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ON INTEGRATION IN BANACH SPACES, X (INTEGRATION WITH RESPECT TO POLYMEASURES)

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INTRODUCTION

In this part we continue our investigation of integration of d-tuples of vector valued functions with respect to the operator valued d-polymeasure Γ, started in Part IX. Throughout the paper we use the notation and concepts from Parts VIII and IX.

From the variety of results obtained: Theorems 1 and 2 are Egoroff-Lusin type theorems of independent interest. In Theorem 3 we obtain the equality $\mathscr{I}(\Gamma) = \mathscr{I}_1(\Gamma)$ under the assumption that the vector d-polymeasures $\Gamma(\cdot)(x_i)$: $\mathsf{X}P_i \to Y$, $(x_i) \in \mathsf{X}X_i$, have locally control d-polymeasures. (The control polymeasure problem, see Problem 2 in Part VIII, is still open). By Corollary of Theorem 5 the same equality holds if each space X_i, $i = 1, \ldots, d$, is finite dimensional. Theorems 6 and 9 give satisfactory descriptions of the range of the integral of a given integrable d-tuple of functions. We introduce the so called *product S-integral* (a generalization of the S-integral of A. Kolmogoroff [19]), and in Theorems 7 and 8 we relate it to integrability. For bimeasures this concept was already used by A. K. Katsaras in [18]. Theorem 11 is a Vitali type convergence theorem. Theorem 14 is a generalization of Theorem I.17, while Theorem 15 generalizes Theorem 3 in [14].

1. PRELIMINARIES

In the following two theorems $\mathscr{P}$ is a δ-ring of subsets of a given set T and X is a Banach space.

From Theorem I.8 in [3] we easily obtain the following general result:

Theorem 1. *Let $f: T \to X$ be a $\mathscr{P}$-measurable function and let its range $f(T) \subset X$ be a relatively σ-compact subset of X. Then there are $\mathscr{P}$-simple functions $f_n \in S(\mathscr{P}, X)$, $n = 1, 2, \ldots$, and sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ such that $f_n(T) \subset f(T)$ for each $n = 1, 2, \ldots$, $F_k \nearrow F = \{t \in T, f(t) \neq 0\}$, and on each F_k, $k = 1, 2, \ldots$ the sequence f_n, $n = 1, 2, \ldots$ converges uniformly to the function f. Hence the function $f: T \to X_1 = \overline{\mathrm{sp}}\, f(T)$ is $\mathscr{P}$-measurable.*

Proof. Let $C_k \subset X$, $k = 1, 2, \ldots$ be a non decreasing sequence of compact subsets such that $\bigcup_{k=1}^{\infty} C_k \supset f(T)$, and put $E_k = F \cap f^{-1}(C_k)$. Then $E_k \in \sigma(\mathscr{P})$ for each $k = 1, 2, \ldots$ by the $\mathscr{P}$-measurability of the function f. Since $F \in \sigma(\mathscr{P})$, there are $F'_k \in \mathscr{P}$, $k = 1, 2, \ldots$ such that $F'_k \nearrow F$. Clearly $F_k = E_k \cap F'_k \nearrow F$, and $F_k \in \mathscr{P}$ for each $k = 1, 2, \ldots$. Since owing to Theorem I.8 $f\,\chi(F_1) \in \bar{S}(F_1 \cap \mathscr{P}, X)$, there is a sequence $f'_{n,F_1} \in S(\mathscr{F}_1 \cap \mathscr{P}, X)$, $n = 1, 2, \ldots$ such that $\|f\,\chi(F_1) - f'_{n,F_1}\|_T < 1/2n$ for each $n = 1, 2, \ldots$. Each f'_{n,F_1}, $n = 1, 2, \ldots$ is of the form $f'_{n,F_1} = \sum_{j=1}^{r_n} x'_{n,j}\,\chi(E_{n,j})$ with $E_{n,j} \in F_1 \cap \mathscr{P}$, and $E_{n,j_1} \cap E_{n,j_2} = \emptyset$ for $j_1 \neq j_2$. Take arbitrary points $t_{n,j} \in E_{n,j}$, and put $f_{n,F_1} = \sum_{j=1}^{r_n} f(t_{n,j})\,\chi(E_{n,j})$. Clearly $\|f\,\chi(F_1) - f_{n,F_1}\|_T < 1/n$. Without loss of generality we may suppose that $F_2 \neq F_1$. Since again by Theorem I.8 $f\,\chi(F_2 - F_1) \in \bar{S}((F_2 - F_1) \cap \mathscr{P}, X)$, similarly as above we obtain a sequence f_{n,F_2-F_1}, $n = 1, 2, \ldots$ such that $f_{n,F_2-F_1}(F_2 - F_1) \subset f(F_2 - F_1) \subset f(T)$, and $\|f\,\chi(F_2 - F_1) - f_{n,F_2-F_1}\|_T < 1/n$ for each n. Continuing in this way we successively obtain a double sequence $f_{n,k} = f_{n,F_k-F_{k-1}}$, $n, k = 1, 2, \ldots$, $F_0 = \emptyset$. Now the sequence $f_n = \sum_{k=1}^{n} f_{n,k}$, $n = 1, 2, \ldots$ evidently has the required properties. The theorem is proved.

In Remark 3 in Part IV we explicitly noted that the Egoroff-Lusin Theorem, see Section 1.4 in Part I, remains valid also for submeasures in the sense of Definition 1 in [12]. Inspecting carefully the usual proof of the Egoroff-Lusin theorem we easily verify the validity of the assertions of the next remark.

Remark 1. The Egoroff-Lusin Theorem remains valid if $\mu: \sigma(\mathscr{P}) \to [0, +\infty]$ is a σ-finite countably additive measure, or if μ is a semimeasure in the sense of Definition 1 in [13], particularly if μ is a submeasure.

We use this facts in the proof of the following theorem.

Theorem 2. (Generalized Egoroff-Lusin Theorem.) *Let $\mu: \sigma(\mathscr{P}) \to [0, +\infty]$ be a σ-finite countably additive measure, or a semimeasure in the sense of Definition 1 in [13]. Further let $f_{n,k}: T \to X$, $n, k = 1, 2, \ldots$ be $\mathscr{P}$-measurable functions, and let $f_{n,k}(t) \to f_n(t) \in X$ as $k \to \infty$ for each $n = 1, 2, \ldots$ and each $t \in T$. Finally, put $F = \bigcup_{n,k=1}^{\infty} \{t \in T,\ f_{n,k}(t) \neq 0\} \in \sigma(\mathscr{P})$. Then there are sets $N \in \sigma(\mathscr{P})$ and $F_j \in \mathscr{P}$, $j = 1, 2, \ldots$ such that $\mu(N) = 0$, $F_j \nearrow F - N$, and on each set F_j, $j = 1, 2, \ldots$ the sequence $f_{n,k}$, $k = 1, 2, \ldots$ converges uniformly to the function f_n for each $n = 1, 2, \ldots$.*

Proof. Since $F \in \sigma(\mathscr{P})$, there are pairwise disjoint $E_r \in \mathscr{P}$, $r = 1, 2, \ldots$ such that $F = \bigcup_{r=1}^{\infty} E_r$. If μ is a measure we suppose without loss of generality that $\mu(E_r) < +\infty$ for each $r = 1, 2, \ldots$. Obviously it is enough to prove the theorem on each E_r, $r = 1, 2, \ldots$ (If we obtain the required $F_{r,j}$, $j = 1, 2, \ldots$, and N_r on E_r, then $F_j =$

$= \bigcup_{r=1}^{j} F_{r,j}$ and $N = \bigcup_{r=1}^{\infty} N_r$, $j = 1, 2, \ldots$ have the required properties.) Let r be fixed. By Remark 1 the Egoroff theorem holds for the convergences $f_{n,k}(t) \to f_n(t)$ as $k \to \infty$, $n = 1, 2, \ldots$, and for the restricted μ: $E_r \cap \sigma(\mathscr{P}) \to [0, +\infty]$. Consequently, for each given $\delta > 0$ and each $n = 1, 2, \ldots$ there is an $A_n \in E_r \cap \sigma(\mathscr{P}) = E_r \cap \mathscr{P}$ such that $\mu(E_r - A_n(\delta)) < \delta/2^n$ and the sequence $f_{n,k}$, $k = 1, 2, \ldots$ converges uniformly to the function f_n on $A_n(\delta)$. Put $A_{\delta,1} = \bigcap_{n=1}^{\infty} A_n(\delta)$. Then $A_{\delta,1} \in E_r \cap \mathscr{P}$, $\mu(E_r - A_{\delta,1}) < \delta$, and on $A_{\delta,1}$ each sequence $\{f_{n,k}\}_{k=1}^{\infty}$, $n = 1, 2, \ldots$ converges uniformly to the function f_n. Similarly, replacing E_r by $E_r - A_{\delta,1}$ and δ by $\frac{1}{2}\delta$, we obtain a set $A_{\delta,2} \in (E_r - A_{\delta,1}) \cap \mathscr{P}$ such that $\mu((E_r - A_{\delta,1}) - A_{\delta,2}) < \frac{1}{2}\delta$ and on $A_{\delta,2}$ each sequence $\{f_{n,k}\}_{k=1}$, $n = 1, 2, \ldots$ converges uniformly to the function f_n. Continuing in this way we obtain a sequence of sets $A_{\delta,s} \in E_r \cap \mathscr{P}$, $s = 1, 2, \ldots$ with the corresponding properties. Now clearly $N_r = E_r - \bigcap_{s=1}^{\infty} A_{\delta,s}$ and $F_{r,j} = \bigcup_{s=1}^{j} A_{\delta,s}$ have the required properties. The theorem is proved.

Using Theorem I.8 we immediately obtain

Corollary. *Let $\mu: \sigma(\mathscr{P}) \to [0, +\infty]$ be as in the theorem and let $f_n: T \to X$, $n = 1, 2, \ldots$ be $\mathscr{P}$-measurable functions. Then there is a set $N \in \sigma(\mathscr{P})$ such that $\mu(N) = 0$ and for each $n = 1, 2, \ldots$ the subset $f_n(T - N) \subset X$ is relatively σ-compact.*

We will also need the following simple consequence of Theorem VIII.9:

Lemma 1. *Let $\gamma: \mathsf{X}\sigma(\mathscr{P}_i) \to Y$ be a uniform vector d-polymeasure, let $A_{i,n} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$, and let $A_{i,n} \to A_i$ for each $i = 1, \ldots, d$. Then*

$$\lim_{n_1, \ldots, n_d \to \infty} \gamma(A_{i,n_i} \cap B_i) = \gamma(A_i \cap B_i)$$

uniformly with respect to $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and

$$\lim_{n_1, \ldots, n_d \to \infty} \bar{\gamma}(A_{i,n_i} \cap B_i) = \bar{\gamma}(A_i \cap B_i)$$

uniformly with respect to $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. For $i = 1, \ldots, d$ put $T_{d+i} = T_i$, $\mathscr{P}_{d+i} = \mathscr{P}_i$, and for $(A_1, \ldots, A_d, B_1, \ldots, B_d) \in \mathscr{P}_1 \times \ldots \times \mathscr{P}_{2d}$ put $\gamma'(A_1, \ldots, A_d, B_1, \ldots, B_d) = \gamma(A_1 \cap B_1, \ldots, A_d \cap B_d)$. Then $\gamma': \mathscr{P}_1 \times \ldots \times \mathscr{P}_{2d} \to Y$ is a uniform vector $2d$-polymeasure, and thus the assertions of the lemma are immediate consequences of assertions 2 and 3 of Theorem VIII.9.

2. FURTHER PROPERTIES OF THE INTEGRAL WITH RESPECT TO THE OPERATOR VALUED d-POLYMEASURE

The following theorem demonstrates the importance of the existence of a control d-polymeasure for a vector d-polymeasure, see Section 3 in Part VIII, for our approach to integration with respect to the operator valued d-polymeasure.

Theorem 3. *Let $\Gamma(\dots)(x_i)\colon \mathsf{X}\mathscr{P}_i \to Y$ have locally control d-polymeasures for each $(x_i) \in \mathsf{X}X_i$, see Section 3 in Part VIII, let $f_i\colon T_i \to X_i$, $i = 1, \dots, d$, be $\mathscr{P}_i$-measurable functions and let $f_{i,n} \in S(\mathscr{P}_i, X_i)$, $i = 1, \dots, d$, $n = 1, 2, \dots$ be such that $f_{i,n}(t_i) \to f_i(t_i)$ as $n \to \infty$ for each $i = 1, \dots, d$ and each $t_i \in T_i$. Further let $X_i' \subset X_i$, $i = 1, \dots, d$, be closed separable linear subspaces such that $f_{i,n}(T_i) \subset X_i'$ for each $i = 1, \dots, d$ and each $n = 1, 2, \dots$. Finally, put $F_i = \bigcup_{n=1}^{\infty} \{t_i \in T_i, f_{i,n}(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$ for $i = 1, \dots, d$. Then there are sets $N_i \in \sigma(\mathscr{P}_i) \cap F_i$ and $F_{i,k} \in \mathscr{P}_i$, $i = 1, \dots, d$, $k = 1, 2, \dots$ such that:*

(i) *$\hat{\Gamma}(F_{i,k}) < +\infty$, and $\|f_i\|_{F_{i,k}} \leqq k$ for each $k = 1, 2, \dots$,*
(ii) *$F_{i,k} \nearrow F_i - N_i$ for each $i = 1, \dots, d$,*
(iii) *on each fixed $F_{i,k}$ the sequence $f_{i,n}$, $n = 1, 2, \dots$ converges uniformly to the function f_i, and*
(iv) *for any $\mathscr{P}_i$-measurable functions $g_i\colon T_i \to X_i'$, $i = 1, \dots, d$ we have $(g_1\, \chi(N_1), g_2, \dots, g_d)$, $(g_1, g_2\, \chi(N_2), g_3, \dots, g_d)$, $\dots$, $(g_1, \dots, g_{d-1}, g_d\, \chi(N_d)) \in \mathscr{I}_1(\Gamma)$, and $\int_{(A_i)} (g_1\, \chi(N_1), g_2, \dots, g_d)\, \mathrm{d}\Gamma = \int_{(A_i)} (g_1, g_2\, \chi(N_2), g_3, \dots, g_d)\, \mathrm{d}\Gamma = \dots = \int_{(A_i)} (g_1, \dots, g_{d-1}, g_d\, \chi(N_d))\, \mathrm{d}\Gamma = 0$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.*

If $(f_i) \in \mathscr{I}(\Gamma)$, then there is a subsequence $\{n_k\} \subset \{n\}$ such that $f_{i,k}'(t_i) = f_{i,n_k}(t_i)\, \chi(F_{i,k} \cup N_i)(t_i) \to f_i(t_i)$ as $k \to \infty$ for each $i = 1, \dots, d$ and each $t_i \in T_i$, $f_{i,k}' \in S(\mathscr{P}_i, X_i')$ for each i and k considered, and

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \lim_{k \to \infty} \int_{(A_i)} (f_{i,k}')\, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Hence $(f_i) \in \mathscr{I}_1(\Gamma)$. Thus $\mathscr{I}(\Gamma) = \mathscr{I}_1(\Gamma)$ under the above given assumption on Γ. If, moreover, the semivariation $\hat{\Gamma}$ is bounded on $\mathsf{X}\mathscr{P}_i$ and each f_i, $i = 1, \dots, d$, is a bounded function, then we can take the functions $f_{i,k}'$ above so that

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \lim_{k_1, \dots, k_d \to \infty} \int_{(A_i)} (f_{i,k}')\, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Let $\Gamma' = \Gamma\colon \mathsf{X}(F_i \cap \mathscr{P}_i) \to L^d(X_i'; Y)$. Obviously we may replace Γ by Γ' in the theorem. According to Theorems 15, 17 and 19 from Part VIII the supremation $\bar{\Gamma}'\colon \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty]$ has a control d-polymeasure, say $\lambda_1 \times \dots \times \lambda_d\colon \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty)$. Applying the Egoroff-Lusin theorem, see Section 1.4 in Part I, coordinate-wise for $i = 1, \dots, d$ and using the σ-finiteness of the semivariation $\hat{\Gamma}'\colon \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty]$ and the fact that $\{t_i \in F_i, |f_i(t_i)| \leqq k\} \nearrow F_i$ as $k \to \infty$ for each $i = 1, \dots, d$, we easily obtain the assertions (i)–(iv) of the theorem.

Now let $(f_i) \in \mathscr{I}(\Gamma')$, and for $(A_i) \in \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i))$ put

$$\gamma(A_i) = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma' = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\,.$$

Then $\gamma\colon \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to Y$ is a vector d-polymeasure $(0 - 0)$ absolutely continuous with respect to the supremation $\bar{\Gamma}'$, see assertion 4 of Theorem IX.3. Hence

$$(1) \qquad \gamma(A_i) = \gamma(A_i - N_i) = \lim_{k_1, \dots, k_d \to \infty} \gamma(A_i \cap F_{i,k_i})$$

by Theorem VIII.1. Clearly (i) and (iii) imply: a) for each $k = 1, 2, \ldots$ there is an $n'_k > k$ such that $\|f_{i,n}\|_{F_{i,k}} \leqq 2k$ for each $n \geqq n'_k$ and each $i = 1, \ldots, d$, and b) for each $k = 1, 2, \ldots$ there is an $n_k \geqq n'_k$ such that

$$(2) \qquad (\|f_1 - f_{1,n}\|_{F_{1,k}} + \ldots + \|f_d - f_{d,n}\|_{F_{d,k}})\,(2k)^{d-1}\,\hat{\Gamma}(F_{i,k}) < 1/k$$

for each $n \geqq n_k$. Evidently we may suppose that $n_{k+1} > n_k$ for each $k = 1, 2, \ldots$. Now, using (1), assertion 1 of Theorem IX.3, (2) and (iv) we easily verify that the subsequence $\{n_k\} \subset \{n\}$ has the required properties.

For the last assertion of the theorem, if $c = \max_{1 \leqq i \leqq d} \|f_i\|_{F_i} < +\infty$, then we take a subsequence $\{n_k\} \subset \{n\}$ such that

$$(\|f_1 - f_{1,n}\|_{F_{1,k}} + \ldots + \|f_d - f_{d,n}\|_{F_{d,k}})\,(2c)^{d-1}\,\hat{\Gamma}'(F_i) < 1/k$$

for $n \geqq n_k$. Similarly as above we verify that $\{n_k\}$ has the required properties. The theorem is proved.

Corollary 1. *Let $\Gamma(\ldots)(x_i)\colon \mathsf{X}\mathscr{P}_i \to Y$ have locally control d-polymeasures for each $(x_i) \in \mathsf{X}X_i$ and let $(f_i) \in \mathscr{I} = \mathscr{I}_1$. Then the indefinite integral γ of (f_i), $\gamma(A_i) = \int_{(A_i)} (f_i)\,\mathrm{d}\Gamma$, $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, has a control d-polymeasure.*

Using Theorem VIII.11 for the general Γ we immediately have the following weaker result.

Corollary 2. *Let $(f_i) \in \mathscr{I}(\Gamma)$, let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and let $(f_{i,n}) \in \mathscr{I}_0 = \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ be such that $f_{i,n} \to f_i$ for each $i = 1, \ldots, d$. Then for any countably generated δ-subrings $\mathscr{P}'_i \subset \mathscr{P}_i$, $i = 1, \ldots, d$, such that $(A_i) \in \mathsf{X}\sigma(\mathscr{P}'_i)$ and $(f_{i,n}) \in \mathsf{X}S(\mathscr{P}'_i, X_i)$, $n = 1, 2, \ldots$ (they always exist) there are $(F'_{i,k}) \in \mathsf{X}\sigma(\mathscr{P}'_i)$, $k = 1, 2, \ldots$, and a subsequence $\{n_k\} \subset \{n\}$ such that $f'_{i,k} = f_{i,n_k}\,\chi(F'_{i,k}) \to f_i$ $(f'_{i,k} \in S(\mathscr{P}'_i, X_i))$ for each $i = 1, \ldots, d$, and*

$$\int_{(A_{i'})} (f_i)\,\mathrm{d}\Gamma = \lim_{k \to \infty} \int_{(A_{i'})} (f'_{i,k})\,\mathrm{d}\Gamma$$

for each $(A'_i) \in \mathsf{X}\sigma(\mathscr{P}'_i)$, particularly for $(A'_i) = (A_i)$.

If, moreover, each f_i, $i = 1, \ldots, d$, is a bounded function and $\hat{\Gamma}(T_i) < +\infty$, then we can take such $(f'_{i,k})$, $k = 1, 2, \ldots$ that

$$\int_{(A_{i'})} (f_i)\,\mathrm{d}\Gamma = \lim_{k_1, \ldots, k_d \to \infty} \int_{(A_{i'})} (f'_{i,k})\,\mathrm{d}\Gamma$$

for each $(A'_i) \in \mathsf{X}\sigma(\mathscr{P}'_i)$.

The next theorem is in a sense a generalization of the previous one. For its proof the Generalized Egoroff-Lusin Theorem is needed, i.e., Theorem 2.

Theorem 4. *Let Γ have a control d-polymeasure $\lambda_1 \times \ldots \times \lambda_d\colon \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty)$ and let $(f_i) \in \mathscr{I} = \mathscr{I}_1$. Further, for each $i = 1, \ldots, d$ let $f_{i,n}\colon T_i \to X_i$, $n = 1, 2, \ldots$ be $\mathscr{P}_i$-measurable functions and let $f_{i,n}(t_i) \to f_i(t_i)$ for λ_i-almost every $t_i \in T_i$. Then there are $(F_{i,k}) \in \mathsf{X}\mathscr{P}_i$, $k = 1, 2, \ldots$ with $\hat{\Gamma}(F_{i,k}) < +\infty$ for each k, and a subsequence $\{n_k\} \subset \{n\}$ such that $(f_{i,n_k}\,\chi(F_{i,k})) \in \mathsf{X}\bar{S}(F_{i,k} \cap \mathscr{P}_i, X_i) \subset \mathscr{I}_1 = \mathscr{I}$ for each $k = 1, 2, \ldots, f'_{i,k}(t_i) = f_{i,n_k}(t_i)\,\chi(F_{i,k})\,(t_i) \to f_i(t_i)$ for λ_i-almost every $t_i \in T_i$,*

$i = 1, \ldots, d$, *and*

$$\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma = \lim_{k \to \infty} \int_{(A_i)} (f'_{i,k}) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. *If, moreover, each* f_i, $i = 1, \ldots, d$, *is a bounded function and* $\hat{\Gamma}(T_i) < +\infty$, *then we can take such* $(f'_{i,k})$ *that*

$$\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma = \lim_{k_1, \ldots, k_d \to \infty} \int_{(A_i)} (f'_{i,k}) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Applying the Egoroff-Lusin Theorem, see Section 1.4 in Part I, to the convergences $f_{i,n} \to f_i$ a.e. λ_i, $i = 1, \ldots, d$ and the measures λ_i we obtain the corresponding sets $F'_{i,k} \in \mathscr{P}_i$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$. For each couple (i, n), $i = 1, \ldots, d$, $n = 1, 2, \ldots$, take a sequence $f_{i,n,j} \in S(P_i, X_i)$, $j = 1, 2, \ldots$ such that $f_{i,n,j} \to f_{i,n}$ as $j \to \infty$. Applying Theorem 2, i.e., the Generalized Egoroff-Lusin Theorem, we obtain sets $F''_{i,k} \in \mathscr{P}_i$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ with the corresponding properties. To prove the theorem, put $F_{i,k} = F'_{i,k} \cap F''_{i,k}$ and proceed as in the proof of Theorem 3 above.

Similarly as Theorem 3, using Theorem 1 one can prove

Theorem 5. *Let* $(f_i) \in \mathscr{I} = \mathscr{I}(\Gamma)$, *and let each* $f_i(T_i) \subset X_i$, $i = 1. \ldots, d$, *be relativesy* σ-*compact. Then* $(f_i) \in \mathscr{I}_1$. *If, moreover, each* f_i, $i = 1, \ldots, d$, *is a bounded function and* $\hat{\Gamma}(T_i) < +\infty$, *then there are* $f_{i,n} \in S(\mathscr{P}_i, X_i)$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ *luch that* $f_{i,n} \to f_i$ *for each* $i = 1, \ldots, d$ *and*

$$\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma = \lim_{n_1, \ldots, n_d \to \infty} \int_{(A_i)} (f_{i,n_i}) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Corollary. *Let each* X_i, $i = 1, \ldots, d$, *be a finite dimensional Banach space. Then* $\mathscr{I} = \mathscr{I}_1$.

For any d-tuple (f_i) of functions $f_i\colon T_i \to X_i$, $i = 1, \ldots, d$ we put

$$\Gamma_\Sigma(f_i) = \{ \sum_{j_1=1}^{r_1} \ldots \sum_{j_d=1}^{r_d} \Gamma(A_{i,j_i}) (f_i(t_{i,j_i})), \; A_{i,j_i} \in \mathscr{P}_i, \; t_{i,j_i} \in A_{i,j_i},$$

$$\text{for fixed } i \text{ the sets } A_{i,j_i}, \; j_i = 1, \ldots, r_i \text{ are pairwise disjoint,}$$

$$r_1, \ldots, r_d = 1, 2, \ldots \} .$$

By $\bar{\Gamma}_\Sigma(f_i)$ we denote the closure of $\Gamma_\Sigma(f_i)$ in Y.

We are now ready to prove

Theorem 6. *Let* $(f_i) \in \mathscr{I}(\Gamma)$. *Then*

$$R(I(f_i)) = \{\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma, \; (A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)\} \subset \bar{\Gamma}_\Sigma(f_i) .$$

Proof. Let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and let $\varepsilon > 0$. For each $i = 1, \ldots, d$ take a sequence $g_{i,n} \in S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $g_{i,n} \to f_i$. Since $\gamma(\cdot, A_2, \ldots, A_d)\colon A_1 \cap \sigma(\mathscr{P}_1) \to Y$, where $\gamma(A_i) = \int_{(A_i)} (f_i) \, \mathrm{d}\Gamma$, is a countably additive vector measure, it has a control measure, say $\lambda_1\colon A_1 \cap \sigma(\mathscr{P}_1) \to [0, +\infty)$. Applying the Egoroff-Lusin

Theorem to the convergence $g_{1,n} \to f_1$, in accordance with Theorem I.8 or Theorem 1 we obtain a set $N_1 \in A_1 \cap \sigma(\mathscr{P}_1)$ such that $\lambda_1(N_1) = 0$ and the range of the function $f_1\, \chi(A_1 - N_1)$ is relatively σ-compact. Clearly $\gamma(A_1, \ldots, A_d) = \gamma(A_1 - N_1, A_2, \ldots, A_d)$. Repeating the above consideration for the convergence $g_{2,n} \to f_2$ and for a control measure of the vector measure $\gamma(A_1 - N_1, A_3, \ldots, A_d)\colon A_2 \cap \sigma(\mathscr{P}_2) \to Y$ we obtain a set $N_2 \in \sigma(\mathscr{P}_2)$ such that the function $f_2\, \chi(A_2 - N_2)$ has a relatively σ-compact range and $\gamma(A_i) = \gamma(A_1 - N_1,\ A_2 - N_2,\ A_3, \ldots, A_d)$. Continuing in this way we obtain sets $N_i \in A_i \cap \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, such that $\gamma(A_i) = \gamma(A_i - N_i)$ and the range of each function $f_i\, \chi(A_i - N_i)$, $i = 1, \ldots, d$, is relatively σ-compact.

Now by Theorem 1 there are sets $F_{i,k} \in \mathscr{P}_i$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ and for each i a sequence $f_{i,n} \in S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $F_{i,k} \nearrow A_i - N_i$ as $k \to \infty$ $f_{i,n}(A_i - N_i) \subset f_i(A_i - N_i)$ for each $n = 1, 2, \ldots$, and on each $F_{i,k}$, $k = 1, 2, \ldots$, the sequence $f_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to the function f_i. According to Theorem VIII.1 there is a k_0 such that

$$\left|\gamma(A_i - N_i) - \gamma(F_{i,k_i})\right| < \tfrac{1}{2}\varepsilon$$

for each $k_1, \ldots, k_d \geqq k_0$. Take $k_1 = k_2 = \ldots = k_d = k_0$. Clearly $c = \sup_{i,n} \|f_{i,n}\|_{F_{i,k0}} < +\infty$ (each $f_{i,n}$ is a simple function and $f_{i,n} \to f_i$ uniformly on F_{i,k_0} for each $i = 1, \ldots, d$), hence $\sup_i \|f_i\|_{F_{i,k0}} < c + 1$. Now

$$\left|\int_{(F_{i,k0})} (f_i)\, \mathrm{d}\Gamma - \int_{(F_{i,k0})} (f_{i,n})\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq d \sup_i \|f_i - f_{i,n}\|_{F_{i,k0}} (c + 1)^{d-1}\, \hat{\Gamma}(F_{i,k_0}) \to 0$$

as $n \to \infty$ by the uniform convergence $f_{i,n} \to f_i$ on F_{i,k_0} (use Theorem IX.3). Hence $\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma \in \bar{\Gamma}_\Sigma(f_i)$. Since $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ was arbitrary, the theorem is proved.

Since $\mathsf{X}\mathscr{P}_i$ is a multiplicative system of sets, we can use the isometric isomorphism between $L^d(X_i; Y)$ and $L(X_1 \otimes^\wedge \ldots \otimes^\wedge X_d, Y)$, see the beginning of Part VIII, to define the S-integral of Kolmogoroff with respect to our polymeasure Γ in the usual way, see [19] and Part VI. However, to the theory of integration with respect to the polymeasure Γ just developed there corresponds a seemingly weaker (it is an open problem whether really weaker) integral, which we call the *product* $\int$ – $\mathsf{X}S$-*integral* and which we now introduce.

Let $(A_i) \in \mathsf{X}\mathscr{P}_i$. By a finite product partition of (A_i) we mean a partition of (A_i) of the form $\pi_1(A_1) \times \ldots \times \pi_d(A_d)$, shortly $\mathsf{X}\pi_i(A_i)$, where $\pi_i(A_i)$, $i = 1, \ldots, d$, is a finite $\mathscr{P}_i$-partition of A_i. If $\mathsf{X}\pi_{i,1}(A_i)$ and $\mathsf{X}\pi_{i,2}(A_i)$ are two finite product partitions of (A_i), then we write $\mathsf{X}\pi_{i,1}(A_i) \leqq \mathsf{X}\pi_{i,2}(A_i)$ if and only if $\pi_{i,1}(A_i) \leqq \pi_{i,2}(A_i)$ for each $i = 1, \ldots, d$, i.e., if $\pi_{i,2}(A_i)$ is a refinement of $\pi_{i;1}(A_i)$ for each i. It is evident that the set $\mathsf{X}\Pi_i(A_i)$ of all finite product partitions of (A_i) is a directed cofinal subset of the set $\Pi(A_i)$ of all finite $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$-partitions of the rectangle $A_1 \times \ldots \times A_d$.

Let us have functions $f_i\colon T_i \to X_i$, $i = 1, \ldots, d$. If $\pi_i(A_i) = (A_{i,j_i})_{j_i=1}^{n_i}$ is a partition of A_i, $i = 1, \ldots, d$, choose points $t_{i,j_i} \in A_{i,j_i}$, $i = 1, \ldots, d$, $j_i = 1, \ldots, n_i$, and, in accordance with the beginning of Section 2 in Part VI, write

$$S_{\mathsf{X}\pi_i(A_i)}(\Gamma, (f_i), (t_{i,j_i})) = \sum_{j_1=1}^{n_1} \ldots \sum_{j_d=1}^{n_d} \Gamma(A_{i,j_i}) (f_i(t_{i,j_i})) .$$

If the net $S_{\mathsf{X}\pi_i(A_i)}(\Gamma, (f_i), (\cdot))$, $\mathsf{X}\pi_i(A_i) \in \mathsf{X}\Pi_i(A_i)$ on Y converges to an element $y \in Y$, we say that the d-tuple of functions (f_i) is product S-integrable, or $\mathsf{X}S$-integrable, on (A_i), and write $\mathsf{X}S_{(A_i)}(f_i)\,\mathrm{d}\Gamma = y$.

For each S-integrable function $f_1 \otimes^{\wedge} \ldots \otimes^{\wedge} f_d \colon \mathsf{X}T_i \to X_1 \otimes^{\wedge} \ldots \otimes^{\wedge} X_d$ on the rectangle $\mathsf{X}A_i$, $A_i \in \mathscr{P}_i$, $i = 1, \ldots, d$, see the beginning of Section 2 in Part VI and the beginning of Section 1 in Part VIII, the d-tuple (f_i) is clearly $\mathsf{X}S$-integrable on (A_i). It is an interesting open problem when the converse is true.

Evidently the $\mathsf{X}S$-integral shares coordinatewise the simple properties of the S-integral which are listed before Lemma 2 in Part VI. If the d-tuple of functions (f_i) is $\mathsf{X}S$-integrable on $(A_i) \in \mathsf{X}\mathscr{P}_i$, then clearly

$$|\mathsf{X}S_{(A_i)}(f_i)\,\mathrm{d}\Gamma| \leqq \|f_1\|_{A_1} \cdot \ldots \cdot \|f_d\|_{A_d} \hat{\Gamma}(A_i) .$$

Further, the following analog of Lemma 2 from Part VI obviously holds.

Lemma 2. *Let $(f_i) \in \mathsf{X}\bar{S}(\mathscr{P}_i, X_i)$, let $(A_i) \in \mathsf{X}\mathscr{P}_i$, and let $\hat{\Gamma}(A_i) < +\infty$. Then $(f_i\chi_{A_i})$ is integrable $(\in I_1)$ as well as $\mathsf{X}S$-integrable and the integrals coincide on each $(A_i) \in \mathsf{X}\mathscr{P}_i$.*

Using Theorem 1 and the ideas of the proofs of Theorems 4, 5 and 6 in Part VI, their generalizations can be easily proved in the following form:

Theorem 7. *Let $f_i \colon T_i \to X_i$, $i = 1, \ldots, d$, be bounded $\mathscr{P}_i$-measurable functions, and let the multiple L_1-gauge $\hat{\Gamma}[(f_i), (\cdot)] \colon \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ be separately continuous on $\mathsf{X}\sigma(\mathscr{P}_i)$, hence bounded by Theorem* VIII.6. *Then the d-tuple (f_i) is $\mathsf{X}S$-integrable on each $(A_i) \in \mathsf{X}\,\mathscr{P}_i$, it is integrable by Theorem* IX.7, *and*

$$\mathsf{X}S_{(A_i)}(f_i)\,\mathrm{d}\Gamma = \int_{(A_i)} (f_i)\,\mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\mathscr{P}_i$.

Theorem 8. *Let $f_i \colon T_i \to X_i$, $i = 1, \ldots, d$, be $\mathscr{P}_i$-measurable functions, let $(A_i) \in \mathsf{X}\mathscr{P}_i$, and let the d-tuple (f_i) be $\mathsf{X}S$-integrable on (A_i). Then $(f_i\chi_{A_i}) \in \mathscr{I}_1$ and*

$$\int_{(B_i)} (f_i)\,\mathrm{d}\Gamma = \mathsf{X}_{(B_i)}S(f_i)\,\mathrm{d}\Gamma$$

for each $(B_i) \in \mathsf{X}(A_i \cap \mathscr{P}_i)$.

Using Theorem 1, Corollary of Theorem 2 and Lemma 1 we easily obtain

Theorem 9. *Let $\Gamma(\cdot)(x_i) \colon \mathsf{X}\mathscr{P}_i \to Y$ have locally a control d-polymeasure for each $(x_i) \in \mathsf{X}X_i$, let $(f_i) \in \mathscr{I}$ $(= \mathscr{I}_1$ by Theorem 3), and let the indefinite integral $\int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma \colon \mathsf{X}\sigma(\mathscr{P}_i) \to Y$ be a uniform vector d-polymeasure, see Definition* VIII.1. *Then:*

1) *There is a sequence of d-tuples of functions $(f_{i,n}) \in \mathscr{I}_0 = \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$ pointwise as $n \to \infty$ for each $i =$*

$= 1, \dots, d$, *and*

$$\lim_{n\to\infty} \int_{(A_i)} (f_{i,n})\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$$

uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. *If each* f_i, $i = 1, \dots, d$, *is a bounded function and* $\hat{\Gamma}(T_i) < +\infty$, *then we can take such a sequence* $(f_{i,n}) \in \mathscr{I}_0$, $n = 1, 2, \dots$ *that*

$$\lim_{n_1,\dots,n_d\to\infty} \int_{(A_i)} (f_{i,n_i})\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$$

uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

2) *For each* $\varepsilon > 0$ *there are sets* $A_{i,j_i} \in \mathscr{P}_i$, $i = 1, \dots, d$, $j_i = 1, \dots, n_i < \infty$ *and points* $t_{i,j_i} \in A_{i,j_i}$ *such that* A_{i,j_i}, $j_i = 1, \dots, n_i$ *are pairwise disjoint for each fixed* $i \in \{1, \dots, d\}$, *and*

$$\left|\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma - \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} \Gamma(A_i \cap A_{i,j_i}) f(t_{i,j_i})\right| < \varepsilon$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. 1) Take $f'_{i,j} \in S(\mathscr{P}_i, X_i)$, $i = 1, \dots, d$, $j = 1, 2, \dots$ such that $f'_{i,j} \to f_i$ and $|f'_{i,j}| \nearrow |f_i|$ pointwise as $j \to \infty$ for each $i = 1, \dots, d$. Put $X'_i = \overline{\mathrm{sp}}\,\{\bigcup_{j=1}^{\infty} f'_{i,j}(T_i)\}$ and $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\}$ for i as above. Then for our consideration we may replace X_i by X'_i. Since each X'_i is a separable closed subspace, by Theorems VIII.17 and VIII.15 we conclude that $\Gamma\colon \mathsf{X}(F_i \cap \mathscr{P}_i) \to L^d(X'_i; Y)$ has a control d-polymeasure, say $\lambda_1 \times \dots \times \lambda_d\colon \mathsf{X}(F_i \cap \mathscr{P}_i) \to [0, +\infty)$. Applying the Egoroff-Lusin Theorem in each coordinate i we obtain sets $N_i \in F_i \cap \sigma(\mathscr{P}_i)$ and $F_{i,k} \in F_i \cap \mathscr{P}_i$, $i = 1, \dots, d$, $k = 1, 2, \dots$ such that $\lambda_i(N_i) = 0$, $F_{i,k} \nearrow F_i - N_i$ as $k \to \infty$, and on each fixed $F_{i,k}$ the sequence $f'_{i,j}$, $j = 1, 2, \dots$ converges uniformly to the function f_i. Without loss of generality we may suppose that $\hat{\Gamma}(F_{i,k}) < +\infty$ for each $k = 1, 2, \dots$. For $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ put $\gamma(A_i) = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$. Since by assumption $\gamma\colon \mathsf{X}\sigma(\mathscr{P}_i) \to Y$ is a uniform vector d-polymeasure, we have

$$\lim_{k_1,\dots,k_d\to\infty} \gamma(A_i \cap (F_{i,k_i} - N_i)) = \gamma(A_i \cap (F_i - N_i)) = \gamma(A_i)$$

uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ by Lemma 1. Hence there is a subsequence $\{k(n)\} \subset \{k\}$ such that

$$|\gamma(A_i) - \gamma(A_i \cap (F_{i,k_i} - N_i))| < \frac{1}{2n}$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $k_1, \dots, k_d \geqq k(n)$.

Since on each fixed $F_{i,k}$ the sequence $f'_{i,j}$, $j = 1, 2, \dots$ converges uniformly to the function f_i, and since $\hat{\Gamma}(F_{i,k}) < +\infty$, there is a subsequence $\{j_n\} \subset \{j\}$ such that

$$\sup_i \|f - f'_{i,j_n}\|\, F_{i,k(n)} < \frac{1}{n}$$

for each $n = 1, 2, \dots$, and

$$\left|\int_{(A_i)} (f_i\, \chi(F_{i,k(n)}))\, \mathrm{d}\Gamma - \int_{(A_i)} (f'_{i,j_n}\, \chi(F_{i,k(n)}))\, \mathrm{d}\Gamma\right| < \frac{1}{2n}$$

for each $(A_i) \in \sigma(\mathscr{P}_i)$ and each n. Now it is clear that

$$f_{i,n} = f'_{i,j_n} \chi(F_{i,k(n)} \cup N_i),$$

$i = 1, \ldots, d$, $n = 1, 2, \ldots$ have the required properties.

2) Is evident if one replaces the values of the simple functions $f'_{i,j_n} \chi(F_{i,k(n)})$ above by suitable $f(t_{i,j_i})$ as in the proof of Theorem 1.

From Theorem 4.4 in [20], see (Y) at the beginning of Part VIII, and from Corollary 2 of Theorem VIII.2 we immediately obtain

Corollary. *The assertions of the theorem are valid for any* $(f_i) \in \mathscr{I}$ *in the following cases:*

1) $d = 2$ *and* $Y = K$ – *the scalars,*
2) $T_i = N = \{1, 2, \ldots\}$ *and* $\mathscr{P}_i = \Phi_1$ = *the collection of all finite subsets of* N *for each* $i = 1, \ldots, d$.

Whether the assertions of the previous theorem are valid in some other cases is an open problem.

Without assuming the existence of local control d-polymeasures $\Gamma(\cdot)(x_i)$: $\mathsf{X}\mathscr{P}_i \to Y$ for each $(x_i) \in \mathsf{X}X_i$ we have the following result.

Theorem 10. *Let* $(f_i) \in \mathscr{I}$ *and let the indefinite integral* $\gamma(\cdot) = \int_{(\cdot)} (f_i) \, d\Gamma$: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$ *be a uniform vector d-polymeasure. Then there are sets* $N_i \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, *such that* $\bar{\gamma}(N_1, T_2, \ldots, T_d) + \ldots + \bar{\gamma}(T_1, \ldots, T_{d-1}, N_d) = 0$, *hence*

$$\int_{(A_i)} (f_i) \, d\Gamma = \int_{(A_i)} (f_i \, \chi(T_i - N_i)) \, d\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *and for the integrable d-tuple of functions* $(f_i \chi_{T_i - N_i})$ *the assertions* 1) *and* 2) *of Theorem* 10 *are valid.*

Proof. The suprenation $\bar{\gamma}$: $\mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty)$, see Definition VIII.2, is separately a subadditive submeasure in the sense of Definition 1 in [12], see Theorem VIII.7. Now we proceed as in the proof of Theorem 9 using either the subadditive submeasures $\bar{\gamma}(\cdot, F_2, \ldots, F_d), \ldots, \bar{\gamma}(F_1, \ldots, F_{d-1}, \cdot)$, see the paragraph before Theorem 2, or their control measures, see Theorem VIII.10.

Theorem 11. *Let* α *and* α_n, $n = 1, 2, \ldots$ *be countable ordinals such that* $\alpha > \alpha_n$ *for each* $n = 1, 2, \ldots$. *Let* $(f_{i,n}) \in \mathscr{I}_{\alpha_n}$ *for each* $n = 1, 2, \ldots$ *and let* $f_{i,n} \to f_i$ *pointwise for each* $i = 1, \ldots, d$. *For* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *put* $\gamma_n(A_i) = \int_{(A_i)} (f_{i,n}) \, d\Gamma$, $n = 1, 2, \ldots$, *and let the supremations* $\bar{\gamma}_n$: $\mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty)$, $n = 1, 2, \ldots$ *be separately uniformly exhaustive (equivalently, continuous). Then* $(f_i) \in \mathscr{I}_\alpha$, *and*

$$\lim_{n \to \infty} \gamma_n(A_i) = \lim_{n \to \infty} \int_{(A_i)} (f_{i,n}) \, d\Gamma = \int_{(A_i)} (f_i) \, d\Gamma = \gamma(A_i)$$

and

$$\lim_{n \to \infty} \bar{\gamma}_n(A_i) = \bar{\gamma}(A_i)$$

both uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. *Hence* γ: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$ *is also a uniform vector d-polymeasure.*

Proof. $(f_i) \in \mathscr{I}_\alpha$ and $\lim_{n\to\infty} \gamma_n(A_i) = \gamma(A_i)$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ by Theorem IX.4-2). Since $\bar{\gamma}_n$: $\mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty)$, $n = 1, 2, \ldots$ are separately uniformly continuous, the set functions $\gamma_0 = \gamma$: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$ and Λ: $\mathsf{X}\sigma(P_i) \to c(Y)$,

$$\Lambda(A_i) = (\gamma_0(A_i), \gamma_1(A_i), \ldots, \gamma_n(A_i), \ldots) \in c(Y),$$

$(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, are uniform vector d-polymeasures.

For $i = 1, \ldots, d$ put $F_i = \bigcup_{n=1}^{\infty} \{t_i \in T_i,\ f_{i,n}(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, and let $\mu_1(B_1) = \bar{\Lambda}(B_1, F_2, \ldots, F_d), \ldots, \mu_d(B_d) = \bar{\Lambda}(F_1, \ldots, F_d, B_d)$ for $B_i \in F_i \cap \sigma(\mathscr{P}_i)$. Then μ_i: $F_i \cap \sigma(\mathscr{P}_i) \to [0, +\infty)$, $i = 1, \ldots, d$, are subadditive submeasures in the sense of Definition 1 in [12]. Hence by the Egoroff-Lusin Theorem, see the paragraph before Theorem 2, there are sets $N_i \in F_i \cap \sigma(\mathscr{P}_i)$ and $F_{i,k} \in F_i \cap \mathscr{P}_i$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ such that $\mu_i(N_i) = 0$, $F_{i,k} \nearrow F_i - N_i$ as $k \to \infty$ for each $i = 1, \ldots, d$, and on each fixed $F_{i,k}$ the sequence $f_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to the function f_i. Since by assumption the semivariation $\hat{\Gamma}$: $\mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty]$ is σ-finite, we may suppose that $\hat{\Gamma}(F_{i,k}) < +\infty$ for each $k = 1, 2, \ldots$. Let k be fixed. Then evidently $\lim_{n\to\infty} \gamma_n(F_{i,k} \cap A_i) = \gamma(F_{i,k} \cap A_i)$ and $\lim_{n\to\infty} \bar{\gamma}_n(F_{i,k} \cap A_i) = \bar{\gamma}(F_{i,k} \cap A_i)$ both uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Since by Lemma 1 $\lim_{k\to\infty} \Lambda(A_i \cap F_{i,k}) = \Lambda(A_i \cap (F_i - N_i)) = \Lambda(A_i)$ and $\lim_{k\to\infty} \bar{\Lambda}(A_i \cap F_{i,k}) = \bar{\Lambda}(A_i \cap (F_i - N_i)) = \bar{\Lambda}(A_i)$ both uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, the theorem is proved.

Before the next theorem let us recall that the Banach space $L^d(X_i; Y)$ is isometrically isomorphic to the Banach space $L(X_1 \otimes^\wedge \ldots \otimes^\wedge X_d, Y)$, see the beginning of Part VIII. Using this identification we have

Theorem 12. *Let Γ^*: $\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_d \to L(X_1 \otimes^\wedge \ldots \otimes^\wedge X_d; Y)$ be an operator valued measure countably additive in the strong operator topology with a locally σ-finite semivariation $\hat{\Gamma}^*$ on $\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_d$. Further let f_i: $T_i \to X_i$, $i = 1, \ldots, d$ be P_i-measurable functions, and let the function $f_1 \otimes^\wedge \ldots \otimes^\wedge f_d$: $\mathsf{X}T_i \to X_1 \otimes^\wedge \ldots \otimes^\wedge X_d$ be integrable with respect to Γ^*. Finally, let Γ: $\mathsf{X}\mathscr{P}_i \to L^d(X_i; Y)$ be the restriction of Γ^* to $\mathsf{X}\mathscr{P}_i$. Then*

1) *$\Gamma(\cdot)(x_i)$: $\mathsf{X}\mathscr{P}_i \to Y$ is a uniform vector d-polymeasure for each $(x_i) \in \mathsf{X}X_i$;*
2) *$\hat{\Gamma}(A_i) \leqq \hat{\Gamma}^*(A_i)$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$;*
3) *$(f_i) \in \mathscr{I} = \mathscr{I}_1$ and*

$$\gamma(A_i) = \int_{(A_i)} (f_i)\, d\Gamma = \int_{A_1 \times \ldots \times A_d} f_1 \otimes^\wedge \ldots \otimes^\wedge f_d\, d\Gamma^* = \gamma^*(A_1 \times \ldots \times A_d)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$;

4) *γ: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$ is a uniform vector d-polymeasure.*

Proof. 1) and 4) are immediate, since the d-polymeasures considered are restrictions of measures. 2) is obvious by the same argument.

3) Applying the first part of Theorem 3 and its proof to Γ and f_i, $i = 1, \ldots, d$ we easily obtain the assertions of 3) for the functions $f_i \chi_{F_i - N} N_i$, $i = 1, \ldots, d$ in the

notation of Theorem 3 and its proof. Under this notation it is evident that the vector measure $\gamma^*: (F_1 \cap \sigma(\mathscr{P}_1)) \times \ldots \times (F_d \cap \sigma(\mathscr{P}_d)) \to Y$ is absolutely continuous with respect to the measure $\lambda_1 \times \ldots \times \lambda_d: (F_1 \cap \sigma(\mathscr{P}_1)) \times \ldots \times (F_d \cap \sigma(\mathscr{P}_d)) \to \to [0, +\infty)$. Hence $\gamma^*(\mathsf{X}A_i) = \gamma^*(\mathsf{X}(A_i \cap (F_i - N_i))) = \gamma(A_i \cap (F_i - N_i)) = \gamma(A_i)$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. The theorem is proved.

The next theorem is a generalization of Theorem VII.4. In fact, it is in a sense a more precise version of Corollary 1 of Theorem IX.4.

Theorem 13. *Let $\Gamma(\cdot)(x_i): \mathsf{X}\mathscr{P}_i \to Y$ have locally control d-polymeasures for each $(x_i) \in \mathsf{X}X_i$ and let $(f_i) \in \mathscr{I}(\Gamma) = \mathscr{I}_1(\Gamma)$. Further let $\mathscr{P}'_i \subset \mathscr{P}_i$, $i = 1, \ldots, d$, be δ-subrings such that each f_i is $\mathscr{P}'_i$-measurable. Then $(f_i) \in \mathscr{I}(\Gamma') = \mathscr{I}_1(\Gamma')$, where $\Gamma' = \Gamma: \mathsf{X}\mathscr{P}'_i \to L^d(X_i; Y)$, and*

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma' = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}'_i)$.

The following theorem is a generalization of Theorem I.17.

Theorem 14. *Let $\Gamma(\cdot)(x_i): \mathsf{X}\mathscr{P}_i \to Y$ have locally control d-polymeasures for each $(x_i) \in \mathsf{X}X_i$ and let $c_0 \not\subset Y$, see [1] and [2]. Then $(f_i) \in \mathscr{I} = \mathscr{I}_1$ if and only if $(f_i) \in \mathscr{I}_1(y^*\Gamma) = \mathscr{I}(y^*\Gamma)$ for each $y^* \in Y^*$.*

Proof. The "only if" part is a consequence of Theorem IX.4 $-$ 3). We prove the sufficiency part using the idea of the proof of Theorem I.17. Let $(f_i) \in \mathscr{I}(y^*\Gamma)$ for each $y^* \in Y^*$, and let us adopt the notation from the proof of Theorem 3. Hence it is now sufficient to show that $(f_i\, \chi(F_i - N_i)) \in \mathscr{I}(\Gamma)$. First we deduce that $(f_1\, \chi(F_{1,k}), \ldots, f_{d-1}\, \chi(F_{d-1,k}), f_d\, \chi(F_d - N_d)) \in \mathscr{I}(\Gamma)$ for each $k = 1, 2, \ldots$. Let k be fixed. According to Theorem IX.4 $-$ 2) it is enough to verify that

$$\lim_{k_d \to \infty} \int_{(A_i)} (f_1\, \chi(F_{1,k}), \ldots, f_{d-1}\, \chi(F_{d,k}), f_d\, \chi(F_d, k_d))\, \mathrm{d}\Gamma \in Y$$

exists for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Suppose the contrary. Then there is an $\varepsilon > 0$, an $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and a subsequence $\{k_{d,j}\} \subset \{k_d\}$ such that $(\mathscr{A}_j = (A_1, \ldots, A_{d-1}, A_d \cap \cap (F_{d,k_{d,j+1}} - F_{d,k_{d,j}}))$

$$\begin{aligned} |y_j| &= \left|\int_{\mathscr{A}_j} (f_1\, \chi(F_{1,k}), \ldots, f_{d-1}\, \chi(F_{d-1}), f_d\, \chi(F_d - N_d))\, \mathrm{d}\Gamma\right| = \\ &= \left|\int_{(A_i)} (f_1\, \chi(F_{1,k}), \ldots, f_{d-1}\, \chi(F_{d-1,k}), f_d\, \chi(F_{d,k_{d,j+1}}))\, \mathrm{d}\Gamma - \right. \\ &\left. - \int_{(A_i)} (f_1\, \chi(F_{1,k}), \ldots, f_{d-1}\, \mathrm{x}(F_{d-1,k}), f_d\, \chi(F_{d,k_{d,j}}))\, \mathrm{d}\Gamma\right| > \varepsilon \end{aligned}$$

for each $j = 1, 2, \ldots$. But this is impossible, since owing to the assumption $(f_i) \in \in \mathscr{I}(y^*\Gamma)$ for each $y^* \in Y^{*k}$ the series $\sum_{j=1}^{\infty} y_j$ is weakly, hence by [1] or [2] also strongly ($c_0 \not\subset Y$) unconditionally convergent. Thus $(f_1\, \chi(F_{1,k}), \ldots, f_{d-1}\, \chi(F_{d-1,k}), f_d\, \chi(F_d - N_d)) \in \mathscr{I}(\Gamma)$ for each $k = 1, 2, \ldots$. Starting with this integrable d-tuple of functions in the same way as above we obtain that also $(f_1\, \chi(F_{1,k}), \ldots \ldots, f_{d-2}\, \chi(F_{d-2,k}), f_{d-1}\, \chi(F_{d-1} - N_{d-1}), f_d\, \chi(F_d - N_d)) \in \mathscr{I}(\Gamma)$ for each $k = 1, 2, \ldots$.

Continuing in this way we finally obtain that $(f_i\, \chi(F_i - N_i)) \in \mathscr{I}(\Gamma)$, which we wanted to show. The theorem is proved.

Our final theorem in this part is a generalization of Theorem 3 in [14]. Its validity is clear from the preceding proof.

Theorem 15. *Let $\Gamma(\cdot)(x_i)\colon \mathsf{X}\mathscr{P}_i \to Y$ have locally control d-polymeasures for each $(x_i) \in \mathsf{X}X_i$. Then $(f_i) \in \mathscr{I} = \mathscr{I}_1$ if and only if $(f_i) \in \mathscr{I}(y^*\Gamma)$ for each $y^* \in Y^*$ and the indefinite integrals $\{\int_{(\cdot)} (f_i)\, \mathrm{d}(y^*\Gamma)\colon \mathsf{X}\sigma(\mathscr{P}_i) \to K\text{-scalars},\ y^* \in Y^*,\ |y^*| \leqq 1\}$ are separately uniformly countably additive.*

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