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SHIFT-AUTOMORPHISM METHODS FOR INHERENTLY NONFINITELY BASED VARIETIES OF ALGEBRAS

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1. INTRODUCTION

In 1954, Lyndon [12] made the unexpected discovery that there exists a finite algebra (algebraic system) whose equations are nonfinitely based, in the sense that there is no finite list of equations true in the algebra from which all equations true in the algebra are deducible. A more recent surprise was the discovery by Murskii [18] and Perkins [22] that some algebras, called *inherently nonfinitely based* by Perkins, are nonfinitely based in a contagious way. For example, if a finite inherently nonfinitely based algebra is a subalgebra or homomorphic image of a larger finite algebra, then the larger algebra is also nonfinitely based (and in fact inherently nonfinitely based). More specifically, Perkins defines an algebra **A** to be inherently nonfinitely based if Var(**A**), the variety determined by the set of all equations true in **A**, is locally finite but **A** is not contained in any finitely based locally finite variety.

(Here a *variety* is a class of algebras that can be defined by a set of equations; a variety is *finitely based* if it can be defined by a finite set of equations and is *locally finite* it its finitely generated members are finite. Thus Var(**A**) is finitely based when **A** is. The instances of contagion mentioned follow from Perkins' definition by the fact that varieties are closed under taking subalgebras and homomorphic images and by the fact that Var(**B**) is locally finite if **B** is finite.)

We present two results that together establish that inherently nonfinitely based algebras, both finite and infinite, constitute a wide class. The first of the two, Theorem 1.1, says that an infinite algebra is inherently nonfinitely based if it has an automorphism with infinite orbits and has nontrivial operations that can be defined, under the action of the automorphism, by finitely many rules. The second result, Theorem

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4.1, shows that many finite algebras, usually with no nontrivial automorphisms, generate infinite algebras fulfilling the conditions of Theorem 1.1 and so are themselves inherently nonfinitely based by contagion. Applications of these results include almost all examples of inherently nonfinitely based finite algebras presently in the literature as well as new examples.

By an *algebra* we mean a nonempty set A endowed with an indexed system of fundamental operations $f_i\colon A^{n(i)} \to A$, where each $n(i)$ is a nonnegative integer; for example, groups, rings, and lattices are algebras. We consider only algebras with finitely many fundamental operations, and we write $\mathbf{A} = \langle A; f_0, \ldots, f_{m-1} \rangle$. In any discussion, it is assumed that all algebras involved have the same type (the same list of $n(i)$).

Just as the commutative law is an equation satisfied by some groups, any algebra may satisfy equations, regarded as formal equalities of pairs of expressions built up from symbols for variables and fundamental operations. An *equational basis* for $\mathbf{A}$ is a set of equations true in $\mathbf{A}$ of which all equations true in $\mathbf{A}$ are logical consequences. $\mathbf{A}$ is said to be *finitely based* if $\mathbf{A}$ has a finite equational basis, and *nonfinitely based* otherwise.

Lyndon's example [12] of a nonfinitely based finite algebra had seven elements, a binary operation, and a constant operation (not essential). An example with three elements and one binary operation was constructed by Murskii [17]. (No smaller example is possible, since by a theorem of Lyndon [11] any two-element algebra is finitely based.) ~~Some examples of nonfinitely based algebras element algebra is finitely based.)~~ Some examples of nonfinitely based algebras relevant for this paper are several specific graph algebras of Shallon [26] and the four-element algebra of Park [21], which has a commutative, idempotent binary operation. These and other examples are discussed by McNulty and Shallon [16].

In contrast to the case of nonfinitely based algebras, every finite group is finitely based (Oates and Powell [19]), as is every finite ring (Kruse [9], L'vov [10]) and every finite lattice or lattice-ordered algebra (McKenzie [14]). Baker [1] proved, in generalization of McKenzie's results, that every finite algebra belonging to a congruence-distributive variety is finitely based. Recently, McKenzie [14] has proved an even more general result: Every finite algebra that generates a residually small congruence-modular variety is finitely based. (As usual, it is assumed that there are only finitely many fundamental operations.) These results are all nontrivial and in some cases, unexpectedly difficult.

Lyndon's example and its successors raise the question of characterizing nonfinitely based finite algebras. A unsolved problem of Tarski asks for an algorithmic solution. To this end, study of both finitely based and nonfinitely based algebras seems to be required; this paper is a contribution to the latter. Interestingly, two major constructions from an algebra are its congruence lattice and its automorphism group; the former is prominent in the finite-basis results mentioned, while the latter is central in our results on inherently nonfinitely based algebras.

Perkins [22] defines a locally finite variety to be *inherently nonfinitely based* if every locally finite variety containing it is nonfinitely based. Thus an algebra **A** is inherently nonfinitely based precisely when $\mathrm{Var}(A)$ is inherently nonfinitely based. It is evident that the property of being inherently nonfinitely based is itself contagious, in that a locally finite variety containing an inherently nonfinitely based algebra or variety must be inherently nonfinitely based.

Murskii [18] has in essence shown that his algebra is inherently nonfinitely based. McNulty and Shallon [16] show that the several graph algebras of Shallon are inherently nonfinitely based. Park's algebra is also inherently nonfinitely based [15]. Ježek [6] lists three new three-element algebras with one binary operation that are inherently nonfinitely based. The theorems presented in this paper have as consequences all these examples.

Not all finite nonfinitely based algebras are inherently nonfinitely based [16].

The algebras considered in this paper usually have an absorbing element, i.e., an element ∞ such that any operation evaluated at a tuple with one or more entries ∞ has the value ∞. (In many familiar algebras an absorbing element is called zero and denoted by 0, but 0 is reserved for other uses in this paper.) However, there are occasional applications to algebras without absorbing element (Examples 4.9, 4.10).

Consider, then, an algebra **A** with an absorbing element ∞ and an automorphism σ. We call the non-∞ elements of A *proper elements*. A tuple of elements is called *proper* if none of them is ∞. It is clear that **A** can have at most one absorbing element, so long as some fundamental operation of **A** has rank at least two, and that ∞ is fixed under σ. Moreover, the automorphism σ acts on A^n for every natural number n, partitioning A^n into orbits. If f is a fundamental operation of **A** and n is the rank of f, then $f \subseteq A^{n+1}$ and f is invariant with respect to the action of σ. We can now formulate the chief result of the first portion of this paper.

1.1. Theorem. *Let* **A** *be an infinite locally finite algebra with finitely many fundamental operations, with an absorbing element* ∞, *and with an automorphism* σ *such that*

(a) $\{\infty\}$ *is the only* σ*-orbit of* **A** *that is finite;*

(b) *the proper part of* f *is partitioned by* σ *into only finitely many orbits, for all fundamental operations* f *of* **A**;

(c) $\phi(a) = \sigma(a)$ *for some proper element* $a \in A$ *and some nonconstant unary polynomial function (unary term function with constants)* $\phi(x)$ *of* **A**.

Then **A** *is inherently nonfinitely based.*

Here an individual algebra is said to be *locally finite* if its finitely generated subalgebras are finite. Not every locally finite algebra generates a locally finite variety (the abelian group $\mathbf{Q}/\mathbf{Z}$ is an example that does not), but the conclusion of Theorem 1.1 asserts that **A** does.

One effect of condition (b) is to engender a concept of "nearness" of elements, whereby only nearby elements interact nontrivially under the operations. This property, which seems to be associated with at least the free algebras in the inherently nonfinitely based varieties that we have seen, is made more explicit in §§ 2 and 3, where Theorem 1.1 is proved.

Condition (c) is essentially a nontriviality condition on the polynomial functions of **A**. Theorem 2.5 will show that a substitute for (c) is simply the requirement that **A** be finitely subdirectly irreducible.

Here are two typical examples to which Theorem 1.1 applies. In § 4, it will be shown that both occur in varieties generated by finite algebras, and that in fact many kinds of finite algebras generate infinite algebras to which Theorem 1.1 applies.

1.2. Example. Let F be the "fence" partially ordered set of Figure 1.

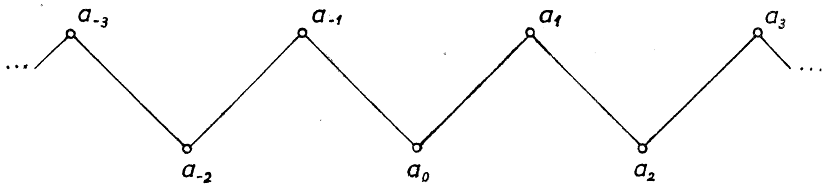

Figure 1

Meets and joins in F are defined for some pairs of elements and not for others; thus F is a partial lattice. Let $F^{\#} = F \cup \{\infty\}$ and make $F^{\#}$ into an algebra $\mathbf{F}^{\#} = \langle F^{\#}; \vee, \wedge \rangle$ by giving $\vee$ and $\wedge$ the value ∞ at all pairs where they are not already defined. Thus, for example,

$$(1.2.1) \qquad a_{2i-1} \wedge a_{2i+1} = a_{2i}, \quad \text{for all} \quad i \in \mathbf{Z}.$$

(In contrast, $a_{2i-1} \vee a_{2i+1} = \infty$.)

$\mathbf{F}^{\#}$ has the automorphism $\sigma(a_j) = a_{j+2}$, and ∞ is an absorbing element. Moreover, $\mathbf{F}^{\#}$ is locally finite. For (a), observe that σ has the orbits $\{\infty\}$, $\{a_i \mid i \text{ odd}\}$, and $\{a_i \mid i \text{ even}\}$. For (b), the proper triples constituting $\wedge$ and $\vee$ form finitely many orbits exemplified by (1.2.1), which can be regarded as describing the triples in the σ-orbit of $(a_{-1}, a_1, a_0) \in \wedge$. Another orbit in $\wedge$ would be that of (a_0, a_1, a_0). For the nontriviality condition (c), observe that the action of σ at some proper element can be duplicated by a nonconstant unary polynomial function, as required. For example, a_0 is carried to $\sigma(a_0)$ not only by σ but by the unary polynomial function ϕ defined by $\phi(x) = (x \vee a_1) \wedge a_2$ in $\mathbf{F}^{\#}$.

In effect, condition (b) states that, by means of the automorphism, the operations of the algebra can be described by finitely many "rules", of which (1.2.1) is an example.

1.3. Example. Let G be the graph shown in Figure 2. Let $\mathbf{G}^{\#} = (G^{\#}, *)$ be the Shallon graph algebra of G. In other words, $a_i * a_{i+1} = a_i$, $a_{i+1} * a_i = a_{i+1}$, and all other products are ∞. Then $\sigma(a_j) = a_{j+1}$ is an automorphism of **G**, and ∞ is an absorbing element. $\mathbf{G}^{\#}$ is locally finite and has a single proper orbit; its operation has two proper orbits with representatives (a_0, a_1, a_0) and (a_1, a_0, a_1); and in (c), a can be a_0 and $\phi(x)$ can be $a_1 * x$.

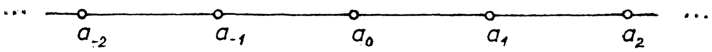

Figure 2

In general, for any graph G (possibly with loops at the vertices but without multiple edges), the (Shallon) *graph algebra* is defined on the set $G^{\#} = G \cup \{\infty\}$ as $\mathbf{G}^{\#} = \langle G^{\#}, *\rangle$, where G is the set of vertices of the graph, $\infty \notin G$, and $x*y = x$ if x and y are joined by an edge, and $x*y = \infty$ otherwise. In [2], we apply our general results to obtain a forbidden-subgraph characterization of those graph algebras that are inherently nonfinitely based, and we provide finite equational bases for those that are not. Thus every nonfinitely based graph algebra is actually inherently nonfinitely based. Graph algebras were first introduced by Shallon [26], where many were shown to be nonfinitely based. Further investigations of graph algebras can be found in [16], [20], and [25].

The methods of this paper evolved from Murskii [17] by way of Murskii [18], Shallon [26], McNulty and Shallon [16], and McNulty [15]. A substantially different method for establishing that a finite algebra is inherently nonfinitely based was originated by Perkins [22]. This technique has been elaborated in McNulty and Shallon [16], and the latest and very interesting contribution is due to Ježek [6]. M. Sapir has indicated, in private correspondence, that a number of finite semigroups are inherently nonfinitely based.

In most respects we follow the terminology and notation of [4]. Additional valuable references are [5] and [7].

2. INFINITE INHERENTLY NONFINITELY BASED ALGEBRAS

Determining whether a locally finite variety $\mathbf{V}$ is inherently nonfinitely based would seem to involve an examination of the varieties that include $\mathbf{V}$. Fortunately, this examination needs to be carried out for only a very restricted collection of varieties. For each natural number n, let $\mathbf{V}^{(n)}$ be the class of algebras defined by the following condition:

$\mathbf{B} \in \mathbf{V}^{(n)}$ if and only if every subalgebra of $\mathbf{B}$ with n or fewer generators belongs to $\mathbf{V}$.

It is easy to check that $\mathbf{V}^{(n)}$ is the variety defined by the equations in no more than n variables that are true in $\mathbf{V}$. Birkhoff [3] pointed out that $\mathbf{V}^{(n)}$ is finitely based if $\mathbf{V}$ is a locally finite variety with only finitely many fundamental operations. As observed by McNulty [15], this leads immediately to the following fact.

2.1. Proposition. *For a locally finite variety with finitely many fundamental operations, the following conditions are equivalent:*

(i) $\mathbf{V}$ *is inherently nonfinitely based;*

(ii) *the variety* $\mathbf{V}^{(N)}$ *is not locally finite for any natural number* N;

(iii) *for infinitely many natural numbers* N, *there is a non-locally-finite algebra* $\mathbf{B}_N$ *whose* N*-generated subalgebras belong to* $\mathbf{V}$. □

An additional observation that will be useful:

2.2. Proposition. *An infinite locally finite algebra generates a locally finite variety if and only if for each natural number* N *there is a finite upper bound on the sizes of its* N*-generated subalgebras.* □

2.3. Outline of the proof of Theorem 1.1. Let there be given an infinite locally finite algebra $\mathbf{A}$ with absorbing element ∞ and automorphism σ, such that the requirements (a), (b), (c) of Theorem 1.1 are satisfied.

In accordance with Proposition 2.1, the main goal is to construct, for infinitely many positive integers N, an algebra $\mathbf{B}_N$ such that $\mathbf{B}_N$ itself is not locally finite but every N-generated subalgebra of $\mathbf{B}_N$ is in the variety Var($\mathbf{A}$). In addition, it must be shown that Var($\mathbf{A}$) is locally finite, which will follow from Proposition 2.2 if it can be shown that the N-generated subalgebras of $\mathbf{A}$ are bounded in size, for infinitely many values of N.

Accordingly, let any N be given, subject only to the requirement that N be larger than the rank $n(f)$ for every fundamental operation f of $\mathbf{A}$. (Here we drop subscripts for fundamental operations and write $n(f)$ for $n(i)$ if $f = f_i$.)

Let $P = A \setminus \{\infty\}$. This definition is more than a notational convenience: It will often be helpful to regard P as a partial algebra $\mathbf{P}$ in its own right, with operations being the restrictions of those of $\mathbf{A}$. In this case, $\mathbf{A} = \mathbf{P}^\# = \mathbf{P} \cup \{\infty\}$, with ∞ supplanting all previously undefined operation values, just as in the construction of the examples $\mathbf{F}^\#$ and $\mathbf{G}^\#$.

The construction of $\mathbf{B}_N$ will be presented in several steps, each an assertion. The detailed proofs are in the following section. First, a more intuitive discussion:

The visual idea of the proof, for the case $\mathbf{A} = \mathbf{G}^\#$ of Example 1.3, is to remove ∞, leaving $\mathbf{G}$, which can be pictured as in Figure 2; then to form $\mathbf{G} \times \mathbf{Z}$, which is then a planar grid; then to wrap $\mathbf{G} \times \mathbf{Z}$ into a cylindrical grid, but with a helical covering map; and then finally to put ∞ back to obtain $\mathbf{B}_N$. If the cylinder is not too tightly wound, it can be shown that any N proper elements of $\mathbf{B}_N$ will lie in a piece of $\mathbf{B}_N$ that is an algebra in the variety of $\mathbf{G}^\#$, and yet $\mathbf{B}_N$, under operations defined via the helical covering, is not locally finite.

More specifically, for $\mathbf{A} = \mathbf{G}^\#$ and $N = 7$ (say), the helical covering could consist of identifying together the elements $\ldots, (a_{i-8}, j+1), (a_i, j), (a_{i+8}, j-1), \ldots,$ for each i, j. Then the resulting cylinder has eight "vertical grid lines" in the $\mathbf{Z}$ direction. Any seven elements on the cylinder must miss one of them. But because in **G** non-adjacent elements do not interact, the cylinder with a vertical line removed is no different from the union of seven consecutive vertical grid lines in the plane, which with ∞ can be shown to be an algebra in the variety of **A**. On the other hand, for the particular case of **G** it can be shown that $(a_i, j) = (a_{i-1}, j) * (a_i, j - 1)$. It follows that in the cylinder the images of $(a_0, 0), \ldots, (a_7, 0)$ generate an infinite subalgebra. Indeed, the image of $(a_0, 0)$ is the same as the image of $(a_8, -1)$, whose product with the image of $(a_7, 0)$ is the image of $(a_8, 0)$, a new element. Similarly, the images of $(a_1, 0)$ and $(a_8, 0)$ generate the image of $(a_9, 0)$, etc., and so on.

A similar pictorial description applies to the case $\mathbf{A} = \mathbf{F}^{\#}$ of Example 1.2 if **F** is pulled straight so that it also resembles Figure 2. However, in this and other examples the combinatorics of the "omitted vertical line" become less trivial, and the relationship of N to the tightness of the helical map needs more careful computation. The function ϕ of condition (c) of the theorem, covert in the preceding paragraph, enters more overtly into the generation of the infinite subalgebra.

A useful device is to let σ itself define the horizontal aspect of the helical map, so that vertical lines on the cylinder are in one-to-one correspondence with σ-orbits. To keep the winding from being too tight, however, it is first necessary to loosen it by changing σ so that the number of orbits is increased.

Step 1. It is no restriction to assume that the number of σ-orbits in **A** is finite.

Step 2. The number of σ-orbits in **A** can be increased by the simple expedient of replacing σ by $\sigma' = \sigma^k$ (for any desired $k > 1$) and ϕ by $\phi' = \sigma^k(\sigma^{-1}\phi)^k$; the number of finite orbits then becomes k times as large and **A**, σ', ϕ' still satisfy the hypotheses of the theorem.

Dropping the primes, then, we see that it is no restriction to suppose that the number of σ-orbits exceeds any preassigned bound.

2.4. Example. Let $\mathbf{F}^{\#}$ be as in Example 1.2, but, taking $k = 4$, let σ now be given by $\sigma(a_j) = a_{j+8}$ and let $\phi(x) = (\ldots((x \vee a_1) \wedge a_2) \vee \ldots) \wedge a_8$. The hypotheses of Theorem 1.1 remain valid, but there are now eight infinite orbits instead of two.

Let us say that $a_1, a_2 \in \mathbf{A}$ are *operationally related* if they both appear in the same tuple in the graph of some fundamental operation of **P**. For example, in $\mathbf{F}^{\#}$, a_{-1} and a_1 are operationally related because $(a_{-1}, a_1, a_0) \in \wedge$. From the same triple, a_{-1} and a_0 are operationally related, as are a_1 and a_0 and even each of these elements and itself. However, a_0 and a_4 are not operationally related.

We shall be interested in subalgebras of **A** that are unions of σ-orbits. One way to obtain such a subalgebra is to take the subalgebra generated by the union of specified σ-orbits. Another way is to start with a subalgebra $\mathbf{S}_0$ no element of which is operationally related to any element of any translate $\sigma^k(\mathbf{S}_0)$, $(k \in \mathbf{Z},\ k \neq 0)$, and then to take $\bigcup_{k \in \mathbf{Z}} \sigma^k(\mathbf{S}_0)$, which will be a subalgebra and a union of σ-orbits. Let us call a subalgebra constructible in this latter way *σ-decomposable*. For example, in 2.4 the subalgebra generated by the orbits of a_{-1} and a_1 is σ-decomposable with $\mathbf{S}_0 =$

$= \{a_{-1}, a_0, a_1, \infty\}$, but $\mathbf{F}^{\#}$ itself is not σ-decomposable. (Note that $\mathbf{S}_0$ contains at most one element from each σ-orbit.)

Step 3. If the number of σ-orbits is increased beyond a certain bound (depending on N) by the method of Step 2, then

($*$) the union of any N σ-orbits generates a σ-decomposable subalgebra of $\mathbf{A}$.

By Step 2, we can and do assume that ($*$) holds. The remainder of the construction therefore depends on N, even though this fact will not be reflected notationally.

As an example, if $N = 3$ then $\mathbf{F}^{\#}$ and σ in 2.4 satisfy ($*$).

If $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are partial algebras of the same type, then $\mathbf{Q}_1 \times \mathbf{Q}_2$ is the partial algebra on the product set for which each fundamental operation f is defined coordinatewise wherever f can be evaluated in both coordinates and is undefined elsewhere.

Let $\mathbf{Z}$ be made into an algebra of the same type as $\mathbf{A}$ by setting $f(k_1, \ldots, k_{n(f)}) = \max(k_1, \ldots, k_{n(f)})$, for each fundamental operation f of the type of $\mathbf{A}$. Then $\mathbf{Z}$ itself may or may not be in Var($\mathbf{A}$); in 1.3, for example, $\mathbf{G}^{\#}$ satisfies the law $x^2 = y^2$ but $\mathbf{Z}$ does not. Nevertheless:

Step 4. $(\mathbf{P} \times \mathbf{Z})^{\#} \in \mathrm{Var}(\mathbf{A})$.

For a partial algebra $\mathbf{Q}$, a *congruence relation* is an equivalence relation ϱ such that for any fundamental operation f of $\mathbf{Q}$ and for any ϱ-blocks $B_1, \ldots, B_{n(f)}$, if some value $f(b_1, \ldots, b_{n(f)})$ ($b_i \in B_i$) is defined then all such values lie in a single ϱ-block ([5], p. 82). This condition is precisely the one needed to have the ϱ-blocks form a partial algebra $\mathbf{Q}/\varrho$ in a natural way.

An instructive example is to let $\mathbf{Q} = \mathbf{F}$ and to let ϱ be the equivalence relation whose blocks are the σ-orbits, with σ as in 2.4. For example, orbit $(a_{-1}) \wedge$ orbit $(a_1) =$ orbit (a_0). $\mathbf{F}/\varrho$ can be pictured as a "crown", a circular version of Figure 1, with four maximal and four minimal elements. In contrast, for σ as in 1.2, the σ-orbits do not give a congruence relation; not even orbit $(a_0) \vee$ orbit (a_0) is well defined.

Now let $\tau: \mathbf{Z} \to \mathbf{Z}$ be the automorphism given by $\tau(k) = k - 1$. Then $\sigma \times \tau$ is an automorphism of $\mathbf{P} \times \mathbf{Z}$.

Step 5. The partition of $\mathbf{P} \times \mathbf{Z}$ into $\sigma \times \tau$-orbits is a congruence relation on $\mathbf{P} \times \mathbf{Z}$.

Let the quotient partial algebra be denoted by $\mathbf{P} \times \mathbf{Z}/\sigma \times \tau$ and let $\pi: \mathbf{P} \times \mathbf{Z} \to \mathbf{P} \times \mathbf{Z}/\sigma \times \tau$ be the natural map.

Let $\mathbf{B}_N = (\mathbf{P} \times \mathbf{Z}/\sigma \times \tau)^{\#}$.

Step 6. $\mathbf{B}_N$ is not locally finite.

Step 7. Each N-generated subalgebra of $\mathbf{B}_N$ is the isomorphic image under π of a subalgebra of $(\mathbf{P} \times \mathbf{Z})^{\#}$.

(Here π is extended by setting $\pi(\infty) = \infty$.)

Step 8. The N-generated subalgebras of $\mathbf{A}$ are bounded in size.

Steps 4 and 7 show that each N-generated subalgebra of $\mathbf{B}_N$ is in Var($\mathbf{A}$), as required. As already remarked, Step 8 (for the infinitely many possible N considered) shows that Var($\mathbf{A}$) is locally finite. The outline of the proof of Theorem 1.1 is complete.

Before proceeding to the details of the steps, let us derive a useful variant of Theorem 1.1:

Condition (c) of Theorem 1.1 is closely related to finitely subdirect irreducibility. Recall that an algebra $\mathbf{A}$ is *finitely subdirectly irreducible* if $\mathbf{A}$ cannot be expressed as a finite subdirect product unless some coordinate projection is an isomorphism, or equivalently, if $0 \in \mathrm{Con}(\mathbf{A})$ is (finitely) meet-irreducible. This theorem differs from Theorem 1.1 only in its third condition:

2.5. Theorem. *Let* $\mathbf{A}$ *be an infinite locally finite algebra with finitely many fundamental operations, with an absorbing element* ∞, *and with an automorphism* σ *such that*

(a) $\{\infty\}$ *is the only* σ*-orbit of* A *that is finite;*

(b) *the proper part of* f *is partitioned by* σ *into only finitely many orbits, for all fundamental operations* f *of* A;

(c′) $\mathbf{A}$ *is finitely subdirectly irreducible.*

Then A *is inherently nonfinitely based.*

Proof. Let Φ be the semigroup of nontrivial unary polynomial functions of $\mathbf{A}$. Here a unary polynomial is considered nontrivial if it is of the form $t(x, c_2, \ldots, c_{n(t)})$ for a term t in which x actually appears. Let $\prec$ be the quasi-order on P given by $a \prec c$ iff $c = \phi(a)$ for some $\phi \in \Phi$. We claim that P is directed under $\prec$. Indeed, by the finite subdirect irreducibility of $\mathbf{A}$, for any $a, b \in P$, $\mathrm{con}(a, \infty) \cap \mathrm{con}(b, \infty)$ contains some $\mathrm{con}(c, d)$ with $c \neq d$ (whence we may assume $c \neq \infty$); since (c, d) is in the transitive symmetric closure of $\{(\phi(a), \infty),\ \phi \in \Phi\}$, either $d = \infty$ or $(c, \infty), (d, \infty) \in \mathrm{con}(a, \infty)$, so that $a \prec c$; similarly, $b \prec c$. Select a finite subset of P that contains more than one element from each of the finitely many σ-orbits; by directedness, its elements will be dominated by some single element e. In particular, $a \prec e$ for some $a \neq e$ in the same orbit as e. Write $e = \sigma^k(a)$. Replacing σ by σ^{-1} if necessary, we may assume $k > 0$. Then replacing σ by σ^k if $k > 1$ (as in Step 3), we may assume $k = 1$. Thus the condition (c) of Theorem 1.1 is satisfied. □

3. DETAILS OF THE STEPS

For *Step* 1: Let $\mathbf{A}'$ consist of ∞ and the set of all elements of $\mathbf{A}$ that appear in proper tuples from fundamental operations. $\mathbf{A}'$ is a σ-invariant subalgebra of $\mathbf{A}$ that has finitely many σ-orbits and still satisfies all the hypotheses of Theorem 1.1. We must argue that if $\mathbf{A}'$ can be shown to be inherently nonfinitely based, then $\mathbf{A}$ itself will be inherently nonfinitely based. Because $\mathbf{A}'$ is a subalgebra of $\mathbf{A}$, to verify this assertion we need only show that Var($\mathbf{A}$) is locally finite, or equivalently, that for each k there is a finite bound b_k on the sizes of the k-generated subalgebras of $\mathbf{A}$. Because $\mathbf{A}'$ is inherently nonfinitely based, $\mathrm{Var}(\mathbf{A}')$ is locally finite and so has a similar bound b'_k for each k. But the elements of $\mathbf{A} \setminus \mathbf{A}'$ can be used to generate only themselves and ∞. Therefore we may choose $b_k = b'_k + k + 1$.

For *Step* 2: Observe that if $\phi(x) = t(x, c_2, \ldots, c_{n(t)})$ for some $n(t)$-ary term t, then $\sigma\phi\sigma^{-1}$ is the unary polynomial function given by $t(x, \sigma(c_2), \ldots, \sigma(c_{n(t)}))$. More generally, $\sigma^r\phi\sigma^{-r}$ is unary polynomial for any integer r. The appropriate ϕ' with $\phi'(a) = \sigma'(a)$ is given by

$$\phi'(x) = (\sigma^{k-1}\phi\sigma^{-(k-1)}) \ldots (\sigma^2\phi\sigma^{-2})(\sigma\phi\sigma^{-1})\,\phi = \sigma^k(\sigma^{-1}\phi)^k .$$

The number of orbits in (b) of Theorem 1.1 also grows by a factor of k.

For *Step* 3: Fix an enumeration $P = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$ for which $\sigma(a_j) = a_{j+m}$ for all j, so that the σ-orbits in P correspond to congruence classes of indices (mod m). The words "consecutive", "interval", "distance", etc. for elements of P are to be understood by reference to subscripts.

Let M be the maximum possible distance between operationally related elements of P. (For example, for $\mathbf{F}^*$ as already enumerated, $M = 2$.) Observe that if X is a subset of P and a value to the left of X is obtained by applying a fundamental operation to certain elements of X, then those elements must all be within a distance $M - 1$ of the leftmost element of X. An induction then yields this "principle of generation to the left": For $X \subseteq P$, all elements to the left of X that are generated by X are already generated by $X \cap \{a_i, a_{i+1}, \ldots, a_{i+M-1}\}$, where a_i is the leftmost element of X.

A claim: There is a positive integer w such that the subalgebra generated by any $X \subseteq P$ extends no further than w elements to the left of the leftmost element of X (if any) and w elements to the right of the rightmost element of X (if any). To verify this claim, consider first the case where X consists of M consecutive elements of P. Because $\mathbf{A}$ is locally finite, the subalgebra generated by X is finite; moreover, up to automorphisms of $\mathbf{A}$ there are only finitely many choices for X. Therefore there is a single w that satisfies the claim for all such X. Next, consider the case where X consists of a semi-infinite interval $\{a_j, j \geqq i\}$ for some i. The "principle of generation to the left" shows that the same w as in the first case satisfies the claim. Next, in the general case, if X has a leftmost element then X is contained in an interval semi-infinite to the right and so generates no elements further to the left than does that interval. The reasoning on the right is similar.

Now, let m and σ be changed to m', σ' by the method of Step 2 so that $m' > N(M + 2w)$, and let Y be the union of N σ'-orbits. By looking at indices (mod m'), we see that there is a jump of length at least $M + 2w + 1$ between some element of Y and the next; therefore Y is the union of pieces, each contiguous (relative to Y), each the σ'-translate of the previous one, and each separated from the previous one by such a jump. Let Y_0 be one such piece and let $\mathbf{S}_0$ be the subalgebra generated by Y_0. By the claim, $\mathbf{S}_0$ extends at most a distance w in either direction beyond Y_0, and similarly for translates by powers of σ'. Then there is a jump of length at least $M + 1$ between consecutive translates of $\mathbf{S}_0$, so that elements from distinct translates are operationally unrelated. It follows that the subalgebra generated by Y is the union of the translates of $\mathbf{S}_0$ by powers of σ' and is therefore σ-decomposable by construction.

3.1. Remark. w can be chosen with $w \leqq (m-1)M$. Indeed, let X be an interval of M elements, as in Case 1 of the claim, and let us work to the left of X. For each subset T of P let $\langle T \rangle$ denote the subalgebra of $\mathbf{A}$ generated by T. Consider these slices of length at most M from X, for $k = 0, 1, \ldots$ in turn:

$$X_k = \langle X \rangle \cap \sigma^{-k}(X).$$

By the "principle of generation to the left", each such slice is generated by the preceding one. Thus $X_k = \langle X_{k-1} \rangle \cap \sigma^{-k}(X)$, and $X_0 = X$. To compare the X_k, move them all to X by setting $U_k = \sigma^k(X_k)$. Then $U_k = \langle \sigma(U_{k-1}) \rangle \cap X$. In other words, the U_k are obtained by iterating the set map $\alpha(T) = \langle \sigma(T) \rangle \cap X$, starting from X. Because $\alpha(X) \subseteq X$ and α is isotonic on subsets, we obtain $X = U_0 \supseteq \supseteq U_1 \supseteq U_2 \supseteq \ldots$. As X has M elements, this sequence must reach a fixed point of α by the M-th term, i.e., $U_M = U_{M+1} = \ldots$. This fixed point must be empty; otherwise, all X_k are nonempty and $\langle X \rangle$ is infinite, in contradiction to the local finiteness of $\mathbf{A}$. Thus $X_k = \emptyset$ for all $k \geqq M$ and so X has no element in or to the left of X_M, or in other words, more than a distance of $(m-1)M$ to the left of X. The reasoning on the right is similar.

For *Step* 4: Let $\mu: (\mathbf{P} \times \mathbf{Z})^\# \to \mathbf{A}^\mathbf{Z} = (\mathbf{P}^\#)^\mathbf{Z}$ by $\mu(p, k) = (\ldots, \infty, \infty, p, p, \ldots)$, where the first p is in the k-th entry, and $\mu(\infty) = (\ldots, \infty, \infty, \ldots)$. Then μ is an isomorphic embedding.

For *Step* 5: Let f be a fundamental operation of $\mathbf{P} \times \mathbf{Z}$, let $n = n(f)$, and let $B_1, \ldots, B_n$ be $\sigma \times \tau$-orbits in $\mathbf{P} \times \mathbf{Z}$. We must show that if $f(\beta_1, \ldots, \beta_n) = \gamma$, $f(\bar{\beta}_1, \ldots, \bar{\beta}_n) = \bar{\gamma}$ for some $\beta_i, \bar{\beta}_i \in B_i$ $(i = 1, \ldots, n)$ then $\gamma, \bar{\gamma}$ are in the same $\sigma \times \times \tau$-orbit. Initially, we know only that for some $k(1), \ldots, k(n) \in \mathbf{Z}$, $(\sigma \times \tau)^{k(i)}(\beta_i) = = \bar{\beta}_i$. We claim that $k(1) = \ldots = k(n)$; then if k is this common value, the fact that $(\sigma \times \tau)^k$ preserves f gives $\bar{\gamma} = (\sigma \times \tau)^k(\gamma)$, and we are done. To verify the claim, write $\beta_i = (p_i, j_i)$, $\bar{\beta}_i = (\bar{p}_i, j_i)$ $(i = 1, \ldots, n)$ and let $\mathbf{S}$ be the subalgebra of $\mathbf{A} = \mathbf{P}^\#$ generated by the orbits of $p_1, \ldots, p_n$ (or equivalently, $\bar{p}_1, \ldots, \bar{p}_n$). By $(*)$ and the assumption $N \geqq n(f) = n$, $\mathbf{S} \setminus \{\infty\}$ decomposes into operationally unrelated translates of a subalgebra $\mathbf{S}_0$ by powers of σ. Since each two p_i are operationally related, all p_i must lie in the same translate, say $\sigma^r(\mathbf{S}_0)$; similarly, all $\bar{p}_i$ lie in some $\sigma^{\bar{r}}(\mathbf{S}_0)$. As each σ-orbit in $\mathbf{P}$ intersects $\mathbf{S}_0$ in exactly one point, $\bar{p}_i = \sigma^k(p_i)$ for $k = = \bar{r} - r$, only, so that $k(i) = k$ for all i.

For *Step* 6. In (c) of the theorem write $\sigma(a) = \phi(a) = t(a, c_2, \ldots, c_{n(t)})$ for a term $t = t(x_1, x_2, \ldots, x_{n(t)})$ and some $c_i \in P$. Because ϕ is a nonconstant function, the variable x_1 actually appears in t. We may assume that variables have been numbered and $n(t)$ chosen so that all of the variables $x_2, \ldots, x_{n(t)}$ also appear. Then for $n(t)$-

tuples from $\mathbf{Z}$, the same term evaluates as $t(i_1, \ldots, i_{n(t)}) = \max(i_1, \ldots, i_{n(t)})$. Recall that for $(p, j) \in \mathbf{P} \times \mathbf{Z}$, $\pi(p, j)$ denotes the $\sigma \times \tau$-orbit of (p, j) as an element of $\mathbf{P} \times \mathbf{Z}/\sigma \times \tau$. Thus $\pi(p, j) = \pi(\sigma(p), j - 1) = \pi(\sigma^2(p), j - 2) = \ldots$.

We show inductively that $\pi(a, 0), \pi(c_2, 0), \ldots, \pi(c_{n(t)}, 0)$ generate $\pi(\sigma^k(a), 0)$ for all $k \geqq 0$, an infinite set: In $\mathbf{P} \times \mathbf{Z}$, $t((a, k), (c_2, 0), \ldots, (c_{n(t)}, 0)) = (\sigma(a), k)$. Then in $\mathbf{P} \times \mathbf{Z}/\sigma \times \tau$,

$$t(\pi(\sigma^k(a), 0), \pi(c_2, 0), \ldots, \pi(c_{n(t)}, 0)) = t(\pi(a, k), \pi(c_2, 0), \ldots, \pi(c_{n(t)}, 0)) = \\ = \pi(\sigma(a), k) = \pi(\sigma^{k+1}(a), 0).$$

In other words, the unary polynomial function $t(\cdot, \pi(c_2, 0), \ldots, \pi(c_{n(t)}, 0))$, iterated from $\pi(a, 0)$, generates an infinite spiral.

For *Step* 7. In $\mathbf{P} \times \mathbf{Z}/\sigma \times \tau$, a fundamental operation f applied to $n(f)$ elements ($\sigma \times \tau$-orbits) yields either a single element (orbit) or nothing, according to Step 5. In $(\mathbf{P} \times \mathbf{Z})^\#$, then, the same operation applied to representatives of the same orbits, for all possible choices, yields either the union of a single orbit and $\{\infty\}$ or just $\{\infty\}$ alone. It follows that for given elements of $B_N = (\mathbf{P} \times \mathbf{Z}/\sigma \times \tau)^\#$, the subalgebra they generate is the image under π of the subalgebra generated by the union of their preimages in $(\mathbf{P} \times \mathbf{Z})^\#$.

Now let N elements of $\mathbf{B}_N = (\mathbf{P} \times \mathbf{Z}/\sigma \times \tau)^\#$ be given. It will be enough to find a subalgebra $\mathbf{T}_0$ of $(\mathbf{P} \times \mathbf{Z})^\#$ that is mapped isomorphically onto a subalgebra of $\mathbf{B}_N$ containing these elements. We may assume none of the elements are ∞. Let U be the union of their corresponding preimages in $\mathbf{P} \times \mathbf{Z}$. The projection of U on the $\mathbf{P}$-coordinate is the union of at most N σ-orbits in $\mathbf{P}$ and so generates a subalgebra $\mathbf{S}$ of $\mathbf{P}^\#$ that is σ-decomposable, by the assumption $(*)$, into translates of a finite subalgebra $\mathbf{S}_0$. For convenience, write $\mathbf{S} = \mathbf{R}^\#$, $\mathbf{S}_0 = \mathbf{R}_0^\#$, where $R = S \setminus \{\infty\}$, $R_0 = S_0 \setminus \{\infty\}$. Consider the subalgebra $(\mathbf{R} \times \mathbf{Z})^\#$ of $(\mathbf{P} \times \mathbf{Z})^\#$, which is $\sigma \times \tau$-invariant and contains U. Observe that $\pi(\mathbf{R} \times \mathbf{Z})^\#$ is exactly the same as $\pi(\mathbf{R}_0 \times \mathbf{Z})$. Take $\mathbf{T}_0 = \mathbf{R}_0 \times \mathbf{Z}$. π is homomorphic on $\mathbf{T}_0$, because elements of $\mathbf{R}_0 \times \mathbf{Z}$ are operationally unrelated to elements of its translates by powers of $\sigma \times \tau$. Since π is one-to-one on $\mathbf{T}_0$ and its image contains the original N generators, we are done.

For *Step* 8. Let N elements of $\mathbf{A}$ be given; we may assume none is ∞. The N elements are contained in N σ-orbits. By $(*)$, these orbits together generate a σ-decomposable subalgebra of $\mathbf{A}$ consisting of σ-translates of a finite subalgebra $\mathbf{S}_0$. The N elements are therefore contained in the union of at most N σ-translates of $\mathbf{S}_0$, which is a subalgebra of cardinality at most Nm, where m is the number of σ-orbits.

4. FINITE INHERENTLY NONFINITELY BASED ALGEBRAS

Finite algebras with an absorbing element can often be shown to be inherently nonfinitely based by using Theorem 1.1.

The method depends on looking for a sequence $\alpha = \ldots b_{-1} b_0 b_1 \ldots$ of proper elements from $\mathbf{B}$ such that α has certain properties. Such a sequence can be regarded as an element $(\ldots, b_{-1}, b_0, b_1, \ldots)$ of the algebra $\mathbf{B}^{\mathbf{Z}}$, as can its translates $\alpha^{(i)}$, where $\alpha^{(i)}$ is α shifted i positions (to the right if $i > 0$, or to the left if $i < 0$, or not at all if $i = 0$).

The following theorem summarizes the case where Theorem 1.1 can be applied with only one infinite σ-orbit.

4.1. Theorem. *Let* $\mathbf{B}$ *be a finite algebra of finite type, with an absorbing element* ∞. *Suppose that a sequence* α *of proper elements of* $\mathbf{B}$ *can be found with these properties*:

(a) *in* $\mathbf{B}^{\mathbf{Z}}$, *any fundamental operation* f *applied to translates of* α *yields as a value either a translated of* α *or a sequence containing* ∞;

(b) *there are only finitely many equations* $f(\alpha^{(i_1)}, \ldots, \alpha^{(i_{n(f)})}) = \alpha^{(j)}$ *in which* f *is a fundamental operation and some arguments is* α *itself.*

(c) *there is at least one equation* $f(\alpha^{(i_1)}, \ldots, \alpha^{(i_{n(f)})}) = \alpha^{(1)}$ *in which some argument is* α *itself, in an entry on which* f *actually depends.*

Then $\mathbf{B}$ *is inherently nonfinitely based.*

4.2. Example. Let $\mathbf{B}$ be the graph algebra P_4 of the graph shown in Figure 3. Let $\alpha = \ldots rsrstutu \ldots$, where the first t is in the 0-th position. Observe that $\alpha^{(i)} * \alpha^{(j)}$ has some coordinates ∞ except when $j = i \pm 1$, where the value is $\alpha^{(i)}$. The requirements of Theorem 4.1 are therefore satisfied.

Figure 3. The graph P_4

P r o o f of Theorem 4.1. First, by (a) the subalgebra $\mathbf{S}$ of $\mathbf{B}^{\mathbf{Z}}$ that is generated by the $\alpha^{(i)}$ consists of the $\alpha^{(i)}$ themselves and some additional elements having one or more entries ∞. Next, let θ be the equivalence relation on $\mathbf{S}$ of which one block consists of all elements with ∞ in some entry and all other blocks are singletons; observe that θ is a congruence relation. In $\mathbf{S}/\theta$, call the nontrivial block ∞. Since $\mathbf{S}/\theta \in \operatorname{Var}(\mathbf{B})$, it will be enough to verify that $\mathbf{S}/\theta$ is inherently nonfinitely based. The right-shift in $\mathbf{B}^{\mathbf{Z}}$ induces an automorphism σ of $\mathbf{S}/\theta$ with a single orbit other than $\{\infty\}$. The conditions (b), (c) of 4.1 then imply (b), (c) of Theorem 1.1 for $\mathbf{S}/\theta$. Condition (b) of 4.1 also implies (a) of 1.1. $\square$

For example, if $\mathbf{B}$ is the graph algebra of the graph of Figure 3, then $\mathbf{S}/\theta$ is $\mathbf{G}^{\#}$ of Example 1.1.

Some additional applications of Theorem 4.1:

4.3. Example. Park's algebra $\mathbf{P}^{\#}$, where $\mathbf{P} = (\{b_1, b_2, b_3\}, \vee)$ with $\vee$ being idempotent, commutative, and satisfying $b_i \vee b_{i+1} = b_{i+1}$ [21]. Here Theorem 4.1 applies with $\alpha = \ldots b_1 b_1 b_2 b_3 b_3 \ldots$.

4.4. Example. Murskii's algebra: the graph algebra of the graph M of Figure 4 [17, 18]. Here $\alpha = \ldots ssssrsrssrsssrssssr \ldots$.

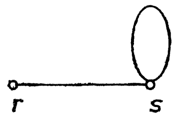

Figure 4. The graph M

4.5. Example. The graph algebra of the graph T of Figure 5 [26]. Here $\alpha =$ $= \ldots rsrsrstrsrsrs \ldots$.

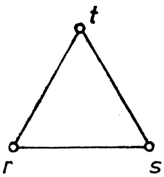

Figure 5. The graph T

4.6. Example. The graph algebra of the graph L_3 of Figure 6 [26]. Here $\alpha =$ $= \ldots rrrsttt \ldots$.

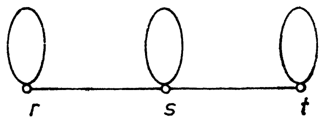

Figure 6. The graph L_3

Other kinds of finite algebras may lead to applications of Theorem 1.1 in which there is more than one σ-orbit:

4.7. Example. Let $\mathbf{F}_0$ be the partially ordered set of Figure 7, regarded as a partial lattice. Then $\mathbf{F}_0^\#$ is inherently nonfinitely based. Indeed, let $\alpha = \ldots rrrsuuu \ldots$ and $\beta = \ldots rrrtuuu \ldots$; then α, β, and their translates generate a subalgebra $\mathbf{S}$ of $(\mathbf{F}_0^\#)^{\mathbf{Z}}$ of which they themselves are the only members that have no ∞ as an entry. As before, the equivalence relation θ that collapses together the remaining members of $\mathbf{S}$ is a congruence relation. $\mathbf{S}/\theta$ is an isomorphic copy of $\mathbf{F}^\#$ of Example 1.2, with a_{2i} and a_{2i+1} being the blocks of $\alpha^{(i)}$ and $\beta^{(i)}$ respectively.

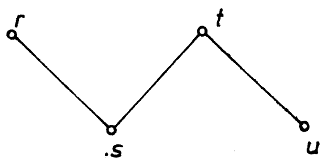

Figure 7. The graph F_0

4.8. Example. Let $\mathbf{H}_0$ be the partial algebra with elements r, s, t and binary operation defined by $rs = t$ and idempotence ($rr = r$, $ss = s$, $tt = t$). Then $\mathbf{H}_0^{\#}$ is an idempotent 4-element algebra. Moreover, $\mathbf{H}_0^{\#}$ is inherently nonfinitely based. Indeed, let $\alpha = \ldots tttrrrr \ldots$ and let $\beta = \ldots tttsrrr \ldots$. Then in $(\mathbf{H}_0)^{\mathbf{Z}}$, $\alpha\beta = \ldots ttttrrr \ldots$, a translate of α. As before, α, β, and their translates generate a subalgebra $\mathbf{S}$ of $(\mathbf{H}_0^{\#})^{\mathbf{Z}}$ of which they are the only members that do not have ∞ as an entry, and the equivalence relation θ that collapses together the remaining members of $\mathbf{S}$ is a congruence relation. $\mathbf{S}/\theta$ is isomorphic to $\mathbf{H}^{\#}$, where $\mathbf{H}$ is an infinite partial algebra satisfying the hypotheses of Theorem 1.1. H has two σ-orbits $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ and $\{\ldots, b_{-1}, b_0, b_1, \ldots\}$, where a_i, b_i are respectively the blocks of $\alpha^{(i)}$ and $\beta^{(i)}$, and the fundamental operation of $\mathbf{H}$ has three σ-orbits, corresponding to the rule $a_i b_i = a_{i+1}$ and to idempotence ($a_i a_i = a_i$, $b_i b_i = b_i$). (A commutative variant of this example can be constructed by changing the operation of $\mathbf{H}_0$ to include $sr = t$ as well.)

4.9. Corollary (Ježek [6], Theorems 4.2 and 4.3). *Let* $\mathbf{G}$ *be a finite idempotent groupoid [or finite idempotent commutative groupoid] such that for any term* t *the equation* $x_1 x_2 = t$ *is satisfied in* $\mathbf{G}$ *if and only if it is a consequence of the idempotent law [idempotent and commutative laws]. Then* $\mathbf{G}$ *is inherently nonfinitely based.* (*A groupoid* is simply an algebra with a single, binary operation.)

Proof. Let $\mathbf{F}_2$ be the free algebra on two generators x_1, x_2 subject to the equations of $\mathbf{G}$. The unbracketed hypothesis implies that $\mathbf{F}_2$ can be mapped homomorphically onto $\mathbf{H}_0^{\#}$ of Example 4.8 by sending $x_1 \mapsto r$, $x_2 \mapsto s$, $x_1 x_2 \mapsto t$, and all other elements to ∞. Therefore $\mathbf{H}_0^{\#} \in \operatorname{Var}(\mathbf{G})$. Since $\mathbf{H}_0^{\#}$ is inherently nonfinitely based, so is $\mathbf{G}$. The bracketed version is proved similarly but uses the commutative variant of Example 4.8.

The method of the proof just concluded presents an interesting contrast to Ježek's syntactic method of proof. Observe that algebras satisfying Ježek's hypotheses need no absorbing element, but they generate varieties that contain some algebras with absorbing element, enabling an application of the present theory.

4.10. Examples. Ježek ([6], Corollary 5.2) lists three idempotent three-element groupoids to which Corollary 4.9 applies. Each of these has elements a, b, c and an idempotent binary operation with $ac = ca = b$, $bc = cb = c$, and $ab, ba \in \{b, c\}$. To derive these examples most directly from the present theory, it can be observed that if $\mathbf{G}$ is any of the three, the subalgebra of $\mathbf{G} \times \mathbf{G}$ generated by (a, c) and (c, a) maps homomorphically onto the commutative variant of $\mathbf{H}_0^{\#}$ of Example 4.8.

4.11. Example. An infinite algebra that satisfies the conditions of Theorem 1.1 somewhat minimally and at first appears not to come from a finite algebra is the following: Let the partial algebra $\mathbf{U}$ have elements $\ldots, a_{-1}, a_0, a_1, \ldots$ and $\ldots, b_{-1}, b_0, b_1, \ldots$, and with a binary operation whose only values are given by $a_{i-1} b_i = a_i$. Then $\mathbf{U}^{\#}$ is inherently nonfinitely based. The intuition gained from other examples would suggest that $\mathbf{U}^{\#}$ might be in the variety generated by the subalgebra $\mathbf{U}_0^{\#}$ for $\mathrm{U}_0 = \{a_0, a_1, b_1\}$, but in fact $\mathbf{U}_0^{\#}$ is easily seen to be finitely based. A resolution lies in the following example.

4.12. Example. Perkins ([22], Example 1) considers any finite algebra $\mathbf{A}$ which has an absorbing element 0 and a unit element 1 and is non-absorptive, non-commutative, and non-associative; he remarks that any such algebra obeys an equation of the form $(\ldots(x_1x_2)\ldots x_n) = u$ only if u is the term on the left, and then derives the fact that such an algebra must be inherently nonfinitely based. For an alternative derivation we proceed as follows: Let $\mathbf{F}_\omega$ be the free algebra on $\mathbf{A}$ on countably many generators $x_1, x_2, \ldots$. Identify all elements other than the generators and the expressions $u_k = (\ldots(x_1x_2)\ldots x_k)$ $(k \geqq 2)$ to a single absorbing element ∞; by the remark about the laws of $\mathbf{A}$, this identification is a congruence relation. Let $\mathbf{F}'_\omega$ be the factor algebra. In $\mathbf{F}'_\omega$ the only non-∞ products are $u_{i-1}x_i = u_i$. This algebra is strongly reminiscent of Example 4.11, but not being "infinite to the left", it has no automorphism, merely an injective endomorphism. However, an ultrapower of $\mathbf{F}'_\omega$ has many subalgebras that are indeed isomorphic to $\mathbf{U}^{\#}$ of Example 4.11 and so are inherently nonfinitely based. Since they are in Var $(\mathbf{A})$, $\mathbf{A}$ is inherently nonfinitely based.

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