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OSCILLATION THEOREMS FOR CERTAIN NEUTRAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Consider the neutral differential equation

$$(1)\qquad L_n[x(t) + p(t)\,x[t-\tau]] + q(t)\,f(x[t-\sigma]) = 0\,, \quad n \text{ is even}\,,$$

where

$$L_0\,x(t) = x(t)\,,\quad L_k\,x(t) = \frac{1}{a_k(t)}\,(L_{k-1}\,x(t))^{\cdot},\quad k = 1, 2, \ldots, n\,,$$

$a_0 = a_n = 1$, $a_i \in C[[t_0, \infty), (0, \infty)]$, $i = 1, 2, \ldots, n-1$, $p, q \in C[[t_0, \infty), R]$, $f \in$ $\in C[R, R]$ and τ and σ are nonnegative constants.

We will assume that

$$(2)\qquad \int^{\infty} a_k(s)\,\mathrm{d}s = \infty\,,\quad k = 1, 2, \ldots, n-1\,;$$

$$(3)\qquad \frac{f(x)}{x} \geqq \gamma > 0 \quad \text{for} \quad x \neq 0\,.$$

The domain of L_n, $D(L_n)$ is defined to be the set of all functions $x\colon [t_0, \infty) \to R$ such that $L_j\,x(t)$, $0 \leqq j \leqq n$ exist and are continuous on $[t_0, \infty)$. Our attention is restricted to those solutions $X \in D(L_n)$ of Eq. (1) which satisfy $\sup\{|x(t)|: t \geqq T\} > 0$ for any $T \geqq t_0$. A solution of Eq. (1) is called *oscillatory* if it has aribtrarily large zeros; otherwise, it is called *nonoscillatory*. Equation (1) is said to be *oscillatory* if all of its solutions are oscillatory.

Recently, Grammatikopoulos et. al. [4] and Ladas and Spicas [6] obtained some interesting criteria for the oscillation of the neutral differential equations of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\,[x(t) + p(t)\,x[t-\tau]] + q(t)\,x[t-\sigma] = 0\,.$$

These results extended some of the known results for the nonneutral differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\,x(t) + q(t)\,x[t-\sigma] = 0\,.$$

The purpose of this paper is to establish some new criteria for the oscillation of Eq. (1). These criteria extend and improve those in [4] and [6]. Our results are new even when $p(t) \equiv 0$.

Neutral delay differential equations have applications in electric networks containing lossless transmission lines. Such networks arise for example, in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see [8] and [9]). Second order neutral delay differential equations appear in the study of vibraring masses attached to an elastic bar. We are not aware of any situation where equations of the type (1) arise for $n > 2$ even when $a_i(t) \equiv 1$, $i = 1, 2, \ldots, n - 1$.

We assume here that solutions of Eq. (1) are defined for all $t \geqq t_0$.

2. MAIN RESULTS

For any functions $p_i \in C[[t_0, \infty), R]$, $i = 1, 2, \ldots, n$, we define

$$I_0 = 1\,, \quad I_i(t, s; p_i, \ldots, p_1) = \int_s^t p_i(u)\, I_{i-1}(u, s; p_{i-1}, \ldots, p_1)\, du\,.$$

It is easy to verify that for $1 \leqq i \leqq n - 1$

$$I_i(t, s; p_1, \ldots, p_i) = (-1)^i I_i(s, t; p_i, \ldots, p_1)$$

and

$$I_i(t, s; p_1, \ldots, p_i) = \int_s^t p_i(u)\, I_{i-1}(t, u; p_1, \ldots, p_{i-1})\, du\,.$$

The following two lemmas will be needed in the proof of our results.

Lemma 1. *If $x \in D(L_n)$, then for $t, s \in [t_0, \infty)$ and $0 \leqq i < k \leqq n$*

(i)
$$L_i\, x(t) = \sum_{j=i}^{k-1} I_{j-1}(t, s; a_{i+1}, \ldots, a_j)\, L_j\, x(s) + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \ldots, a_{k-1})\, a_k(u)\, L_k\, x(u)\, du\,.$$

(ii)
$$L_i\, x(t) = \sum_{j=i}^{k-1} (-1)^{j-i} I_{j-i}(s, t; a_j, \ldots, a_{i+1})\, L_j\, x(s) + (-1)^{k-i} \int_t^s I_{k-i-1}(u, t; a_{k-1}, \ldots, a_{i+1})\, a_k(u)\, L_k\, x(u)\, du\,.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

Lemma 2. *Suppose condition (2) holds. If $x \in D(L_n)$ is of constant sign and not identically zero for all large t, then there exist a $t_x \geqq t_0$ and an integer l, $0 \leqq l \leqq n$ with $n + l$ even for $x(t)\, L_n\, x(t)$ nonnegative or $n + l$ odd for $x(t)\, L_n\, x(t)$ nonpositive and such that for every $t \geqq t_x$*

$$l > 0 \quad \textit{implies} \quad x(t)\, L_k\, x(t) > 0\,, \quad (k = 0, 1, \ldots, l)$$

and

$$l \leqq n - 1 \quad \textit{implies} \quad (-1)^{l-k}\, x(t)\, L_k\, x(t) > 0\,, \quad (k = l, l + 1, \ldots, n)\,.$$

This lemma generalizes a well known lemma of Kiguradze and can be proved similarly.

It will be convenient to make use of the following notation in the remainder of this paper. For any $T \geqq t_0 \geqq 0$ and all $t \geqq T$ we let

$$\omega_l[t, T] = \int_T^t I_{l-1}(t, s; a_1, \ldots, a_{l-1})\, a_l(s)\, I_{n-l-1}(t, s;\ a_{n-1}, \ldots, a_{l+1})\, \mathrm{d}s$$

for

$$1 \leqq l \leqq n - 1\,.$$

Theorem 1. *Let conditions* (2) *and* (3) *hold, and*

$$q(t) \geqq 0 \quad and \quad 0 \leqq p(t) \leqq 1 \quad for \quad t \geqq t_0\,. \tag{4}$$

If, for each $l \in \{1, 3, \ldots, n - 1\}$

$$\liminf_{t\to\infty} \int_{t-\sigma}^t \omega_l[s - \sigma, T]\, q(s)\, [1 - p(s - \sigma)]\, \mathrm{d}s > \frac{1}{\gamma \mathrm{e}} \tag{5}$$

for all large T, $t > T + \sigma$, *then Eq.* (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1) say $x(t) > 0$ for $t \geqq t_0$. There exists $t_1 \geqq t_0$ such that $x[t - \tau] > 0$ and $x[t - \sigma] > 0$ for all $t \geqq t_1$. Set

$$z(t) = x(t) + p(t)\, x[t - \tau]\,.$$

Then

$$z(t) > 0 \quad \text{for all} \quad t \geqq t_1\,,$$

and

$$L_n\, z(t) \leqq 0 \quad \text{for} \quad t \geqq t_1\,.$$

The hypotheses of Lemma 2 are satisfied for $t \geqq t_1$ and hence there exist $t_2 \geqq t_1$ and $l \in \{1, 3, \ldots, n - 1\}$ such that

$$L_k\, z(t) > 0 \quad (k = 0, 1, \ldots, l)\,, \quad t \geqq t_2 \tag{6}$$

and

$$(-1)^{l-k}\, L_k\, z(t) > 0 \quad (k = l, l + 1, \ldots, n)\,, \quad t \geqq t_2\,.$$

Clearly

$$\dot{z}(t) > 0 \quad \text{and} \quad L_{n-1}\, z(t) > 0 \quad \text{for} \quad t \geqq t_2\,.$$

In view of condition (3), Eq. (1) becomes

$$L_n\, z(t) + \gamma\, q(t)\, x[t - \sigma] \leqq 0\,; \quad t \geqq t_0\,,$$

or

$$L_n\, z(t) + \gamma\, q(t)\, \{z[t - \sigma] - p(t - \sigma)\, x[t - \tau - \sigma]\} \leqq 0\,. \tag{7}$$

Since $z(t) > x(t)$, we obtain

$$L_n\, z(t) + \gamma\, q(t)\, \{z[t - \sigma] - p(t - \sigma)\, z[t - \tau - \sigma]\} \leqq 0\,.$$

Now, using the fact that $z(t)$ is increasing on $[t_2, \infty)$, we have

$$L_n\, z(t) + \gamma\, q(t)\, \{1 - p(t - \sigma)\}\, z[t - \sigma] \leqq 0 \quad \text{for} \quad t \geqq t_2\,. \tag{8}$$

By Lemma 1 (ii) we get

$$L_l\, z(s) = \sum_{j=l}^{n-2} (-1)^{j-l} I_{j-l}(t, s; a_j, \dots, a_{l+1})\, L_j\, z(t) +$$
$$+ (-1)^{n-l-1} \int_s^t I_{n-l-2}(u, s; a_{n-2}, \dots, a_{l+1})\, a_{n-1}(u)\, L_{n-1} z(u)\, du$$
$$\text{for} \quad t_2 \leqq s \leqq t\,.$$

Using (6) and the fact that $L_{n-1}\, z(t)$ is decreasing on $[t_2, \infty)$ we obtain

(9) $$L_l\, z(s) \geqq I_{n-l-1}(t, s; a_{n-1}, \dots, a_{l+1})\, L_{n-1}\, z(t) \quad \text{for} \quad t_2 \leqq s \leqq t\,.$$

Next, we use Lemma 1 (i), and obtain

$$z(t) = \sum_{j=0}^{l-1} I_j(t, t_2; a_1, \dots, a_j)\, L_j\, z(t_2) +$$
$$+ \int_{t_2}^t I_{l-1}(t, s; a_1, \dots, a_{l-1})\, a_l(s)\, L_l\, z(s)\, ds\,, \quad t \geqq t_2\,.$$

By (6) we get

(10) $$z(t) \geqq \int_{t_2}^t I_{l-1}(t, s; a_1, \dots, a_{l-1})\, a_l(s)\, L_l\, z(s)\, ds\,, \quad \text{for} \quad t \geqq t_2\,.$$

By combining (9) and (10) we obtain

$$z(t) \geqq \int_{t_2}^t I_{l-1}(t, s; a_1, \dots, a_{l-1})\, a_l(s)\, I_{n-l-1}(t, s; a_{n-1}, \dots, a_{l+1})\, ds\, L_{n-1}\, z(t)$$

or

$$z(t) \geqq \omega_l[t, t_2]\, L_{n-1}\, z(t) \quad \text{for} \quad t \geqq t_2\,.$$

There exists a $t_3 \geqq t_2$ so that $t > t_2 + \sigma$ for all $t \geqq t_3$. Then

$$z[t - \sigma] \geqq \omega_l[t - \sigma, t_2]\, L_{n-1}\, z[t - \sigma] \quad \text{for} \quad t > t_3\,,$$

and (8) becomes

(11) $$L_n\, z(t) + \gamma\, q(t)\, \omega_l[t - \sigma, t_2]\, [1 - p(t - \sigma)]\, L_{n-1}\, z[t - \sigma] \leqq 0$$
$$\text{for} \quad t \geqq t_3\,.$$

Now, set $y(t) = L_{n-1}\, z(t)$. Thus (11) takes the form

(12) $$\dot{y}(t) + \gamma\, q(t)\, \omega_l[t - \sigma, t_2]\, [1 - p(t - \sigma)]\, y[t - \sigma] \leqq 0\,, \quad t \geqq t_3\,.$$

In view of (5) and Theorem 1 in [7], Inq. (12) has no eventually positive solutions. This contradicts the fact that $z(t) > 0$ for $t \geqq t_1$, and the proof si complete.

The following examples are illustrative.

Example 1. The neutral differential equation

(E_1)
$$\left(t\left(t\left(\frac{1}{t}\left(x(t) + \tfrac{1}{2}x[t - 2\pi]\right)^{\bullet}\right)^{\bullet}\right)^{\bullet}\right)^{\bullet} + \frac{3e^{-\sin t}}{2(t - \pi)\, t^2}\, x[t - \pi]\, e^{\sin x[t - \pi]} = 0 \quad \text{for} \quad t > \pi$$

has a nonoscillatory solution $x(t) = t$. All conditions of Theorem 1 are satisfied except condition (5).

Example 2. Consider the neutral differential equation

$$(\mathrm{E}_2) \qquad (\mathrm{e}^{-t}(x(t) + \tfrac{1}{2}x[t - 2\pi])^{\cdot})^{\cdot} + (2(1 + \tfrac{1}{2}\mathrm{e}^{-2\pi})\,\mathrm{e}^{\pi/4})\,\mathrm{e}^{-t}\,x[t - \pi/4] = 0 \quad \text{for} \quad t \geqq 2\pi\,.$$

All conditions of Theorem 1 are satisfied and hence every solution of Eq. (E_2) is oscillatory. One such solution is $x(t) = \mathrm{e}^t \sin t$.

Remark. Theorem 1 is new even if $p \equiv 0$.

Theorem 2. *Let conditions* (2)–(4) *hold. If*

$$(13) \qquad \int^{\infty} q(s)\,[1 - p(s - \sigma)]\,\mathrm{d}s = \infty\,,$$

then Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Assume $x(t) > 0$ for $t \geqq t_0$. As in the proof of Theorem 1, we get (8). Integrating (8) from t_2 to t we have

$$L_{n-1}\,z(t) \leqq L_{n-1}\,z(t_2) - \gamma\,z[t_2 - \sigma]\int_{t_2}^{t} q(s)\,[1 - p(s - \sigma)]\,\mathrm{d}s$$

which, as $t \to \infty$, leads to a contradiction. The proof is complete.

Example 3. The neutral differential equation

$$(\mathrm{E}_3) \qquad (\mathrm{e}^{3t/2}(x(t) + \tfrac{1}{2}x[t - \pi])^{\cdot})^{\cdot} + \tfrac{1}{4}(1 + \tfrac{1}{2}\mathrm{e}^{\pi})\,\mathrm{e}^{\pi}\mathrm{e}^{3t/2}\,x[t - \pi] = 0\,, \quad t \geqq 0$$

has a bounded nonoscillatory solution $x(t) = \mathrm{e}^{-t}$. Only condition (2) of Theorem 2 is violated.

Remark. If $a_i \equiv 1$, $i = 1, \ldots, n - 1$ and $f(x) = x$, then Theorem 10 in [4] and Theorem 2 are the same.

For convenience of notation we define

$$\beta^n(t) = -\frac{\gamma\,q(t)}{p[t - \sigma + \tau]} > 0 \quad \text{and} \quad \delta = \frac{\sigma - \tau}{n} > 0\,.$$

Theorem 3. *Let conditions* (2) *and* (3) *hold and let there exist constants* p_1, p_2 *and* q *such that*

$$(14) \qquad -1 \leqq p_1 \leqq p(t) \leqq p_2 < 0 \quad \textit{for} \quad t \geqq t_0$$

and

$$(15) \qquad q(t) \geqq q > 0 \quad \textit{for} \quad t \geqq t_0\,.$$

If

$$(16) \qquad a_{n-i}[t - i\delta] - \left(\frac{\beta[t - \delta]}{\beta(t)}\right)^{i-1} \geqq (i - 1)\,\frac{\dot\beta(t)}{\beta^2(t)} \geqq 0\,, \quad i = 1, 2, \ldots, n\,, \quad t \geqq t_0$$

and

$$(17) \qquad \liminf_{t \to \infty} \int_{t-\delta}^{t} \beta(s)\,\mathrm{d}s > 1/\mathrm{e}\,,$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Assume $x(t) > 0$ for $t \geqq t_0$. Then there exists a $t_1 \geqq t_0$ so that $x[t - \tau] > 0$ and $x[t - \sigma] > 0$ for $t \geqq t_1$. Set

$$z(t) = x(t) + p(t)\, x[t - \tau].$$

Then

$$L_n z(t) \leqq 0 \quad \text{for} \quad t \geqq t_1.$$

We claim that $z(t)$ is eventually a negative solution of Eq. (1) From Eq. (1) we have

$$\text{(18)} \qquad L_n z(t) = -q(t) \frac{f(x[t - \sigma])}{x[t - \sigma]}\, x[t - \sigma] \leqq -\gamma\, q(t)\, x[t - \sigma] \leqq$$
$$\leqq -\gamma q x[t - \sigma] \quad \text{for} \quad t \geqq t_1,$$

which implies that $L_{n-1} z(t)$ is strictly decreasing for $t \geqq t_1$, while $L_i z(t)$, $n = 0, 1, \ldots, n - 2$ ate strictly monotone functions of t. Therefore, either

$$\text{(19)} \qquad \lim_{t \to \infty} L_{n-1} z(t) = -\infty$$

or

$$\text{(20)} \qquad \lim_{t \to \infty} L_{n-1} z(t) \equiv m \quad \text{is finite}.$$

Assume that (19) holds. Then we can easily obtain

$$\text{(21)} \qquad \lim_{t \to \infty} L_i z(t) = -\infty, \quad i = 0, 1, \ldots, n - 1.$$

Next, assume that (20) is satisfied. Then integrating (18) from t_1 to t and letting $t \to \infty$, we have

$$\int_{t_1}^{\infty} q x[s - \sigma]\, ds \leqq L_{n-1} z(t_1) - m$$

which implies that $x \in L_1[t_1, \infty)$. Thus, by (14), $z \in L_1[t_1, \infty)$. Since $z(t)$ is monotonic, it follows that

$$\text{(22)} \qquad \lim_{t \to \infty} z(t) = 0$$

and so also $m = 0$. From (22) we conclude that for each $i = 0, 1, \ldots, n - 1$

$$\text{(23)} \qquad L_i z(t)\, L_{i+1} z(t) < 0 \quad \text{for} \quad t \geqq t_1.$$

Clearly, either (21) or (22) and (23) implies that

$$z(t) < 0 \quad \text{for} \quad t \geqq t_1.$$

Now,

$$x(t) < -p(t)\, z[t - \tau] \leqq -p_1 x[t - \tau] \leqq x[t - \tau]$$

which implies that $x(t)$ is a bounded function.

Since

$$z[t - \sigma + \tau] = p[t - \sigma + \tau]\, x[t - \sigma] + x[t - \sigma + \tau] \geqq$$
$$\geqq p[t - \sigma + \tau]\, x[t - \sigma] \quad \text{for} \quad t \geqq t_1.$$

Thus

$$\frac{q(t)}{p[t-\sigma-\tau]} z[t-\sigma+\tau] \leqq q(t)\, x[t-\sigma], \quad \text{for} \quad t \geqq t_1 .$$

Using the above inequality in (18) we get

$$L_n z(t) - \left(\frac{-\gamma\, q(t)}{p[t-\sigma+\tau]}\right) z[t-(\sigma-\tau)] \leqq 0 \quad \text{for} \quad t \geqq t_1$$

or

$$L_n z(t) - \beta^n(t)\, z[t-n\delta] \leqq 0 \quad \text{for} \quad t \geqq t_1 . \tag{24}$$

By Theorem 2 in [1] it follows that the above inequality has no eventually negative bounded solution. This contradicts the fact that $z(t) < 0$ for $t \geqq t_1$ and the proof of theorem is complete.

Theorem 4. *Let condition* (14) *in Theorem* 3 *be replaced by*

$$p_1 \leqq p(t) \leqq p_2 < 0 \quad \textit{for} \quad t \geqq t_0 . \tag{14}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of Eq. (1) say $x(t) > 0$ for $t \geqq t_0$. As in the proof of Theorem 3 we have $z(t) < 0$ for $t \geqq t_1$. The rest of the proof is similar to that of Theorem 3 and hence is omitted.

Theorem 5. *Let condition* (16) *in Theorem* 3 (*resp. Theorem* 4) *be replaced by*

$$a_{n-i}[t-i\delta] \geqq \beta(t) \quad \textit{for} \quad i = 1, 2, \ldots, n . \tag{25}$$

Then the conclusion of Theorem 3 (*resp. Theorem* 4) *holds.*

Proof. The proof of Theorem 5 is similar to that of Theorem 3 (resp. Theorem 4) except that we use Theorem 3 in [1] instead of Theorem 2 in [1]. The details are omitted.

Remark. Theorems 3–5 include as special cases Theorems 8 and 11 in [4]. They also extend our Theorems 2 and 3 in [1] to more general equations of the form of Eq. (1). We also mertion that conditions (16) and (25) are discarded if $a_i \equiv 1$, $i = 1, \ldots, n-1$.

The following examples are illustrative.

Example 4. The neutral differential equation

$$\left(\frac{1}{t}\,[x(t) - x[t-1]]^{\cdot}\right)^{\cdot} + \frac{e-1}{e^2}\left[\frac{1}{t} + \frac{1}{t^2}\right] x[t-2] = 0, \quad t > 0, \tag{E$_4$}$$

has a nonoscillatory solution $x(t) = e^{-t}$. All conditions of Theorem 5 are satisfied except condition (17).

Example 5. Consider the neutral differential equation

$$L_n[x(t) - \tfrac{1}{2}x[t-1]] + \tfrac{1}{2}x[t-(n+1)] = 0, \quad n \text{ even}, \quad t > n, \tag{E$_5$}$$

where $L_0\, x(t) = x(t)$, $L_k\, x(t) = (1/t)\,(L_{k-1}\, x(t))^{\cdot}$, $k = 1, 2, \ldots, n$, $a_0 = a_n = 1$.

Here $a_i(t) = t$, $i = 1, 2, \ldots, n-1$, $\beta(t) = 1$ and $\delta = 1$. All conditions of Theorem 3 (or 5) are satisfied and hence all solutions of Eq. (E_5) are oscillatory.

We note that none of the results in [1]–[6] can be used to investigate the oscillatory character of Eq. (E_5).

The following theorem includes Theorem 12 in [4] as a special case.

Theorem 6. *Let conditions* (2) *and* (3) *hold, and*

$$-1 \leqq p(t) \leqq 0 \quad \text{and} \quad q(t) \geqq 0 \quad \text{for} \quad t \geqq t_0 . \tag{26}$$

If

$$\int^\infty q(s)\, \mathrm{d}s = \infty , \tag{27}$$

then every unbounded solution of Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be an unbounded nonoscillatory solution of Eq. (1) say $x(t) > 0$ for $t \geqq t_0$. There exists a $t_1 \geqq t_0$ so that $x[t-\sigma] > 0$ and $x[t-\tau] > 0$ for $t \geqq t_1$. Set

$$z(t) = x(t) + p(t)\, x[t-\tau] .$$

We have

$$L_n\, z(t) \leqq -\gamma\, q(t)\, x[t-\sigma] \leqq 0 \quad \text{for} \quad t \geqq t_1 , \tag{28}$$

and so $L_i\, z(t)$ for $i = 0, 1, \ldots, n-1$, are monotone functions. We claim that $z(t) \geqq 0$ for $t \geqq t_1$. Otherwise $z(t) < 0$ for $t \geqq t_1$ and hence

$$x(t) < -p(t)\, x[t-\tau] \leqq x[t-\tau] \quad \text{for} \quad t \geqq t_1$$

which is impossible since $x(t)$ is unbounded. Thus $x(t) \geqq 0$ for $t \geqq t_1$ and by Lemma 2, there exists a $t_2 \geqq t_1$ so that $L_{n-1}\, z(t) \geqq 0$ and $\dot{z}(t) \geqq 0$ for $t \geqq t_2$.

Integrating (28) from t_2 to t we have

$$L_{n-1}\, z(t) \leqq L_{n-1}\, z(t_2) - \gamma \int_{t_2}^{t} q(s)\, x[s-\sigma]\, \mathrm{d}s \leqq$$
$$\leqq L_{n-1}\, z(t_2) - \gamma\, z[t_2 - \sigma] \int_{t_2}^{t} q(s)\, \mathrm{d}s \to -\infty \quad \text{as} \quad t \to \infty ,$$

a contradiction. This completes the proof.

For illustration we consider the following example.

Example 6. The neutral differential equation

$$\left(t\left(t\left(\frac{1}{t}\left(x(t) - \tfrac{1}{2}x[t-2]\right)\right)^{\cdot}\right)^{\cdot}\right)^{\cdot} + \frac{\mathrm{e}^{-\sin t}}{2(t-\pi)\, t^2}\, x[t-\pi]\, \mathrm{e}^{\sin x[t-\pi]} = 0 \quad \text{for } t > \pi , \tag{E_6}$$

has an unbounded nonoscillatory solution $x(t) = t$. Only condition (27) of Theorem 5 is violated. However, all unbounded solutions of the equation

$$\left(t\left(t\left(\frac{1}{t}\left(x(t) - \tfrac{1}{2}x[t-2\pi]\right)\right)^{\cdot}\right)^{\cdot}\right)^{\cdot} + \frac{\mathrm{e}^{-\sin t}}{t}\, x[t-\pi]\, \mathrm{e}^{\sin x[t-\pi]} = 0 \tag{E_7}$$

are oscillatory by Theorem 6. It is easy to check that Theorems 3–5 fail to apply to Eq. (E_7).

References

[1] *S. R. Grace* and *B. S. Lalli:* Oscillation theorems for certain delay differential inequalities, J. Math. Anal. Appl., *106* (1985), 414—426.

[2] *S. R. Grace* and *B. S. Lalli:* A note on Ladas' paper: Oscillatory effects of retarded actions, J. Math. Anal. Appl., *88* (1982), 257—264.

[3] *S. R. Grace* and *B. S. Lalli:* Oscillation of odd order nonlinear functional differential equations with deviating arguments, Appl. Anal., *20* (1985), 189—199.

[4] *M. K. Grammatikopoulos, G. Ladas* and *A. Meimaridou:* Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients, Chines Ann. Math. (to appear).

[5] *M. K. Grammatikopoulos, G. Ladas* and *A. Meimaridou:* Oscillation and asymptotic behavior of second order neutral differential equations, Annal. Mat. Pura ed Appli. (to appear).

[6] *G. Ladas* and *Y. Sficas:* Oscillations of higher-order neutral equations, J. Austral. Math. Soc. (series B), *27* (1986), 502—511.

[7] *G. Ladas* and *I. P. Stavroulakis:* Oscillations caused by several retarded and advanced arguments, J. Differential Equations, *44* (1982), 134—152.

[8] *M. Slemrod* and *E. F. Infante:* Asymptotic stability criteria for linear systems of difference-differential equations of neutral type and their discrete analogues, J. Math. Anal. Appl. *38* (1972), 399—415.

[9] *W. Snow:* Existence, uniqueness and stability for nonlinear differential-difference equations in the neutral case, N.Y.U. Currant Inst. Math. Sci. Rep. IMM-NYU 328 (February 1965).

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