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r) f_i(y(r)) \, dr \, ds, \quad k \geqq 2,$$

and satisfies

$$|(Fy)'(t)| \leqq \textstyle\int_T^\infty r^{n-2} \sum_{i=1}^{N} |q_i(r)| \, f_i(\alpha(r+1)^k) \, dr, \quad k = 1,$$

$$|(Fy)'(t)| \leqq \frac{\alpha t^{k-2}}{(k-2)!} + \frac{t^{k-1}}{(k-1)!} \int_T^\infty r^{n-k-1} \sum_{i=1}^{N} |q_i(r)| \, f_i(\alpha(r+1)^k) \, dr, \quad k \geqq 2.$$

Therefore the Schauder-Tychonoff fixed point theorem is applicable to F and there exists an element $y \in Y$ such that $y = Fy$. This function $y = y(t)$ satisfies (13), so that it is a positive solution of equation (1) on $[T, \infty)$.

To study the asymptotic behavior of $y(t)$ we note from (13) that

$$y^{(k-1)}(t) = \alpha + \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^N q_i(r) f_i(y(r)) \, dr \, ds \,, \quad t \geqq T$$

and

$$y^{(k)}(t) = \int_t^\infty \frac{(r-t)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^N q_i(r) f_i(y(r)) \, dr \,, \quad t \geqq T.$$

It is obvious that $y^{(k)}(t) \to 0$ as $t \to \infty$. Use of (20) shows that

$$y^{(k-1)}(t) = \alpha - \sum_{i=1}^N \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)_- (r) f_i(y(r)) \, dr \, +$$

$$+ \sum_{i=1}^N \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)_+ (r) f_i(y(r)) \, dr \geqq$$

$$\geqq \frac{\alpha}{2} + \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_{i_0})_+ (r) f_{i_0}(y(r)) \, dr \, ds \geqq$$

$$\geqq \frac{\alpha}{2} + \int_T^t \frac{(r-T)^{n-k}}{(n-k)!} (q_{i_0})_+ (r) f_{i_0}\left(\frac{\alpha r^{k-1}}{2(k-1)!}\right) dr \,, \quad t \geqq T \,,$$

which, combined with (9), implies that $y^{(k-1)}(t) \to \infty$ as $t \to \infty$. Thus, the solution $y(t)$ satisfies $\lim_{t\to\infty} y^{(k)}(t) = 0$ and $\lim_{t\to\infty} y^{(k-1)}(t) = \infty$ which is equivalent to (5) by L'Hospital's rule. This completes the proof of (i).

(ii) Let k, $1 \leqq k \leqq n-1$, be an integer such that $n \equiv k \pmod 2$. Let α, $0 < \alpha < \min\{a, b\}$ be fixed, take $T > 0$ so large that

$$\sum_{i=1}^N \int_T^\infty t^{n-k-1} (q_i)_- (t) f_i(\alpha(t+1)^k) \, dt \leqq \alpha \,,$$

$$\sum_{i=1}^N \int_T^\infty t^{n-k} (q_i)_+ (t) f_i(\alpha(t+1)^k) \, dt \leqq \tfrac{1}{2}\alpha$$

and define the operator F by

$$Fy(t) = \frac{\alpha t^{k-1}}{(k-1)!} - \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^N q_i(r) f_i(y(r) \, dr \, ds \,, \quad t \geqq T.$$

Then, applying the same argument as in the preceding case, we obtain a fixed point y of F in the set Y defined by (16), which gives rise to a positive solution of equation (1) existing on $[T, \infty)$ and having the asymptotic behavior (5). This finishes the proof.

Example 1. Consider the mixed sublinear-superlinear equation

$$y^{(n)} + \varphi(t) \, y^\gamma + \psi(t) \, y^\delta = 0 \,, \tag{22}$$

where $n \geqq 2$, $0 < \gamma < 1$, $\delta > 1$ and $\varphi, \psi \colon [0, \infty) \to \mathbb{R}$ are continuous.

Conditions (7) and (8) reduce, respectively, to

$$(23) \qquad \int^{\infty} t^{n-k-1+k\gamma}\,\varphi_{+}(t)\,\mathrm{d}t < \infty\,, \quad \int^{\infty} t^{n-k-1+k\delta}\,\psi_{+}(t)\,\mathrm{d}t < \infty$$

and

$$(24) \qquad \int^{\infty} t^{n-k+k\gamma}\,\varphi_{-}(t)\,\mathrm{d}t < \infty\,, \quad \int^{\infty} t^{n-k+k\delta}\,\psi_{-}(t)\,\mathrm{d}t < \infty\,,$$

and condition (9) is equivalent to requiring that either

$$(25) \qquad \int^{\infty} t^{n-k+(k-1)\gamma}\,\varphi_{+}(t)\,\mathrm{d}t = \infty$$

or

$$(26) \qquad \int^{\infty} t^{n-k+(k-1)\delta}\,\psi_{+}(t)\,\mathrm{d}t = \infty\,.$$

It is easily seen that (26) and the second condition of (23) are not consistent because $(n - k - 1 + k\delta) - (n - k + (k - 1)\,\delta) = \delta - 1 > 0$. Therefore, by (i) of Theorem 1 we conclude that if k is an integer such that $1 \leqq k \leqq n - 1$ and $n \not\equiv k \pmod 2$, then conditions (23), (24) and (25) are sufficient for equation (22) to have a positive solution $y(t)$ satisfying (5). Similarly, the conditions

$$(27) \qquad \int^{\infty} t^{n-k-1+k\gamma}\,\varphi_{-}(t)\,\mathrm{d}t < \infty\,, \quad \int^{\infty} t^{n-k-1+k\delta}\,\psi_{-}(t)\,\mathrm{d}t < \infty\,,$$

$$(28) \qquad \int^{\infty} t^{n-k+k\gamma}\,\varphi_{+}(t)\,\mathrm{d}t < \infty\,, \quad \int^{\infty} t^{n-k+k\delta}\,\psi_{+}(t)\,\mathrm{d}t < \infty$$

and

$$(29) \qquad \int^{\infty} t^{n-k+(k-1)\gamma}\,\varphi_{-}(t)\,\mathrm{d}t = \infty$$

guarantee the existence of a positive solution $y(t)$ satisfying (5). It should be noticed that Theorem 1 cannot be applied to the purely superlinear case of (22).

Theorem 1 is a local existence theorem in that the solution is guaranteed to exist on an interval $[T, \infty)$, $T > 0$ being sufficiently large, that is, in a "small" neighborhood of infinity. Under stronger sublinearity hypotheses on $f_i(y)$ the global existence of a solution satisfying (5) can be established as the following theorem shows.

Theorem 2. *Suppose that, for each i, $1 \leqq i \leqq N$, $y^{-1} f_i(y)$ is nonincreasing and satisfies*

$$(30) \qquad \lim_{y\to\infty} y^{-1} f_i(y) = 0\,.$$

(i) *Let k be an integer such that $1 \leqq k \leqq n - 1$ and $n \not\equiv k \pmod 2$. Then, condition (7), (8) and (9) are sufficient for equation (1) to have infinitely many positive solutions $y(t)$ which exist on $[0, \infty)$ and satisfy (5)*

(ii) *Let k be an integer such that $1 \leqq k \leqq n - 1$ and $n \equiv k \pmod 2$. Then conditions (10), (11) and (12) are sufficient for equation (1) to have infinitely many positive solutions $y(t)$ which exist on $[0, \infty)$ and satisfy (5).*

Proof. We only prove the statement (i), since the statement (ii) is proved similarly. Suppose that k satisfies $1 \leqq k \leqq n - 1$ and $n \not\equiv k \pmod 2$. Let α, $0 < \alpha < \min\{a, b\}$, be fixed. Then, by (7), the function $\sum_{i=1}^{N} t^{n-k-1}(q_i)_+\,(t)\,f_i(\alpha(t + 1)^k)$

is integrable on $[0, \infty)$, and, in view of the nonincreasing nature of $y^{-1} f_i(y)$, $\beta > \alpha$ implies

$$\beta^{-1} \sum_{i=1}^{N} t^{n-k-1}(q_i)_+ (t) f_i(\beta(t + 1)^k) \leqq \alpha^{-1} \sum_{i=1}^{N} t^{n-k-1}(q_i)_+ (t) f_i(\alpha(t + 1)^k) , \quad t \geqq 0 .$$

Moreover, by (30), the left hand side of the above tends to zero as $\beta \to \infty$ pointwise on $[0, \infty)$. So, the Lebesgue dominated convergence theorem shows that

$$\lim_{\beta \to \infty} \beta^{-1} \sum_{i=1}^{N} \int_0^\infty t^{n-k-1}(q_i)_+ (t) f_i(\beta(t + 1)^k) \, dt = 0 , \tag{31}$$

and similarly

$$\lim_{\beta \to \infty} \beta^{-1} \sum_{i=1}^{N} \int_0^\infty t^{n-k}(q_i)_- (t) f_i(\beta(t + 1)^k) \, dt = 0 . \tag{32}$$

Because of (31) and (32), a constant $\beta_0 > 0$ can be chosen so that

$$\sum_{i=1}^{N} \int_0^\infty t^{n-k-1}(q_i)_+ (t) f_i(\beta(t + 1)^k) \, dt \leqq \beta$$

and

$$\sum_{i=1}^{N} \int_0^\infty t^{n-k}(q_i)_- (t) f_i(\beta(t + 1)^k \, dt \leqq \tfrac{1}{2}\beta$$

for all $\beta \geqq \beta_0$. If we define, for a fixed $\beta \geqq \beta_0$,

$$Y = \left\{ y \in C[0, \infty) : \frac{\beta t^{k-1}}{2(k - 1)!} \leqq y(t) \leqq \frac{\beta t^{k-1}}{(k - 1)!} + \frac{\beta t^k}{k!} , \quad t \geqq 0 \right\}$$

$$Fy(t) = \frac{\beta t^{k-1}}{(k - 1)!} + \int_0^t \frac{(t - s)^{k-1}}{(k - 1)!} \int_s^\infty \frac{(r - s)^{n-k-1}}{(n - k - 1)!} \sum_{i=1}^{N} q_i(r) f_i(y(r)) \, dr \, ds , \quad t \geqq 0 ,$$

and proceed as in the proof of (i) of Theorem 1, then we can show that F has a fixed element $y \in Y$ and that this element gives a solution of equation (1) existing on $[0, \infty)$ and satisfying (5). Since any positive number β greater then β_0 can be taken in defining Y and F, there exist infinitely many such solutions of (1). This completes the proof of the first statement of the theorem.

Example 2. Consider the differential equation

$$y^{(n)} + q(t) \, y^\gamma [\log (1 + y)]^\delta = 0 , \tag{33}$$

where $n \geqq 2$, $0 < \gamma < 1$, $0 < \gamma + \delta < 1$, and $q : [0, \infty) \to \mathbb{R}$ is continuous. This is a special case of (1) in which $N = 1$, $q_i(t) = q(t)$, $f_1(y) = y^\gamma [\log (1 + y)]^\delta$. Clearly, $y^{-1} f_1(y)$ is nonincreasing for $y > 0$ and satisfies $\lim_{y \to \infty} y^{-1} f_1(y) = 0$. Conditions (7), (8) and (9) for equation (33) reduce, respectively, to

$$\int^\infty t^{n-k-1+\gamma k}(\log t)^\delta \, q_+(t) \, dt < \infty , \tag{34}$$

$$\int^\infty t^{n-k+\gamma k}(\log t)^\delta \, q_-(t) \, dt < \infty \tag{35}$$

and

(36) $$\int^{\infty} t^{n-k+\gamma(k-1)}((k-1)\log t)^{\delta}\, q_+(t)\, \mathrm{d}t = \infty\,;$$

conditions (10), (11) and (12) can be formulated by interchanging the role of $q_+(t)$ and $q_-(t)$ in (34), (35) and (36).

If in particular, $n = 2$, $k = 1$, and $q(t)$ is given by

(37) $$q(t) = \begin{cases} \dfrac{\sin(\log t)}{t^2} & \text{for } \sin(\log t) \geqq 0\,, \text{ i.e., for } t \in \bigcup_{i=0}^{\infty} [\mathrm{e}^{2i\pi}, \mathrm{e}^{(2i+1)\pi}] \\ \dfrac{\sin(\log t)}{t^3} & \text{for } \sin(\log t) \geqq 0\,, \text{ i.e., for } t \in \bigcup_{i=1}^{\infty} [\mathrm{e}^{(2i-1)\pi}, \mathrm{e}^{2i\pi}]\,, \end{cases}$$

then conditions (34) – (36) are satisfied, because

$$\int_{\mathrm{e}}^{\infty} t^{\gamma}(\log t)^{\delta}\, q_+(t)\, \mathrm{d}t < \int_{\mathrm{e}}^{\infty} \frac{(\log t)^{\delta}}{t^{2-\gamma}}\, \mathrm{d}t < \infty\,,$$

$$\int_{\mathrm{e}}^{\infty} t^{\gamma+1}(\log t)^{\delta}\, q_-(t)\, \mathrm{d}t < \int_{\mathrm{e}}^{\infty} \frac{(\log t)^{\delta}}{t^{2-\gamma}}\, \mathrm{d}t < \infty$$

and

$$\int_1^{\infty} t\, q_+(t)\, \mathrm{d}t = \sum_{i=0}^{\infty} \int_{\mathrm{e}^{2i\pi}}^{\mathrm{e}^{(2i+1)\pi}} \frac{\sin(\log t)}{t}\, \mathrm{d}t = \sum_{i=0}^{\infty} \int_{2i\pi}^{(2i+1)\pi} \sin s\, \mathrm{d}s = \infty\,.$$

Consequently, by (i) of Theorem 2, equation (33) with $n = 2$ and $q(t)$ defined by (37) possesses infinitely many positive solution $y(t)$ which exist on $[\mathrm{e}, \infty)$ and have the asymptotic behavior

$$\lim_{t\to\infty} \frac{y(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = \infty\,.$$

Remark 1. Condition (7) and (8) [resp. (10) and (11)] clearly imply

$$\sum_{i=1}^{N} \int_0^{\infty} t^{n-k-1} |q_i(t)|\, f_i(\alpha t^k)\, \mathrm{d}t < \infty$$

for k, $1 \leqq k \leqq n-1$, such that $n \not\equiv k \pmod 2$ [resp. $n \equiv k \pmod 2$], where $\alpha = \min\{a, b\}$. It follows that, under the hypotheses of Theorems 1 and 2, equation (1) also has a positive solution $y(t)$ satisfying (4):

$$\lim_{t\to\infty} y(t)/t^k = \text{const} > 0\,.$$

Remark 2. If assumption (2 – c) is replaced by

(2) (c′) each $f_i\colon \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing and $y f_i(y) > 0$ for $y \neq 0$,

then one can easily formulate criteria for the existence of a negative solution $y(t)$ of (1) with the property

$$\lim_{t\to\infty} \frac{y(t)}{t^k} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{y(t)}{t^{k-1}} = -\infty\,.$$

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