Bohdan Zelinka
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A REMARK ON SIGNED POSETS AND SIGNED GRAPHS

BOHDAN ZELINKA, Liberec

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In [1] signed posets and signed graphs corresponding to them are studied. The concept of a signed partially ordered set (shortly poset) is defined by means of the Möbius function on a poset.

Let P be a finite poset. The Möbius function μ on P is a mapping of $P \times P$ into the set of integers defined in such a way that $\mu(x, y) = 1$ for $x = y$, $\mu(x, y) = -\sum_{x \leqq z < y} \mu(x, z)$ for $x < y$ and $\mu(x, y) = 0$ otherwise. A poset P is called signed, if the Möbius function on P attains only the values $-1, 0$ and 1.

A signed graph is an undirected graph together with a mapping of its edge set into the set $\{1, -1\}$. In other words, a graph is signed, if its edge set is partitioned into two disjoint sets; the set of positive edges and the set of negative edges.

To a signed poset P a signed graph $S(P)$ is assigned in such a way that the vertex set of $S(P)$ is P and two distinet vertices x, y are joined by a positive (or negative) edge if and only if $\mu(x, y) = 1$ (or $\mu(x, y) = -1$, respectively).

In [1] a problem is posed, for which signed graphs S do we have $S(P) = S$ for some poset P. We shall consider a particular case when all edges of $S(P)$ are negative. Some results concerning this case are given in Theorem 1 in [1]. If all edges of $S(P)$ are negative, then $S(P)$ contains no triangle, is isomorphic to the Hasse diagram of P, and each interval $[x, y]$ for $x \leqq y$ in P is a chain. The condition that S contains no triangle is not sufficient for S to be $S(P)$ for some P.

Before formulating a theorem, we introduce some concepts concerning trees.

A rooted tree is an ordered pair (T, r), where T is a tree and r is one of its vertices, called the root. (It may be chosen arbitrarily.) If a rooted tree (T, r) is given and v is a vertex of T, then the subtree of (T, r) rooted at v is the rooted tree (T', v), where T' is the subtree of T whose vertex set is the set of all vertices x or T with the property that v lies on the path connecting r and x in T.

Now we can prove a theorem.

Theorem. *Let S be a finite undirected signed graph, all of whose edges are negative. Then the following two assertions are equivalent:*

(i) *There exists a signed poset P such that $S \cong S(P)$.*

(ii) *There exist two-empty disjoint subsets X, Y of V(S) and a system $\mathcal{T}$ of subtrees of S with the following properties:*

(a) *The graph S is the union of all trees of $\mathcal{T}$.*

(b) *There exists a one-to-one correspondence between the elements $x \in X$ and trees $T(x) \in \mathcal{T}$ such that $x \in V(T(x))$ and $x \notin \bigcup_{y \in X - \{x\}} V(T(y))$ for all $x \in X$.*

(c) *All terminal vertices of all trees of $\mathcal{T}$ are in Y.*

(d) *If two trees $T(x_1)$, $T(x_2)$ from $\mathcal{T}$ have a common vertex v, then there exists a tree T_0 such that (T_0, v) is a subtree of both $(T(x_1), x_1)$ and $(T(x_2), x_2)$ rooted at v.*

Remark. If G is a graph, then $V(G)$ denotes (as usual) the vertex set of G.

Proof. (i) $\Rightarrow$ (ii). Let there exist a poset P such that $S \cong S(P)$. According to [1], S is isomorphic to the Hasse diagram of P. We may consider it directly as the Hasse diagram of P; thus we take $V(S) = P$. Two vertices x, y of S are adjacent if and only if x covers y or y covers x. We introduce an orientation in S in such a way that an edge joining x and y is directed from x to y if and only if y covers x. Let S_0 be the directed graph obtained in this way from S. The graph S_0 is evidently acyclic. We shall prove that for any two vertices u, v of S_0 there exists at most one directed path from u to v. Suppose that there exist two such paths P_1, P_2. If we go along P_1 from u to v, then let x_0, x_1 be the vertices of P_1 such that x_0x_1 is the first edge of P_1 not belonging to P_2. The vertex x_0 belongs to both P_1 and P_2. If also x_1 belongs to P_2, then there exists a subpath of P_2 from x_0 to x_1 of a length at least 2. In the ordering of P the inner vertices of this path are greater than x_0 and less than x_1, hence x_1 does not cover x_0 and this is a contradiction with the assumption that S is the Hasse diagram of P. Therefore x_1 does not belong to P_2. Let x_2 be such a vertex that x_0x_2 is an edge of P_2; the vertex x_2 does not belong to P_1 for the same reason as x_1 does not belong to P_2. The set $\{y \in P \mid y \geqq x_1 \ \& \ y \geqq x_2\}$ is non-empty, because it contains v. Let x_3 be a minimal element of this set. Then $x_3 > x_0$ and $\mu(x_0, x_3) = -\sum_{x_0 \leqq z < x_3} \mu(x_0, z)$. We have $\mu(x_0, x_0) = 1$, $\mu(x_0, x_1) = \mu(x_0, x_2) = -1$ and $\mu(x_0, y) \leqq 0$ for all $y \neq x$. Hence $\sum_{x_0 \leqq Z < x_3} \mu(x_0, z) \leqq \mu(x_0, x_0) + \mu(x_0, x_1) + \mu(x_0, x_2) = -1$ and $\mu(x_0, x_3) \geqq 1$, which is a contradiction. Thus we have proved that there exists at most one directed path from u to v in S_0. Let X (or Y) be the set of all sources (or sinks, respectively) of S_0. As S_0 is finite and acyclic, $X \neq \emptyset$ and $Y \neq \emptyset$. Let $x \in X$ and let $T_0(x)$ be the subgraph of S_0 whose vertex set is the set of all vertices of S_0 to which directed paths from x go. Suppose that $T_0(x)$ contains a subgraph H which (considered as undirected) is a circuit. The graph H, being a subgraph of an acyclic graph, is acyclic and contains a sink y. Then it contains vertices y_1, y_2 such that y_1y, y_2y are edges of H. As y_1, y_2 belong to $T_0(x)$, there exist a directed path P_1 from x to y_1 and a directed path P_2 from x to y_2. If we add the vertex y and the edge y_1y (or y_2y) to P_1 (or to P_2), we obtain a path P'_1 (or P'_2, respectively) from x to y. The paths P'_1, P'_2 are distinct, because their last edges are distinct; this is a contradiction with the above proved assertion. Hence $T_0(x)$, considered as undirected, is a tree. For each $x \in X$ let $T(x)$ be the tree $T_0(x)$ considered as undirected, i.e. as a subtree of S. Let x_1, x_2 be two distinct vertices from X and consider $T_0(x_1)$ and $T_0(x_2)$. Suppose that they have a common vertex v and let $T_0'(v)$ be the subgraph of S_0 induced by the set of all vertices to which directed paths from v go. As directed paths go to v from both x_1 and x_2, there are also directed paths from x_1 and x_2 to all vertices of $T_0'(v)$, and $T_0'(v)$ is a common subgraph of $T_0(x_1)$ and $T_0(x_2)$. If $T'(v)$ is the graph $T_0'(v)$ considered as undirected, then evidently $(T'(v), v)$ is a subtree of both $(T(x_1), x_1)$ and $(T(x_2), x_2)$ rooted at v. Therefore (d) holds. The validity of (a) is evident. Each tree $T(x)$ evidently contains x and cannot contain any $y \in X - \{x\}$, because all vertices of X are sources of S_0; this implies (b). The condition (c) follows from the fact that Y is the set of all sinks of S_0.

(ii) $\Rightarrow$ (i). Let (ii) hold. We direct any tree $T(x)$ in such a way that x becomes its unique source (this can be done in exactly one way). Such an orientation causes that each subtree of $(T(x), x)$ rooted at a vertex v is directed so that v is its unique source. Hence if some edge of S belongs to more than one tree from $\mathcal{T}$, by (d) it is directed in the same way in the orientations of all of them. The graph obtained by such an orientation from S will be denoted by S_0; evidently it is acyclic. Let u, v be two vertices of S_0 such that there exists a directed path from u to v. Each directed path from u must lie in the tree $T'(u)$ with the property that $(T'(u), u)$ is a subtree of $(T(x), x)$ rooted at v for any $x \in X$ such that v is in $T(x)$. This implies that the directed path from u to v is unique. Now on $P = V(S)$ we can define a partial ordering $\leqq$ in such a way that $x \leqq y$ if and only if there exists a directed path from x to y in S_0. In the poset P with this ordering every interval $[x, y]$ is a chain; by Theorem 1 from [1] this implies (i). □

From this result some corollaries easily follow.

Corollary 1. *A finite undirected graph S satisfies* (ii) *if and only if it can be directed in such a way that for any two vertices x, y there exists at most one directed path from x to y.*

Corollary 2. *Let a finite undirected graph contain two subsets X, Y of its vertex set such that* (ii) *holds for them. Then* (ii) *holds also in the case when we interchange X and Y.*

Corollary 3. *Let P be a finite signed poset for which all edges of $S(P)$ are negative. Let P have the greatest element or the least element. Then $S(P)$ is a tree.*

Corollary 4. *Let P be a finite signed poset for which all edges of $S(P)$ are negative. Let P have the greatest element and the least element. Then P is a chain and $S(P)$ is a path.*

Corollary 5. *Every finite signed bipartite graph S in which all edges are negative is isomorphic to $S(P)$ for some signed poset P.*

Note the in this case X and Y may be the bipartition classes of S and all trees of $\mathscr{T}$ may be stars.

Corollary 6. *If a finite signed graph in which all edges are negative is isomorphic to $S(P)$ for some signed poset P, then every graph obtained from S by subdividing its edges has the same property.*

At the end we remark that Theorem enables us to construct graphs with the mentioned property, but another theorem would be needed which would enable us to decide whether a given graphs has this property.

Reference

[1] *Harary, F. - Sagan, B.:* Signed posets, C.M.S. D.J.C. v. 3—10 (1983).

Author's address: 461 17 Liberec I, Studentská 1292, Czechoslovakia (katedra tváření a plastů VŠST).