Alexander Haščák

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DISCONJUGACY AND MULTIPOINT BOUNDARY VALUE PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS WITH DELAY

ALEXANDER HAŠČÁK, Bratislava

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Boundary value problems play an important role in the theory of differential equations, both ordinary and ordinary with delay. There are many papers devoted to their study (see e.g. [2], [10], [15] and references therein). Boundary value problems for ordinary linear differential equations are closely related to the study of their disconjugacy. There are many papers devoted to this topic (see [1] and references theirein). However, the corresponding theory for differential equations with delay has not yet been built up. The purpose of this paper is to give a generalization of the notion of a disconjugate linear differential equation for linear differential equations with delay and then to give the relation between the new notion and multipoint boundary value problems. This will enable us to treat multipoint boundary value problems by methods analogous to those known from the theory of ordinary differential equations.

I

Consider the following n-th order linear homogeneous differential equation with (for simplicity) a single delay

$$(\mathrm{E}_n) \quad x^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, x^{(k)}(t) + \sum_{k=0}^{n-1} b_k(t)\, x^{(k)}(t - \Delta(t)) = 0$$

having continuous (in the interval $t_0 \leqq t < T \leqq +\infty$) coefficients $a_k(t)$, $b_k(t)$ and a delay $\Delta(t) \geqq 0$.

The underlying initial value problem (IVP) for the equation (E_n) is defined as follows: On the initial set

$$E_{t_0} = \{t - \Delta(t) \colon t - \Delta(t) < t_0,\ t \in \langle t_0, T)\} \cup \{t_0\}$$

let a continuous initial vector function $\phi(t) = (\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t))$ be given.

We have to find the solution $x(t) \in C^n(\langle t_0, T))$ of (E_n) satisfying

$$\text{(IV)} \qquad \begin{aligned} &x^{(k)}(t_0) = \phi_k(t_0)\,, \quad k = 0, 1, \ldots, n-1\,, \\ &x^{(k)}(t - \Delta(t)) = \phi_k(t - \Delta(t)) \quad \text{if} \quad t - \Delta(t) < t_0\,. \end{aligned}$$

Under the above assumptions, the initial value problem (E_n), (IV) has exactly one solution on the interval $\langle t_0, T)$.

Definition 1. The set of solutions $x(t)$ of (E_n) satisfying

$$(1) \qquad \begin{aligned} &x(t_0) = 0\,, \\ &x(t - \Delta(t)) \equiv 0 \quad \text{if} \quad t - \Delta(t) < t_0 \end{aligned}$$

is called a *band of solutions* (or shortly *a band*) at the point t_0.

Definition 2. The set of solutions $x(t)$ of (E_n) which, in addition to the condition (1), satisfy also

$$(2) \qquad \begin{aligned} &x^{(k)}(t_0) = \phi_k(t_0)\,, \quad k = 1, 2, \ldots, n-1\,, \\ &x^{(k)}(t - \Delta(t)) = \phi_k(t - \Delta(t)) \equiv \phi_k(t_0) \quad \text{if} \quad t - \Delta(t) < t_0 \end{aligned}$$

is called the *principal band of solutions of* (E_n) *at* t_0. It will be denoted by $B(E_n, t_0)$.

Theorem 1. *$B(E_n, t_0)$ is an $(n-1)$-dimensional vector space.*

Proof. It is easy to see that $B(E_n, t_0)$ is a vector space. We shall show that its dimension is $(n-1)$. Denote by $x_i(t, t_0)$ the solution in $B(E_n, t_0)$ which satisfies (2) with the initial function $\phi = (\phi_0, \phi_1, \ldots, \phi_{n-1})$ defined by

$$(3) \qquad \begin{aligned} &\phi_i(t) \equiv 1\,, \quad t \in E_{t_0}\,, \quad i = 1, 2, \ldots, n-1\,, \\ &\phi_j(t) \equiv 0\,, \quad t \in E_{t_0}\,, \quad j \neq i\,. \end{aligned}$$

Then every solution $x(t)$ in $B(E_n, t_0)$ has the form

$$x(t) = \alpha_1\, x_1(t, t_0) + \alpha_2\, x_2(t, t_0) + \ldots + \alpha_{n-1}\, x_{n-1}(t, t_0)\,.$$

The proof of Theorem 1 is complete.

Let $x(t) \in B(E_n, t_0)$, $x(t) \not\equiv 0$. The n-th consecutive zero (including multiplicity) of $x(t)$ to the right of t_0 will be denoted by $\eta(x, t_0)$.

Definition 3. Let $a \in \langle t_0, T)$. By the *adjoint point to the point a* (with respect to (E_n)) we mean the point

$$\alpha(a) = \inf\{\eta(x, a) \colon x(t) \in B(E_n, a),\, x(t) \not\equiv 0\}\,.$$

Definition 4. The equation (E_n) is said to be *disconjugate in an interval I*, iff

$$a \in I \Rightarrow \alpha(a) \notin I\,.$$

Theorem 2. *Let $I = \langle \alpha, \beta \rangle$ be a compact interval. Then for some $\delta > 0$, (E_n) is disconjugate on every subinterval J of I whose length is less than δ.*

Proof. We prove this theorem by contradiction. Let

$$M = \max_{0 \leqq i \leqq n-1} \max_{t \in I} \left(|a_i(t)| + |b_i(t)|\right)$$

and

$$\delta = \min\left(1, \frac{1}{nM}\right).$$

Assume that $J \subset I$, the length of J is less than $\delta > 0$, and (E_n) is not disconjugate in J. Then there exists a point $t_0 \in J$ and a solution $x(t) \in B(E_n, t_0)$ which has at least n zeros (including multiplicity) in $J_1 = \langle t_0, +\infty) \cap J$. Thus $x^{(k)}(t)$ has at least $(n - k)$ zeros in J_1 $(k = 1, 2, \ldots, n - 1)$. Denote

$$\mu_k = \max_{t \in J_1} |x^{(k)}(t)|.$$

From (2) we have

$$\max_{t \in J_1} |x^{(k)}(t)| \geqq \max_{t \in J_1} |x^{(k)}(t - \Delta(t))|.$$

From this fact and from the existence of zeros in J_1 we obtain by the Mean-Value Theorem

$$\mu_k \leqq \mu_{k+1}\delta, \quad k = 0, 1, \ldots, n - 1.$$

Now, if $\mu_k > 0$, then $\mu_k < \mu_{k+1}\delta$. Since $\mu_0 > 0$ we get

$$0 < \mu_k < \delta^{n-k}\mu_n, \quad k = 0, 1, \ldots, n - 1.$$

On the other hand, from (E_n) we get

$$\mu_n \leqq \sum_{k=0}^{n-1} \left(|a_k(t)| + |b_k(t)|\right) \mu_k \leqq M \sum_{k=0}^{n-1} \mu_k <$$

$$< M(\delta^n + \delta^{n-1} + \ldots + \delta)\, \mu_n \leqq nM\delta\mu_n$$

i.e.

$$1 < nM$$

which is a contradiction.

Let us now define an n-point boundary value problem (BVP) for (E_n).

Let

$$\tau_0, \tau_1, \ldots, \tau_m \in I = \langle t_0, T), \quad \tau_0 < \tau_1 \leqq \ldots \leqq \tau_m, \quad m + 1 \leqq n,$$

$$r_0, r_1, \ldots, r_m \in N, \quad r_0 + r_1 + \ldots + r_m = n,$$

$$\beta_0^{(1)}, \ldots, \beta_0^{(r_0)}, \beta_1^{(1)}, \ldots, \beta_m^{(r_m)} \in R,$$

and let $\phi_0(t), \phi_1(t), \ldots, \phi_{r_0-1}(t)$ be continuous functions defined on E_{τ_0} such that

$$\phi_{i-1}(\tau_0) = \beta_0^{(i)}, \quad i = 1, 2, \ldots, r_0.$$

The problem is to find the solution $x(t)$ of the equation (E_n) which satisfies the conditions

(BV) $$x^{(v_j-1)}(\tau_j) = \beta_j^{(v_j)}, \quad v_j = 1, 2, \ldots, r_j, \quad j = 0, 1, \ldots, m,$$

$$x^{(v-1)}(t - \Delta(t)) = \phi_v(t - \Delta(t)), \quad v = 1, 2, \ldots, r_0 \quad \text{if} \quad t - \Delta(t) < \tau_0.$$

Definition 5. The *principal band of solutions for the boundary value problem* (E_n), (BV) is the set of solutions from the principal band of solutions at τ_0 which satisfy

(4) $$x^{(v-1)}(t - \Delta(t)) \equiv 0 \quad \text{if} \quad t - \Delta(t) < \tau_0 \,, \quad v = 1, 2, \ldots, r_0 \,.$$

We shall denote this band by $B(E_n, \tau_0, r_0)$.

It is easy to prove

Theorem 3. *$B(E_n, \tau_0, r_0)$ is an $(n - r_0)$ dimensional vector space.*

Finally, we shall define the adjoint boundary value problem (ABVP) to the boundary value problem (BVP).

Let

$$\tau_0, \tau_1, \ldots, \tau_m \,, \quad r_0, r_1, \ldots, r_m \,,$$
$$\beta_1^{(1)}, \ldots, \beta_1^{(r_1)}, \beta_2^{(r_2)}, \ldots, \beta_m^{(r_m)}$$

be such as in (BVP).

The problem is to find a solution $x(t)$ of the equation (E_n) which is from $B(E_n, \tau_0, r_0)$ and satisfies the conditions

$$x^{(v_j-1)}(\tau_j) = \beta_j^{(v_j)} \,, \quad v_j = 1, \ldots, r_j \,, \quad j = 1, \ldots, m \,.$$

Theorem 4. *The equation (E_n) is disconjugate on an interval I iff the adjoint boundary value problem (ABVP) to each boundary value problem (BVP) has exactly one solution.*

Proof. Each solution $x(t) \in B(E_n, \tau_0, r_0)$ can be written in the form

(5) $$x(t) = \alpha_1 x_{r_0}(t, \tau_0) + \alpha_2 x_{r_0+1}(t, \tau_0) + \ldots + \alpha_{n-r_0} x_{n-1}(t, \tau_0) \,.$$

Let

$$A = \begin{bmatrix} x_{r_0}(\tau_1, \tau_0) & \ldots & x_{n-1}(\tau_1, \tau_0) \\ \ldots & \ldots & \ldots \\ x_{r_0}^{(r_1-1)}(\tau_1, \tau_0) & \ldots & x_{n-1}^{(r_1-1)}(\tau_1, \tau_0) \\ x_{r_0}(\tau_2, \tau_0) & \ldots & x_{n-1}(\tau_2, \tau_0) \\ \ldots & \ldots & \ldots \\ x_{r_0}^{(r_m-1)}(\tau_m, \tau_0) & \ldots & x_{n-1}^{(r_m-1)}(\tau_m, \tau_0) \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_{n-r_0} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1^{(1)} \\ \vdots \\ \beta_1^{(r_1)} \\ \beta_2^{(1)} \\ \vdots \\ \beta_m^{(r_m)} \end{bmatrix}.$$

Then we have to choose α such that

$$A\alpha = \beta \,.$$

This, however, is possible for each β if and only if the corresponding homogeneous equation

(6) $$A\alpha = 0$$

has only the trivial solution. This occurs if and only if the differential equation (E_n) is disconjugate in I (since if (E_n) is disconjugate on I, then the trivial solution is the only solution $x(t) \in B(E_n, \tau_0)$ which has n zeros (including multiplicity) in I).

Definition 6. Let a continuous initial vector function $\phi(t) = (\phi_0(t), \phi_1(t), \ldots \ldots, \phi_{n-1}(t))$ be defined on an initial set $E_{\tau 0}$. Then we define

$$H_{\phi,ro} = \{(\phi_0(t), \ldots, \phi_{r_0-1}(t); c_1 + \phi_{r_0}(t), \ldots, c_{n-r_0} + \phi_{n-1})t)): c_i \in R\} .$$

Theorem 5. *Let the coefficients a_i, b_i $(i = 0, 1, \ldots, n - 1)$ of the equation (E_n) be continuous on an interval I. Then the equation (E_n) is disconjugate on I if and only if every boundary value problem (E_n), (BV) has exactly one solution $x(t)$ such that $x(t - \Delta(t)) \in H_{\phi,r_0}$ provided $t - \Delta(t) < \tau_0$.*

Proof. Denote by $x(t; \tau_0, \phi_0, \ldots, \phi_{n-1})$ a solution of (E_n) satisfying (IV). Now, Theorem 5 follows from the uniqueness of the solution of the initial value problem (IVP), from Theorem 4 and from

$$\begin{gathered} x(t; \tau_0, \phi_0, \ldots, \phi_{n-1}) = \\ = x(t; \tau_0, \phi_0, \ldots, \phi_{r_0}, \phi_{r_0+1} - \phi_{r_0+1}(\tau_0), \ldots, \phi_{n-1} - \phi_{n-1}(\tau_0)) + \\ + x(t; \tau_0, 0, \ldots, 0, \phi_{r_0+1}(\tau_0), \ldots, \phi_{n-1}(\tau_0)) , \\ \phi_i = \phi_i(t) , \quad i = 0, 1, \ldots, n - 1 . \end{gathered}$$

Corollary 1. *The differential equation*

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)\, x^{(i)}(t) + b_0(t)\, x(t - \Delta(t)) = 0$$

is disconjugate on I if and only if the boundary value problem (E_n), (BV) has exactly one solution.

Corollary 2. *The differential equation (E_n) is disconjugate on I if and only if every boundary value problem (E_n), (BV) has exactly one solution $x(t)$ such that $x^{(r_0)}(t - \Delta(t)), \ldots, x^{(n-1)}(t - \Delta(t))$ are constant for $t - \Delta(t) < \tau_0$.*

Now we will show two examples which will clarify and illustrate the new notions. For this purpose let us consider the following differential equation

$$(E_2) \qquad \begin{gathered} x''(t) + N(t)\, x(t) + M(t)\, x(t - \Delta(t)) = 0 , \\ N(t), M(t), \Delta(t) \in C((t_0, T), R) \end{gathered}$$

which is a special case of (E_n). By Sturm's Theorem we have that in the case $\Delta(t) \equiv 0$ the function $\alpha(t)$ (which assigns to $t \in (t_0, T)$ its adjoint point $\alpha(t)$ if such a point exists, otherwise we put $\alpha(t) = T$) is an increasing function. However, this is not valid if $\Delta(t) \not\equiv 0$ as the following example shows.

Example 1. Let the functions $N(t)$, $M(t)$ and $\Delta(t)$ be defined by the following formulas

$$\Delta(t) = \begin{cases} 0 , & t < 0 , \\ t , & t \geqq 0 , \end{cases}$$

$$N(t) = 1 , \qquad t \in (-\infty, +\infty) ,$$

and

$$M(t) = \left\langle \begin{array}{ll} -1\,, & t < 0\,, \\ -(t+1)\,, & t \geqq 0\,. \end{array} \right.$$

Then (E_2) becomes

(7) $$x''(t) = 0\,, \quad t < 0\,,$$
$$x''(t) + x(t) - (t+1)\,x(0) = 0\,, \quad t \geqq 0\,.$$

It is easy to see that $B(7, -1)$ is the one-dimensional vector space with the basis $x_0(t, -1) = t + 1$. Thus $\alpha(-1) = +\infty$. On the other hand, $B(7, 0)$ is the one-dimensional vector space with the basis $x_0(t, 0) = \sin t$ and thus $\alpha(0) = \pi$.

Example 2. It is easy to see that if the coefficients $N(t)$ and $M(t)$ in (E_2) are nonnegative functions, then (E_2) is disconjugate on (t_0, T).

II

In the condition (BV) we have $r_0 \in N$, i.e. $r_0 \geqq 1$. If r_0 in (BV) is equal to zero we have an other boundary value problem:

Let

$$\tau_1, \tau_2, \ldots, \tau_m \in I = (t_0, T)\,, \quad \tau_1 \leqq \tau_2 \leqq \ldots \leqq \tau_m\,, \quad m \leqq n\,,$$
$$r_1, r_2, \ldots, r_m \in N\,, \quad r_1 + r_2 + \ldots + r_m = n$$

and

$$\beta_1^{(1)}, \beta_1^{(2)}, \ldots, \beta_m^{(r_m)} \in R\,.$$

The problem is to find the solution of the equation (E_n) which satisfies the conditions

(BV_2) $$x^{(v_j - 1)}(\tau_j) = \beta_j^{(v_j)}\,, \quad v_j = 1, 2, \ldots, r_j\,, \quad j = 1, 2, \ldots, m\,.$$

Now the question arises: is there any relation between the zeros of some subset of solutions of (E_n) and the existence and uniqueness of solution of the problem (E_n), (BV_2)?

To give an answer to this question, we shall proceed as in the first part of this paper.

For $\tau_0 \in I$ let us denote by $B'(E_n, \tau_0)$ the set of all solutions of (E_n) with constant initial vector functions which are defined on the initial set E_{τ_0}. It is easy to see that $B'(E_n, \tau_0)$ is an n-dimensional vector space and

(8) $$B(E_n, \tau_0) \subset B'(E_n, \tau_0)\,.$$

Let $x(t) \in B'(E_n, \tau_0)$, $x(t) \not\equiv 0$. The n-th consecutive zero (including multiplicity) of $x(t)$ to the right of τ_0 will be denoted by $\eta(x, t_0)$.

Definition 7. Let $a \in I$. By the *first adjoint point to the point a* (with respect to (E_n)) we mean the point

$$\alpha_1(a) = \inf\{\eta(x, a) \colon x \in B'(E_n, a),\ x \not\equiv 0\}\,.$$

Corollary 3. $\alpha_1(a) \leqq \alpha(a)$.

Definition 8. The equation (E_n) is said to be *strictly disconjugate on an interval I* iff

$$a \in I \Rightarrow \alpha_1(a) \notin I .$$

Corollary 4. *If the equation (E_n) is strictly disconjugate on I, then it is disconjugate on I.*

Now we can prove the following theorems (the proofs are analogous to those of the corresponding theorems in the first part of the paper):

Theorem 6. *Let $I = \langle \alpha, \beta \rangle$ be a compact interval. Then for some $\delta > 0$, (E_n) is strictly disconjugate on every subinterval J of I whose length is less than δ.*

Theorem 7. *The equation (E_n) is strictly disconjugate on an interval I iff for each $\tau_0 \in I$, $\tau_0 < \tau_1$, the boundary value problem (E_n), (BV_2) has exactly one solution in $B'(E_n, \tau_0)$.*

Theorem 8. *The equation (E_n) is strictly disconjugate on an interval I iff for each $\tau_0 \in I$, $\tau_0 < \tau_1$, and for each continuous vector function $\phi(t)$ defined on the initial set E_{τ_0}, the boundary value problem (E_n), (BV_2) has exactly one solution $x(t)$ such that*

$$x(t - \Delta(t)) \in H_{\phi,0} \quad \textit{provided} \quad t - \Delta(t) < \tau_0 .$$

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Author's address: 842 15 Bratislava, Mlynská dolina, Czechoslovakia (Katedra matematickej analýzy MFF UK).