Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 4, 623–641

Persistent URL: http://dml.cz/dmlcz/102259

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

NON-LINEAR PERTURBATIONS OF LINEAR NON-INVERTIBLE BOUNDARY VALUE PROBLEMS IN FUNCTION SPACES OF TYPE $B^s_{p,q}$ AND $F^s_{p,q}$

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(Received July 23, 1986)

1. INTRODUCTION

This paper is an application and extension of the results obtained in our previous papers [4, 15]. Here, we shall develop the theory of existence of solutions of non-linear partial differential equations of the type

$$L u(x) = f(x) + N u(x)$$

where L is a linear operator and N denotes special non-linear operators. If Ω is a smooth bounded domain in R_n (the Euclidean n-space) with boundary $\partial\Omega$, then we are looking for solutions of the above equation which satisfy boundary conditions and where the corresponding linear problem is non-invertible.

We consider these problems in the framework of the theory of the function spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$, respectively. These two scales of spaces include many well-known classical spaces. We use essentially the fact that these spaces are admissible in the sense of Klee [8], mapping properties of non-linear operators acting in these spaces, and results of Landesman-Lazer- and Kazdan-Warner-type, respectively.

The plan of the paper is the following. In Section 2 we describe the preliminaries (function spaces in R_n and on smooth domains, regular elliptic differential equations, admissibility of spaces, mapping properties of non-linear operators generated by C^∞-functions). In Section 3 we consider in contrast to [4] non-linear partial differential equations with boundary conditions where the corresponding linear problem is non-invertible. For this purpose we use the topological method of the Leray-Schauder degree.

2. PRELIMINARIES

2.1. Spaces. Let R_n be the Euclidean n-space. The theory of the spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ was developed in Triebel [16, 17]. We do not need the full theory, but only some properties, which we list in the sequel.

The spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$. Let $S(R_n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on R_n and let $S'(R_n)$ be the set of all tempered distributions on R_n. Let $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset S(R_n)$ be a smooth resolution of the unity in R_n with the following properties:

(i) $\{\varphi_j(x)\}_{j=0}^{\infty}$ is a system of real-valued even functions with respect to the origin, i.e.

$$\varphi_j(x) = \varphi_j(-x) \quad \text{if} \quad x \in R_n .$$

(ii) $0 \leqq \varphi_j(x) \leqq 1$ if $x \in R_n$ and $j = 0, 1, 2, \ldots$, and $\sum_{j=0}^{\infty} \varphi_j(x) \equiv 1$.

(iii) $\operatorname{supp} \varphi_0 \subset \{x, |x| \leqq 2\}$,
$\operatorname{supp} \varphi_j \subset \{x, 2^{j-1} \leqq |x| \leqq 2^{j+1}\}$ if $j = 1, 2, \ldots$.

(iv) For any multi-index α there exists a constant c_α such that

$$|D^\alpha \varphi_j(x)| \leqq c_\alpha 2^{-j|\alpha|} \quad \text{if} \quad x \in R_n \quad \text{and} \quad j = 0, 1, 2, \ldots .$$

Let $\Phi(R_n)$ be the collection of all systems $\varphi \subset S(R_n)$ satisfying the properties (i)–(iv).

Remark 1. In general (cf. Triebel [17, 2.31]), $\Phi(R_n)$ is defined without property (i). We use (i) to introduce the real part of the spaces $B^s_{p,q}(R_n)$, etc. (for definition see [4, 3.2.]).

Let F and F^{-1} be the Fourier transform and its inverse on $S'(R_n)$, respectively. If $-\infty < s < \infty$, $0 < p < \infty$, $0 < q < \infty$ and $\varphi \in \Phi(R_n)$ then

(1)
$$B^s_{p,q}(R_n) = \{f \mid f \in S'(R_n),\ \|f \mid B^s_{p,q}(R_n)\|^\varphi = \\ = (\sum_{j=0}^{\infty} 2^{jsq} (\int_{R_n} |F^{-1}[\varphi_j Ff](x)|^p \,dx)^{q/p})^{1/q} < \infty\}$$

and

(2)
$$F^s_{p,q}(R_n) = \{f \mid f \in S'(R_n),\ \|f \mid F^s_{p,q}(R_n)\|^\varphi = \\ = (\int_{R_n} (\sum_{j=0}^{\infty} 2^{jsq} \mid F^{-1}[\varphi_j Ff](x) \mid^q)^{p/q} \,dx)^{1/p} < \infty\} .$$

If $q = \infty$ then one has to replace $(\sum_{j=0}^{\infty} |\cdot|^q)^{1/q}$ by $\sup_j |\cdot|$ in (1) and (2). If $p = \infty$ (and $0 < q \leqq \infty$) then one has to replace $(\int_{R_n} |\cdot|^p \,dx)^{1/p}$ in (1) by $\sup_{x \in R_n} |\cdot|$. Then the spaces $B^s_{p,q}(R_n)$ are defined for $-\infty < s < \infty$, $0 < p \leqq \infty$, $0 < q \leqq \infty$ and the spaces $F^s_{p,q}(R_n)$ are defined for $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leqq \infty$.

The following facts are well-known (see, e. g. Triebel [16, 17]). $\|f \mid B^s_{p,q}(R_n)\|^\varphi$ and $\|f \mid F^s_{p,q}(R_n)\|^\varphi$ are quasi-norms (norms if $\min(p, q) \geqq 1$). If $\varphi \in \Phi(R_n)$ and $\psi \in \Phi(R_n)$ then $\|f \mid B^s_{p,q}(R_n)\|^\varphi$ and $\|f \mid B^s_{p,q}(R_n)\|^\psi$ are equivalent quasi-norms in $B^s_{p,q}(R_n)$ and we write $\|f \mid B^s_{p,q}(R_n)\|$ in the sequel. Similarly, $\|f \mid F^s_{p,q}(R_n)\|$. All these spaces are quasi-Banach spaces (Banach spaces if $\min(p, q) \geqq 1$).

As above mentioned these two scales of function spaces include many well-known classical spaces. Equivalent quasi-norms for these spaces may be found in Triebel [16, 17].

The spaces $B^s_{p,q}(\Omega)$, $F^s_{p,q}(\Omega)$, $B^s_{p,q}(\partial\Omega)$ and $F^s_{p,q}(\partial\Omega)$. Let Ω be a bounded C^∞-domain with the boundary $\partial\Omega$. Then one can introduce the spaces $B^s_{p,q}(\partial\Omega)$ and $F^s_{p,q}(\partial\Omega)$ by standard procedure via local charts, cf. Triebel [17, 3.2.2.]. The spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ are defined as usually by the restriction method, cf. Triebel [17, 3.2.2.].

Let $-\infty < s < \infty$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. Then

$$B^s_{p,q}(\Omega) = \{f \mid f \in D'(\Omega),\ \exists g \in B^s_{p,q}(R_n) \text{ with } g \mid \Omega = f\}\,, \tag{3}$$

$$\|f \mid B^s_{p,q}(\Omega)\| = \inf \|g \mid B^s_{p,q}(R_n)\| \tag{4}$$

where the infimum is taken over all $g \in B^s_{p,q}(R_n)$ in the sense of (3). Similarly one can define the spaces $F^s_{p,q}(\Omega)$.

Traces. Let Ω be a bounded C^∞-domain in R_n and let $f(x)$ be a function defined in Ω belonging to some function spaces of the above type. R denotes the restriction operator, given by $Rf = f \mid \partial\Omega$. The following results are well-known. If $0 < p \leqq \infty$, $0 < q \leqq \infty$ and

$$s > (n-1)\left(\frac{1}{\min(p,1)} - 1\right) + \frac{1}{p}$$

then R is a linear and continuous mapping

$$\text{from } B^s_{p,q}(\Omega) \text{ onto } B^{s-1/p}_{p,q}(\partial\Omega)\,. \tag{5}$$

If $0 < p < \infty$, $0 < q \leqq \infty$ and

$$s > (n-1)\left(\frac{1}{\min(p,1)} - 1\right) + \frac{1}{p}$$

then R is a linear and continuous mapping

$$\text{from } F^s_{p,q}(\Omega) \text{ onto } B^{s-1/p}_{p,p}(\partial\Omega)\,, \tag{6}$$

cf. [2] and Triebel [17, 3.3.3.].

Imbeddings. Let Ω be a bounded C^∞-domain in R_n. In the sequel, we need the following imbedding theorems. If $0 < p \leqq \infty$ (with $p < \infty$ in the case of the spaces $F^s_{p,q}(\Omega)$), $0 < q \leqq \infty$ and $s > n/p$, then

$$B^s_{p,q}(\Omega) \hookrightarrow C(\bar{\Omega}) \quad \text{and} \quad F^s_{p,q}(\Omega) \hookrightarrow C(\bar{\Omega})\,. \tag{7}$$

Here "$\hookrightarrow$" denotes the continuous imbedding and $C(\bar{\Omega})$ is the collection of all complex-valued continuous functions on $\bar{\Omega}$. Furthermore,

$$B^0_{\infty,1}(\Omega) \hookrightarrow C(\bar{\Omega}) \hookrightarrow B^0_{\infty,\infty}(\Omega) \tag{8}$$

holds. If $0 < p < \infty$, $0 < q \leqq \infty$, then

$$B^s_{p,\min(p,q)}(\Omega) \hookrightarrow F^s_{p,q}(\Omega) \hookrightarrow B^s_{p,\max(p,q)}(\Omega)\,. \tag{9}$$

The following assertions may be found in Triebel [17, Theorem 3.3.1]. Let $0 < p_0 \leqq \infty$, $0 < p_1 \leqq \infty$, $0 < q_0 \leqq \infty$, $0 < q_1 \leqq \infty$ and $-\infty < s_1 < s_0 < \infty$. Then

(10) $$B^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1,q_0}(\Omega) \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

and

(11) $$B^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1,q_1}(\Omega) \quad \text{if} \quad s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}.$$

Let $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 \leqq \infty$, $0 < q_1 \leqq \infty$ and $-\infty < s_1 < s_0 < \infty$. Then

(12) $$F^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow F^{s_1}_{p_1,q_1}(\Omega) \quad \text{if} \quad s_0 - \frac{n}{p_0} \geqq s_1 - \frac{n}{p_1}.$$

Admissibility of the spaces of type $B^s_{p,q}$ and $F^s_{p,q}$. We use the notation „admissible" in the sense of Klee [8], cf. also Riedrich [14] with respect to quasi-normed spaces. Consequently, a quasi-normed space A is said to be *admissible* if for every compact subset $K \subset A$ and for every $\varepsilon > 0$ there exists a continuous mapping $T: K \to A$ such that $T(K)$ is contained in a finite-dimensional subset of A and $x \in K$ implies $\|Tx - x \mid A\| \leqq \varepsilon$. In [4] it was proved that the spaces $B^s_{p,q}(R_n)$, $B^s_{p,q}(\Omega)$, $B^s_{p,q}(\partial\Omega)$, $F^s_{p,q}(R_n)$, ..., etc. are admissible.

Remark 2. In the following, cf. also [4], we use essentially the fact that the spaces considered here are admissible. Then it is possible (as in the case of Banach spaces) to define the Leray-Schauder degree and to apply the Leray-Schauder theory (cf. Riedrich [14]).

In [4, 3.2] the real part of the spaces $B^s_{p,q}(R_n)$, etc. was introduced. We denote it by $\tilde{B}^s_{p,q}(R_n)$, etc. The above mentioned results hold also for these spaces.

2.2. Linear elliptic differential operators. As in the previous subsection, Ω denotes always a bounded C^∞-domain in R_n with boundary $\partial\Omega$. We recall here some well-known notations (for exact definitions see e.g. Triebel [17, 4.1.2]). Let A,

(1) $$(Au)(x) = \sum_{|\alpha| \leqq 2} a_\alpha(x) D^\alpha u(x), \quad x \in \Omega, \quad a_\alpha \in \tilde{C}^\infty(\bar{\Omega}) \quad \text{if} \quad |\alpha| \leqq 2,$$

be a properly elliptic self-adjoint differential operator in $\bar{\Omega}$, and let B,

(2) $$(Bu)(y) = \sum_{|\alpha| \leqq d} b_\alpha(y) D^\alpha u(y), \quad y \in \partial\Omega, \quad b_\alpha \in \tilde{C}^\infty(\partial\Omega) \quad \text{if} \quad |\alpha| \leqq d \leqq 1,$$

be a boundary operator such that $\{A, B\}$ is regular elliptic. The corresponding boundary value problem has the form

(3) $$(Au)(x) = f(x) \quad \text{if} \quad x \in \Omega,$$
$$(Bu)(y) = g(y) \quad \text{if} \quad y \in \partial\Omega.$$

In [2], the following results were proved (see also Triebel [17, 4.3.3]). Let $\{A, B\}$

be the above system such that (3) with $u(x) \in C^\infty(\bar{\Omega})$ has only the trivial solution $u(x) \equiv 0$ if $f = g \equiv 0$.

(i) Let $0 < p \leqq \infty$, $0 < q \leqq \infty$ and

$$s > (n-1)\left(\frac{1}{\min(p,1)} - 1\right) + \frac{1}{p} + d - 2 .$$

Then $\{A, B\}$ yields an isomorphic mapping

$$\text{from} \quad B^{s+2}_{p,q}(\Omega) \quad \text{onto} \quad B^s_{p,q}(\Omega) \times B^{s+2-d-1/p}_{p,q}(\partial\Omega) .$$

(ii) Let $0 < p < \infty$, $0 < q \leqq \infty$ and

$$s > (n-1)\left(\frac{1}{\min(p,1)} - 1\right) + \frac{1}{p} + d - 2 .$$

Then $\{A, B\}$ yields an isomorphic mapping

$$\text{from} \quad F^{s+2}_{p,q}(\Omega) \quad \text{onto} \quad F^s_{p,q}(\Omega) \times B^{s+2-d-1/p}_{p,p}(\partial\Omega) .$$

In this paper, we consider the case where (3) has a finite-dimensional $\ker(\{A, B\})$. It follows from the known regularity theorems that

$$\ker(\{A, B\}) \subset C^\infty(\bar{\Omega}) .$$

(see, e.g. Nečas [11, Théorème 4.2.2] or Agmon, Douglis, Nirenberg [1]).

Using the results in [2] we obtain the following:

Let $0 < p \leqq \infty, 0 < q \leqq \infty, s > (n-1)\, 1/(\min(p, 1)) - 1) + 1/p + d - 2$ and let $\{g_1, \ldots, g_M\}$ be an orthonormal basis of $\ker(\{A, B\})$. Then P defined by

$$Pz = z - \sum_{j=1}^{M} g_j \int_\Omega z\bar{g}_j \, dx \tag{4}$$

is a projection in $B^s_{p,q}(\Omega)$. Furthermore, there exists a linear and bounded operator L

$$L\colon B^s_{p,q}(\Omega) \to \{z \in B^{s+2}_{p,q}(\Omega) \mid Bz = 0, \ \int_\Omega z(x)\, \bar{g}_j(x)\, dx = 0\}$$

with the property

$$ALz = Pz .$$

Then the problem

$$Au = f , \quad Bu = 0 \tag{5}$$

has a solution $u \in B^{s+2}_{p,q}(\Omega)$ if

$$Pf = f \tag{6}$$

holds. If (6) is satisfied then

$$g = Lf + \sum_{j=1}^{M} \lambda_j g_j , \quad \lambda_j \in \mathbb{C} , \quad j = 1, \ldots, M ,$$

is the general solution of (5).

Remark 3. In 3.2 we apply P and $P^c = I - P$ (projection onto $\ker(\{A, B\})$ to define a bifurcation system.

If $0 < p < \infty$ then the above result holds in the case $F^s_{p,q}(\Omega)$, too.

2.3. Non-linear operators generated by C^∞-functions. In this subsection we list up some results which may be found in [15, 5.4]. In the following, let Ω be a bounded C^∞-domain in R_n and C^ϱ denotes as usual the classical Hölder space if $\varrho > 0$ is not an integer and the well-known Banach space of differentiable functions if $\varrho > 0$ is an integer. For real s we put $s = [s]_- + \{s\}_+$, $[s]_-$ integer, $0 < \{s\}_+ \leqq 1$.

Theorem 1. (*i*) *Let* $0 < p < \infty, 0 < q \leqq \infty$,

$$s > n\left(\frac{1}{\min(p, 1)} - 1\right),$$

$\varrho > \max(1, s)$ *and* $\Phi \in \tilde{C}^\varrho(u(R_n))$. *Then*

$$\|\Phi(u) \mid F^s_{p,q}\| \leqq c\|\Phi(u) \mid L_p\| + $$
$$+ c\Big(\sum_{l=1}^{[\varrho]_-} \sup_{x\in R_n} |\Phi^{(l)}(u(x))| \; \|u \mid F^s_{p,q}\| \; \|u \mid L_\infty\|^{l-1} +$$
$$+ \|\Phi \mid C^\varrho(u(R_n))\| \; \|u \mid F^{s}_{p,q}\| \; \|u \mid L_\infty\|^{\varrho-1} +$$
$$+ \|\Phi \mid C^\varrho(u(R_n))\| \; \|u \mid F^s_{p,q}\|\Big)$$

and

$$\|\Phi(u) \mid L_p\| \leqq \|\Phi \mid C^1(\overline{u(R_n)})\| \; \|u \mid L_p\| .$$

(ii) *Let* $0 < p \leqq \infty$, $0 < q \leqq \infty$,

$$s > n\left(\frac{1}{\min(p, 1)} - 1\right),$$

$\varrho > \max(1, s)$ *and* $\Phi \in \tilde{C}^\varrho(u(R_n))$. *Then*

$$\|\Phi(u) \mid B^s_{p,q}\| \leqq \|\Phi(u) \mid L_p\| + $$
$$+ c\Big(\sum_{l=1}^{[\varrho]_-} \sup_{x\in R_n} |\Phi^{(l)}(u(x))| \; \|u \mid B^s_{p,q}\| \; \|u \mid L_\infty\|^{l-1} +$$
$$+ \|\Phi \mid C^\varrho(u(R_n))\| \; \|u \mid B^s_{p,q}\| \; \|u \mid L_\infty\|^{\varrho-1} +$$
$$+ \|\Phi \mid C^\varrho(u(R_n))\| \; \|u \mid B^s_{p,q}\|\Big)$$

and

$$\|\Phi(u) \mid L_p\| \leqq \|\Phi \mid C^1(\overline{u(R_n)})\| \; \|u \mid L_p\| .$$

The following result is a consequence of Theorem 1 and the fact that the imbeddings $F^{s+\varepsilon}_{p,q}(\Omega) \hookrightarrow F^s_{p,q}(\Omega)$ and $B^{s+\varepsilon}_{p,q}(\Omega) \hookrightarrow B^s_{p,q}(\Omega)$, $\varepsilon > 0$, are compact (cf. Triebel [19, 3.1. (30)]).

Corollary. *Let* $\Phi: R_1 \to R_1$ *be a* $\tilde{C}^\infty$*-function and let* $\varepsilon > 0$. *Then* $u \to \Phi(u)$ *is a completely continuous mapping*

(1) *from* $\tilde{B}^{s+\varepsilon}_{p,q}(\Omega) \cap \tilde{L}_\infty(\Omega)$ *into* $\tilde{B}^s_{p,q}(\Omega) \cap \tilde{L}^\infty(\Omega)$

(*from* $\tilde{F}^{s+\varepsilon}_{p,q}(\Omega) \cap \tilde{L}_\infty(\Omega)$ *into* $\tilde{F}^s_{p,q}(\Omega) \cap \tilde{L}^\infty(\Omega)$)

if $0 < p \leqq \infty$ $(0 < p < \infty)$, $0 < q \leqq \infty$ *and*

$$s > n\left(\frac{1}{\min(p, 1)} - 1\right).$$

Furthermore, there exists a function g, $g\colon [0, \infty) \to [0, \infty)$, *which is independent of* u, *such that*

(2)
$$\|\Phi(u) \mid B^s_{p,q}\| \leqq g(\|u \mid L_\infty\|)\, \|u \mid B^s_{p,q}\|$$
$$(\|\Phi(u) \mid F^s_{p,q}\| \leqq g(\|u \mid L_\infty\|)\, \|u \mid F^s_{p,q}\|).$$

3. SOLVABILITY OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

3.1. Introduction. Let $\Omega \subset R_n$ be a bounded C^∞-domain. We consider semilinear Dirichlet problems

(1)
$$-\Delta u - \lambda u + \Phi(u) = f \quad \text{in} \quad \Omega\,, \qquad u \mid \partial\Omega = 0$$

of the following types, where Φ is a real C^∞-function on R_1.

(i) **Non-coercitive problems** with constant asymptotics:

$$\Phi(t) \to \Phi_\pm \quad \text{if} \quad t \to \pm\infty\,.$$

(1) is solvable in certain Besov spaces if f satisfies the Landesman-Lazer condition:

$$\Phi_+ \int_{\{x\in\Omega \mid w(x)>0\}} w(x)\,\mathrm{d}x + \Phi_- \int_{\{x\in\Omega \mid w(x)<0\}} w(x)\,\mathrm{d}x < \int_\Omega f(x)\, w(x)\,\mathrm{d}x$$

for all $0 \neq w \in \tilde{C}^\infty(\bar{\Omega})$ with $-\Delta w - \lambda w = 0$, $w \mid \partial\Omega = 0$. If λ is not an eigenvalue of the corresponding Dirichlet problem

(1′)
$$-\Delta u - \lambda u = 0\,, \qquad u \mid \partial\Omega = 0\,,$$

then (1) is solvable provided Φ is bounded. Similar results hold for sublinear problems. Details may be found in 3.2 and 3.3, respectively.

(ii) **Results of Kazdan-Warner type.** A typical example is

(2)
$$-\Delta u - \lambda_1 u + \mathrm{e}^u = f \quad \text{in} \quad \Omega\,, \qquad u \mid \partial\Omega = 0\,,$$

where λ_1 is the first eigenvalue of the Dirichlet problem (1′) with $\lambda = \lambda_1$. Let φ be the unique positive solution of

$$-\Delta\varphi = \lambda_1\varphi \text{ in } \Omega\,, \quad \varphi \mid \partial\Omega = 0\,, \quad \|\varphi \mid L_2(\Omega)\| = 1\,.$$

Then (2) is solvable if and only if

$$\int_\Omega f(x)\, \varphi(x)\,\mathrm{d}x > 0\,.$$

This is a special case of a general result, which is proved in Fučík [5, Chapter 34] for Hölder spaces and in 3.4 for Besov spaces.

3.2. Results of Landesman-Lazer type, bounded non-linearities. Let

(1)
$$A\,u(x) = \sum_{|\alpha| \leqq 2} a_\alpha(x)\, D^\alpha u(x) \text{ in } \Omega\,, \quad a_\alpha \in \tilde{C}^\infty(\bar{\Omega})\,,$$
$$B\,u(y) = \sum_{|\alpha| \leqq d} b_\alpha(y)\, D^\alpha u(y) \mid \partial\Omega\,, \quad b_\alpha \in \tilde{C}^\infty(\partial\Omega)$$

be a second order elliptic boundary value problem of type $d + 1 \leqq 2$. We consider the semilinear problem

(2)
$$Au + \Phi(u) = f \quad \text{in} \quad \Omega\,,$$
$$Bu = 0 \quad \text{on} \quad \partial\Omega\,,$$

where $\Phi: (-\infty, \infty) \to (-\infty, \infty)$ is a real C^∞-function.

Theorem 1. *Suppose that $\{A, B\}$ is invertible and that Φ is bounded.*

(i) *If $0 < p < \infty$, $0 < q \leqq \infty$, $s > \max(n/p, (1/p) + d)$, $t > d - 2$, and $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^t_{\infty,\infty}(\Omega)$, then (2) has a solution $u \in \tilde{F}^s_{p,q}(\Omega)$.*

(ii) *If $0 < p \leqq \infty$, $0 < q \leqq \infty$, $s > \max(n/p, (1/p) + d)$, $t > d - 2$, and $f \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^t_{\infty,\infty}(\Omega)$, then (2) has a solution $u \in \tilde{B}^s_{p,q}(\Omega)$.*

Theorem 2. *Let the following assumptions be satisfied:*

(a) *$\{A, B\}$ is self-adjoint, i.e.,*

$$\int_\Omega (Au)(x)\, v(x)\, dx = \int_\Omega u(x)\, A\, v(x)\, dx$$

if $u, v \in \tilde{C}^\infty(\bar{\Omega})$, $Bu = Bv = 0$.

(b) $\Phi(t) \to \Phi_\pm$, $\Phi_\pm \in \mathbb{R}$, if $t \to \pm\infty$.

We formulate the condition

$$\text{(L)} \quad \Phi_+ \int_{\{x\in\Omega \mid w(x)>0\}} w(x)\, dx + \Phi_- \int_{\{x\in\Omega \mid w(x)<0\}} w(x)\, dx > \int_\Omega f(x)\, w(x)\, dx\,,$$
$$0 \neq w \in \ker(\{A, B\})\,.$$

Let s, t and q be the same as in Theorem 1.

(i) *Let $0 < p \leqq \infty$ and $f \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^t_{\infty,\infty}(\Omega)$. We suppose that the Landesman-Lazer condition (L) is satisfied for all $0 \neq w \in \ker(\{A, B\})$. Then (2) has a solution $u \in \tilde{B}^s_{p,q}(\Omega)$.*

(ii) *If $0 < p < \infty$, $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^t_{\infty,\infty}(\Omega)$ and if (L) is satisfied for all $0 \neq w \in \ker(\{A, B\})$, then (2) has a solution $u \in \tilde{F}^s_{p,q}(\Omega)$.*

We mention that the right-hand side of (L) makes sense: cf. [2, 6.1, Lemma 2 (iii)] and 2.2.

In the following we prove Theorem 2, the (afterwards almost trivial) proof of Theorem 1 is left to the reader.

Proof of Theorem 2. We prove (ii), the proof of (i) is almost the same.

Let P^c denote the orthogonal projection from $\tilde{F}^{s-2}_{p,q}(\Omega)$ onto $\ker(\{A, B\})$ and $P = I - P^c$ (I: identity). We mention that

$$P: \tilde{F}^t_{p,q}(\Omega) \to \tilde{F}^t_{p,q}(\Omega) \quad \text{is bounded if} \quad t \geqq s - 2$$

(cf. [2, 6.1, Lemma 2 (iv)]). We also refer to 2.2. There exists an unique continuous linear operator (the so-called right inverse of A)

$$L: \tilde{F}^{s-2}_{p,q}(\Omega) \to \tilde{F}^{s}_{p,q}(\Omega)$$

with

$$BLz = P^c Lz = 0 \quad \text{and} \quad ALz = Pz\,, \quad z \in \tilde{F}^{s-2}_{p,q}(\Omega)\,.$$

Now we define a family of completely continuous mappings

$$T_\lambda: \tilde{F}^{s}_{p,q}(\Omega) \to \tilde{F}^{s}_{p,q}(\Omega)\,, \quad 0 \leqq \lambda \leqq 1\,,$$

with

$$T_\lambda u = \lambda\{P^c u - P^c[\Phi(u) - f] + L(P[f - \Phi(u)])\}\,. \tag{3}$$

$\{T_\lambda\}_{0 \leqq \lambda \leqq 1}$ is a family of completely continuous mappings because of the properties of L, P, P^c and Φ and the compact imbedding from $\tilde{F}^{\varrho+\varepsilon}_{p,q}(\Omega)$ in $\tilde{F}^{\varrho}_{p,q}(\Omega)$ if $\varepsilon > 0$ (cf. Triebel [19, 3.1 (30)]).

We consider the fixed point problem

$$T_\lambda u = u\,, \quad 0 \leqq \lambda \leqq 1\,. \tag{4}$$

For $u \in \tilde{F}^{s}_{p,q}(\Omega)$ we write $u = v + w$, $w \in \ker(\{A, B\})$, $v \in \ker(\{A, B\})^\perp$, i.e. $P^c u = w$, $Pu = v$. Applying P to (4) we get

$$v = \lambda L(P[f - \Phi(v + w)])\,. \tag{5}$$

An application of P^c to (4) yields

$$w = \lambda w - \lambda P^c[\Phi(u) - f]\,. \tag{6}$$

In particular, $T_1 u = u$ is equivalent to (2). The system (5), (6) is usually called the bifurcation system or the alternative system for the equation (2). Now we shall study the solvability of (5), (6). For this purpose we use the Leray-Schauder theory.

Furthermore, there exists an l with $2 > l > d$ such that $f \in \tilde{B}^{l-2}_{\infty,\infty}(\Omega)$. Since Φ is bounded this assertion and (5) yield

$$\|v \mid L_\infty\| \leqq C_0\,. \tag{7}$$

Let $0 \neq w_0 \in \ker(\{A, B\})$ then we get by (L)

$$\begin{aligned} \int_\Omega \{P^c[\Phi(v + kw_0) - f]\}(x)\, w_0(x)\,dx &= \\ = \int_\Omega [\Phi(v + kw_0) - f](x)\, P^c w_0(x)\,dx &= \\ = \int_\Omega [\Phi(v + kw_0) - f](x)\, w_0(x)\,dx &> 0 \end{aligned} \tag{8}$$

for all v with $\|v \mid L_\infty\| \leqq C_0$ if k is large enough. Let $k(w_0)$ be the smallest number such that (8) holds for $k > k(w_0)$. Since $w_0 \to k(w_0)$ is upper semicontinuous on $\ker(\{A, B\}) \setminus \{0\}$, $k(\cdot)$ is bounded on the unit sphere of $\ker(\{A, B\})$. In other words,

$$\int_\Omega \{P^c[\Phi(v + w) - f]\}(x)\, w_0(x)\,dx > 0 \tag{9}$$

if $0 \neq w_0 \in \ker(\{A, B\})$ and

$$\|v \mid L_\infty\| \leqq C_0\,, \quad \|w \mid L_\infty\| > C_1\,. \tag{10}$$

If $u = v + w$ is a solution of (4), the following condition holds

(11) $$\|w \mid L_\infty\| \leqq C_1 .$$

Otherwise we would get by (6), (7), (9) and (10)

(12) $$\langle w, w \rangle_{L_2} = \lambda \langle w, w \rangle_{L_2} - \lambda \langle P^c[\Phi(v + w) - f], w \rangle_{L_2} < \lambda \langle w, w \rangle_{L_2}$$

where $0 \leqq \lambda \leqq 1$.

Let $2 > \varepsilon > 0$ be small enough such that $s - \varepsilon > n/p$. (7) and (11) yield

(13) $$\|\Phi(v + w) \mid \tilde{F}^{s-2}_{p,q}(\Omega)\| \leqq c_2(1 + \|v + w \mid \tilde{F}^{s-\varepsilon}_{p,q}(\Omega)\|) .$$

Here we used the results from 2.3. Since ker $(\{A, B\})$ is finite-dimensional and consists of C^∞-functions (11) shows

$$\|w \mid \tilde{F}^{s-\varepsilon}_{p,q}(\Omega)\| \leqq c_1' .$$

Thus (5) and (13) imply

(14) $$\|v \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq c_3(1 + \|v \mid \tilde{F}^{s-\varepsilon}_{p,q}(\Omega)\| .$$

Because of (7), the imbedding $L_\infty(\Omega) \subset F^0_{p,2}(\Omega)$ and the inequality

$$\|G \mid F^{\Theta s_0 + (1-\Theta)s_1}_{p,q}(\Omega)\| \leqq C\|G \mid F^{s_0}_{p,2}(\Omega)\|^\Theta \, \|G \mid F^{s_1}_{p,q}(\Omega)\|^{1-\Theta} ,$$

$0 < \Theta < 1$, we get from (14)

(15) $$\|v \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq c_4(1 + \|v \mid \tilde{F}^s_{p,q}(\Omega)\|^\Theta) .$$

We conclude

(16) $$\|v \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq C_5$$

and

(17) $$\|w \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq C_6 .$$

Now we apply the Leray-Schauder theory to prove that $T_1 u$ has a fixed point.

Let

$$K = \{u \in \tilde{F}^s_{p,q}(\Omega) \mid \|u \mid \tilde{F}^s_{p,q}(\Omega)\| < 2(C_5 + C_6)\}$$

where C_5 and C_6 has the meaning of (16) and (17), respectively. By the definition of K, (4) has no solution on ∂K. Now we can apply the Leray-Schauder theory in the admissible quasi-Banach space $\tilde{F}^s_{p,q}(\Omega)$, cf. 2.1. We have shown that there does not exist an $u \in \partial K$ and an $\lambda \in [0, 1]$ such that $T_\lambda u = u$ holds. Then by the properties of the Leray-Schauder degree (for definition and properties see e.g. Fučík [5] or Fučík, Kufner [6]) it follows that there exists a solution $u \in K$ such that $T_1 u = u$ holds. Consequently, u is also a solution of (2). Our proof is finished.

Remark 1. Theorem 1 and 2 are results of the so-called „Fredholm alternative for non-linear operators". The Fredholm alternative for non-linear operators was probably first formulated independently by Nečas [10] and Pokhozhaev [13]. The result is the following (cf. Zeidler [21, 28]). If the linear equation $Lu = 0$ has only the trivial solution $u = 0$, then the non-linear equation $Lu + Nu = f$ has at least one solution for arbitrary right-hand side f if N is sublinear, cf. Fučík [5, Definition 7.4].

If $Lu = 0$ has a nontrivial solution $u = 0$, then the equation $Lu + Nu = f$ has only a solution u, if f satisfies some conditions of solvability. Here we used a so-called asymptote with respect to $\ker(L)$ (Theorems of Landesman-Lazer type).

Remark 2. Problems in this direction were considered by many authors. We refer to Fučík [5] and the references given there. Results of the Landesman-Lazer type may be found in Fučík [5], Hess [7], Landesman, Lazer [9], Williams [20] and Zeidler [21].

Remark 3. If $\Phi_- < 0 < \Phi_+$, $\Phi_- < \Phi(t) < \Phi_+$ then (L) is also necessary for the solvability of (2): If u is a solution of (2) then

$$\int_\Omega f(x)\, w(x)\, \mathrm{d}x = \int_\Omega \Phi(u(x))\, w(x)\, \mathrm{d}x <$$
$$< \Phi_+ \int_{\{x\in\Omega | w(x)>0\}} w(x)\, \mathrm{d}x + \Phi_- \int_{\{x\in\Omega | w(x)<0\}} w(x)\, \mathrm{d}x\,.$$

Modifications and generalizations of the results given here in the framework of Sobolev and Hölder spaces may be found in Fučík [5, Chapter 11, 13, 18 and 23], Fučík, Kufner [6, 34] and Zeidler [21, 28]. Examples are also contained in Nečas [12].

Remark 4. (L) shows that

$$\int_\Omega f(x)\, w(x)\, \mathrm{d}x < \Phi_+ \int_{\{x\in\Omega | w(x)>0\}} w(x)\, \mathrm{d}x + \Phi_- \int_{\{x\in\Omega | w(x)<0\}} w(x)\, \mathrm{d}x$$

for all $0 \neq w \in \ker(\{A, B\})$. In the sense of Fučík [5, Definition 11.1] the function

$$\Psi\colon w \to \int_\Omega f(x)\, w(x)\, \mathrm{d}x - \Phi_+ \int_{\{x\in\Omega | w(x)>0\}} w(x)\, \mathrm{d}x - \Phi_- \int_{\{x\in\Omega | w(x)<0\}} w(x)\, \mathrm{d}x$$

is an asymptote of the operator $Su = \Phi(u) - f$ with respect to $\ker(\{A, B\})$, cf. also Fučík [5, 11.2].

Remark 5. If $d = 0$ in Theorem 1 and 2, respectively, then the condition of f is the following:

$$f \in \tilde{B}^{s-2}_{p,q}(\Omega)\,, \quad s > n/p$$

and

$$f \in \tilde{F}^{s-2}_{p,q}(\Omega)\,, \quad s > n/p\,,$$

respectively.

3.3. Results of Landesman-Lazer typ, sublinear non-linearities. In contrast to 3.2 we consider (3.2/2) where Φ is a real sublinear C^∞-function. We formulate our result.

Let

(1)
$$A\, u(x) = \sum_{|\alpha|\leq 2} a_\alpha(x)\, D^\alpha u(x) \text{ in } \Omega\,, \quad a_\alpha \in \tilde{C}^\infty(\bar{\Omega})\,,$$
$$B\, u(y) = \sum_{|\alpha|\leq d} b_\alpha(y)\, D^\alpha u(y) \,\big|\, \partial\Omega\,, \quad b_\alpha \in \tilde{C}^\infty(\partial\Omega)$$

be a second order elliptic boundary value problem of type $d + 1 \leq 2$ and let

(2)
$$Au + \Phi(u) = f \quad \text{in} \quad \Omega\,,$$
$$Bu = 0 \quad \text{on} \quad \partial\Omega\,,$$

where $\Phi: (-\infty, \infty) \to (-\infty, \infty)$ is a real C^∞-function related to the corresponding semilinear problem.

Theorem 1. *Let the following assumptions be satisfied:*
(a) *$\{A, B\}$ is self-adjoint in the sense of Theorem* 3.2/2.
(b) *$\Phi(t) \to \pm\infty$ if $t \to \pm\infty$.*
(c) *There exists a monotone and positive function γ with $\gamma(t)/t \to 0$ if $t \to \infty$ such that*

$$|\Phi(t)| < \gamma(|t|) \tag{3}$$

and there exist a positive number δ, $0 < \delta < 1$, and a $t_0 > 0$ such that

$$|\Phi(t)| > \delta\, \gamma(|t|) \quad \text{if} \quad |t| > t_0 . \tag{4}$$

Let $s > \max(n/p, (1/p) + d)$, $t > d - 2$ and $0 < q \leqq \infty$.

(i) *Let $0 < p \leqq \infty$ and $f \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^t_{\infty,\infty}(\Omega)$. Then* (2) *has a solution $u \in \tilde{B}^s_{p,q}(\Omega)$.*

(ii) *If $0 < p < \infty$ and $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^t_{\infty,\infty}(\Omega)$, then* (2) *has a solution $u \in \tilde{F}^s_{p,q}(\Omega)$.*

Proof. We prove (ii), the proof of (i) is almost the same. The operators P, P^c, L and T_λ, $0 \leqq \lambda \leqq 1$, have the same meaning as in the proof of Theorem 3.2/2. Then $\{T_\lambda\}_{0 \leqq \lambda \leqq 1}$ is a family of completely continuous mappings because of the properties of L, P, P^c and Φ and the compact imbedding from $F^{\varrho+\varepsilon}_{p,q}(\Omega)$ in $F^{\varrho}_{p,q}(\Omega)$ if $\varepsilon > 0$. According to the proof of Theorem 3.2/2 we get

$$v = \lambda L(P[f - \Phi(v + w)]) \tag{5}$$

and

$$w = \lambda w - \lambda P^c[\Phi(v + w) - f] \tag{6}$$

where for $u \in \tilde{F}^s_{p,q}(\Omega)$ we put

$$u = v + w\,, \quad v \in \ker(\{A, B\})^\perp\,, \quad w \in \ker(\{A, B\})\,.$$

Furthermore, there exists an l with $2 > l > d$ such that $f \in \tilde{B}^{l-2}_{\infty,\infty}(\Omega)$. Using (3) and (5), we get for every $\tilde{\varepsilon} > 0$ and suitable chosen $\tilde{c}_\varepsilon = \tilde{c}_\varepsilon(\gamma)$ such that

$$\|v \mid L_\infty\| \leqq c(1 + \gamma(\|v + w \mid L_\infty\|) \leqq \tilde{c}_\varepsilon + \tilde{\varepsilon}(\|v \mid L_\infty\| + \|w \mid L_\infty\|) \tag{7}$$

hold.

For $\tilde{\varepsilon} < \frac{1}{2}$ we get

$$\|v \mid L_\infty\| \leqq 2\tilde{c}_\varepsilon + 2\tilde{\varepsilon}\|w \mid L_\infty\|\,. \tag{8}$$

This yields that for every $\varepsilon > 0$ there exists a $c_\varepsilon = c_\varepsilon(\gamma)$ such that

$$\|v \mid L_\infty\| \leqq c_\varepsilon + \varepsilon\|w \mid L_\infty\| \tag{9}$$

holds.

In the following let $w_0 \in \ker(\{A, B\})$ with $\|w_0 \mid L_\infty\| = 1$ and v satisfies (9), such that

$$\|v \mid L_\infty\| \leqq C_\varepsilon + \varepsilon t\,.$$

Furthermore let $\varepsilon < \frac{1}{4}$ and let t_ε be large enough such that $\Phi(\alpha + t\beta)\,\beta > 0$ holds, if $|\beta| > 2\varepsilon$, $t > t_\varepsilon$ and $|\alpha| \leqq C_\varepsilon + \varepsilon t$.

Then we obtain

$$\begin{aligned} I &= \int_\Omega \{P^c[\Phi(v + tw_0)]\}(x)\, w_0(x)\,\mathrm{d}x = \\ &= \int_\Omega [\Phi(v + tw_0)](x)\, w_0(x)\,\mathrm{d}x = \\ &= \int_{\{x\in\Omega \mid |w_0(x)| > 2\varepsilon\}} [\Phi(v + tw_0)](x)\, w_0(x)\,\mathrm{d}x + \\ &+ \int_{\{x\in\Omega \mid |w_0(x)| \leqq 2\varepsilon\}} [\Phi(v + tw_0)](x)\, w_0(x)\,\mathrm{d}x\,. \end{aligned}$$

Hence we get for t (large enough), i.e. $t > t_\varepsilon$,

$$\begin{aligned} &\int_{\{x\in\Omega \mid |w_0(x)| > 2\varepsilon\}} [\Phi(v + tw_0)](x)\, w_0(x)\,\mathrm{d}x > \\ &> \int_{\{x\in\Omega \mid |w_0(x)| > 1/2\}} [\Phi(v + tw_0)](x)\, w_0(x)\,\mathrm{d}x > c'\delta\gamma(\tfrac{1}{2}t - C_\varepsilon - \varepsilon t)\,. \end{aligned}$$

Here we used the properties of γ, (4) and $t > t_\varepsilon$.

Furthermore, we get

$$\begin{aligned} \int_{\{x\in\Omega \mid |w_0(x)| \leqq 2\varepsilon\}} [\Phi(v + tw_0)](x)\, w_0(x)\,\mathrm{d}x &< 2\varepsilon c''\gamma(C_\varepsilon + \varepsilon t + 2\varepsilon t) \leqq \\ &\leqq 2\varepsilon c''\gamma(C_\varepsilon + 3\varepsilon t) \end{aligned}$$

i.e.,

$$I > c\gamma(\tfrac{1}{2}t - C_\varepsilon - \varepsilon t) - 2c''\varepsilon\gamma(C_\varepsilon + 3\varepsilon t)$$

if $t > t_\varepsilon$.

Now we choose $\varepsilon < \frac{1}{8}$ such that $c > 2c''\varepsilon$. Let $t^\varepsilon > t_\varepsilon$ sufficiently large such that for $t > t^\varepsilon$ the following inequality holds. $\frac{1}{2}t - C_\varepsilon - \varepsilon t > C_\varepsilon + 3\varepsilon t$.

Then we obtain

$$I = \int_\Omega [\Phi(v + tw_0)](x)\, w_0(\mathrm{d}x) > (c - 2c''\varepsilon)\,\gamma(\tfrac{1}{2}t - C_\varepsilon - \varepsilon t) > K_1 > 0\,,$$

where K_1 is an arbitrary positive number provided $t = t(K_1)$ is large enough. Here we used the properties of γ.

Therefore we get the following result:

$$\int_\Omega \{P^c[\Phi(v + w) - f]\}(x)\, w(x)\,\mathrm{d}x = \int_\Omega [\Phi(v + w) - f](x)\, w(x)\,\mathrm{d}x > 0 \tag{10}$$

for all v with $\|v \mid L_\infty\| \leqq c_\varepsilon + \varepsilon\|w \mid L_\infty\|$ and all $w \in \ker(\{A, B\})$ if $\|w \mid L_\infty\| > t_0(f)$. By (9) and (10) we obtain the following estimate if $u = v + w$ is a solution of $T_\lambda u = u$:

$$\|w \mid L_\infty\| \leqq C_1\,, \quad \|v \mid L_\infty\| \leqq C_2 \tag{11}$$

Otherwise it would follow by (6), (9) and (10)

$$\langle w, w\rangle_{L_2} = \lambda\langle w, w\rangle_{L_2} - \lambda\,\langle P^c[\Phi(v + w) - f], w\rangle_{L_2} < \lambda\langle w, w\rangle_{L_2} \tag{12}$$

where $0 \leqq \lambda \leqq 1$.

Now we can prove the assertion of Theorem in analogy to the proof of Theorem 3.2/2 (ii).

It holds also the following result:

Theorem 2. *Suppose that $\{A, B\}$ is invertible and $\Phi\colon(-\infty, \infty) \to (-\infty, \infty)$ is a real C^∞-function with the following properties*:

There exists a monotone and positive function γ with $\gamma(t)/t \to 0$ if $t \to \infty$ such that

(13) $$|\Phi(t)| < \gamma(|t|) .$$

Let $s > \max(n/p, (1/p) + d)$, $t > d - 2$ *and* $0 < q \leqq \infty$.

(i) *Let* $0 < p \leqq \infty$ *and* $f \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^{t}_{\infty,\infty}(\Omega)$. *Then* (2) *has a solution* $u \in \tilde{B}^{s}_{p,q}(\Omega)$.

(ii) *If* $0 < p < \infty$ *and* $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^{t}_{\infty,\infty}(\Omega)$, *then* (2) *has a solution* $u \in \tilde{F}^{s}_{p,q}(\Omega)$.

Remark 1. Non-linearities of this type are usually denoted to be sublinear, cf. Fučík [5, Definition 7.4]. (10) shows that

$$\int_\Omega [\Phi(v + w) - f](x)\, w_0(x)\, dx > 0 .$$

In the sense of Fučík [5, Definition 11.1] every positive number $N > 0$ is a sub-asymptote of the operator $Su = \Phi(u) - f$ with respect to $\ker(\{A, B\})$, cf. also Fučík [5, Lemma 14.4].

Remark 2. An operator $A + \Phi$, where A has the meaning of (1) and Φ is of the type described in 3.2 and 3.3, respectively, is said to be an *asymptotic linear operator*, cf. Zeidler [21, 28]. The results obtained in 3.2 and 3.3 are examples of the so-called „Fredholm-alternative for non-linear operators", cf. Remark 3.2/1.

Remark 3. Results in this direction in the framework of Sobolev spaces may be found in Fučík [5, 14].

Remark 4. In Fučík [5, 14] the following estimate for Φ must be satisfied

$$|\Phi(t)| \leqq c_1(1 + |t|^\delta) , \quad c_1 > 0 , \quad \delta \in (0, 1) .$$

Remark 5. If $d = 0$ in Theorem 1 and Theorem 2, respectively then f has only to satisfy the following condition:

$$f \in \tilde{B}^{s-2}_{p,q}(\Omega) , \quad s > n/p ,$$

and

$$f \in \tilde{F}^{s-2}_{p,q}(\Omega) , \quad s > n/p ,$$

respectively.

3.4. Kazdan-Warner results. We present a generalization of S. Fučík [5, Chapter 34]. Let $\Omega \subset R_n$ be a bounded C^∞-domain and let A be a second order elliptic operator:

(1) $$A = - \sum_{1 \leqq |\alpha| \leqq 2} a_\alpha(x)\, D^\alpha ,$$

(2) $$a_\alpha \in \tilde{C}^\infty(\bar{\Omega}) \quad \text{(i.e., the } a_\alpha \text{ are real } C^\infty\text{-functions)} ,$$

(3) $$\sum_{|\alpha| = 2} a_\alpha(x)\, y^\alpha \geqq C|y|^2 > 0 , \quad y \in R_n , \quad y \neq 0 .$$

Let $\lambda_1 > 0$ be the smallest eigenvalue of the homogeneous Dirichlet problem $A \mid B^1_{2,0}(\Omega)$.

Denote by φ^* the (unique) positive eigenfunction to λ_1 with

(4) $$\int_\Omega \varphi^*(x)^2 \, dx = 1 ,$$

(5) $$\varphi^* \in \tilde{C}^\infty(\bar{\Omega}) , \quad \varphi^*(x) \geqq 0 \quad \text{if} \quad x \in \bar{\Omega} , \quad \varphi^* \mid \partial\Omega = 0 , \quad A\varphi^* = \lambda_1 \varphi^* .$$

Let $f \in \tilde{L}_\infty(\Omega)$, $g \in \tilde{C}^\infty(\bar{\Omega} \times R)$. We consider the problem

(6) $$Au - \lambda_1 u = f - g(\cdot, u(\cdot)) , \quad u \mid \partial\Omega = 0 .$$

The following conditions ensure the existence of u in the Hölder-space theory:

(A_+) There is a real number s_+ and a bounded C^∞-function $h_+ : \bar{\Omega} \times R \to R$ such that if $v \in \tilde{C}^\infty(\bar{\Omega})$, $v > s_+ \varphi^*$ in $\bar{\Omega}$, then

(7) $f(x) - g(x, v(x)) \leqq h_+(x, v(x))$, $x \in \bar{\Omega}$

and

(8) $\int_\Omega h_+(x, v(x)) \, \varphi^*(x) \, dx \leqq 0.$

(A_-) There is a real number s_- and a bounded C^∞-function $h_- : \bar{\Omega} \times R \to R$ with

(9) $f(x) - g(x, v(x)) \geqq h_-(x. v(x))$

and

(10) $\int_\Omega h_-(x, v(x)) \, \varphi^*(x) \, dx \geqq 0$ if $v \in \tilde{C}^\infty(\bar{\Omega})$, $v < s_- \varphi^*$ in $\bar{\Omega}$.

The following conditions ensure the validity of A_+ and A_-:

(B_+) There exists a $h_+ \in \tilde{C}^\infty(\bar{\Omega})$ with $\int_\Omega h_+(x) \, \varphi^*(x) \, dx < 0$ and $f(x) - \overline{\lim}_{t \to \infty} g(x, t) < h_+(x)$ uniformly in x.

(B_-) There exists a $h_- \in \tilde{C}^\infty(\bar{\Omega})$ with $\int_\Omega h_-(x) \, \varphi^*(x) \, dx > 0$ and $f(x) - \underline{\lim}_{t \to -\infty} g(x, t) > h_-(x)$ uniformly in x.

Theorem 1. *Let $0 < p \leqq \infty$, $0 < q \leqq \infty$, and $n/p < s < \infty$.*

(i) *If $f \in \tilde{B}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^{-1}_{\infty,1}(\Omega)$ and if (A_+) and (A_-) are satisfied, then (6) has a solution $u \in \tilde{B}^s_{p,q}(\Omega)$.*

(ii) *If $p < \infty$, $f \in \tilde{F}^{s-2}_{p,q}(\Omega) \cap \tilde{B}^{-1}_{\infty,1}(\Omega)$, and if ($A_+$) and ($A_-$) are both satisfied, then (6) has a solution $u \in \tilde{F}^s_{p,q}(\Omega)$.*

Examples. (cf. S. Fučík [5, 34.17]) Let $h \in \tilde{C}^\infty(\bar{\Omega})$, $h(x) \geqq 0$, $h \not\equiv 0$. Then $-\Delta u - \lambda_1 u = f(x) - h(x) \, e^{u(x)}$ in Ω, $u \mid \partial\Omega = 0$ is solvable if and only if

$$\int_\Omega f(x) \, \varphi^*(x) \, dx > 0 .$$

In order to prove Theorem 1 we need some results about subsolutions and supersolutions. We consider only $\tilde{F}^s_{p,q}$-spaces, the case of $\tilde{B}^s_{p,q}$ can be treated in a similar manner.

A distribution $f \in \tilde{D}'(\Omega)$ is said to be *non-negative* if and only if $f(\varphi) \geqq 0$ for any

$\varphi \in \tilde{C}_0^\infty(\Omega)$ with $\varphi \geqq 0$. The set of non-negative distributions is $\sigma(\tilde{D}'(\Omega), \tilde{C}^0_\infty(\Omega))$-closed. A function $u \in \tilde{C}(\bar{\Omega})$ is said to be a *subsolution* (*supersolution*) if (6) iff $Au - \lambda_1 u \leqq \leqq f - g(\cdot, u(\cdot))$ $(Au - \lambda_1 u \geqq f - g(\cdot, u(\cdot)))$ in the above sense and $u \mid \partial\Omega = 0$.

The following theorem is a generalization of S. Fučík [5, 34.7].

Theorem 2. *Let u_- be a subsolution of (6) and u_+ be a supersolution of (6), $u_-(x) \leqq u_+(x)$ in Ω. If $0 < p < \infty$, $0 < q \leqq \infty$, $s > n/p$, $u_+ \in \tilde{F}^s_{p,q}(\Omega)$, $u_- \in \tilde{F}^s_{p,q}(\Omega)$, and $f \in \tilde{F}^{s-2}_{p,q}(\Omega)$. Then there exists a solution u of (6) with $u_- \leqq u \leqq \leqq u_+$ and $u \in \tilde{F}^s_{p,q}(\Omega)$.*

Proof. The above conditions yield $u_\pm \in \tilde{C}(\bar{\Omega})$. Let $\omega > \lambda_1$ and

$$\omega - \frac{\partial g}{\partial t}(x, t) > 0$$

if $u_-(x) < t < u_+(x)$. Let T be the operator which assigns to each $u \in \tilde{C}(\bar{\Omega})$ the unique solution $v \in \bigcup_{\varepsilon > 0} \tilde{B}^\varepsilon_{\infty,\infty}(\Omega)$ of

$$Av + (\omega - \lambda_1)\, v = f(x) - g(x, u(x)) + \omega\, u(x)\,, \quad v \mid \partial\Omega = 0\,. \tag{11}$$

We put $u_k^{(+)} = T^k u_+$ and $u_k^{(-)} = T^k u_-$ $(T^1 = T,\ T^{k+1} = TT^k)$. The following lemma and the arguments in S. Fučík [5, 34.7] prove

$$u_-(x) \leqq u_1^{(-)}(x) \leqq u_2^{(-)}(x) \leqq \ldots \leqq u_2^{(+)}(x) \leqq u_1^{(+)}(x) \leqq u_+(x)\,. \tag{12}$$

Lemma 1. *Let $v \in \bigcup_{\varepsilon > 0} \tilde{B}^\varepsilon_{\infty,\infty}(\Omega)$ and $\lambda > 0$. If $v \mid \partial\Omega = 0$ and $Av + \lambda v \geqq 0$ in the distribution-sense* (cf. *the above remarks*), *then $v(x) \geqq 0$.*

From 2.2, 2.3, (11), and $u_\pm \in L_\infty$ we deduce

$$\|u_k^{(\pm)} \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq A + B\|u_k^{(\pm)} \mid \tilde{F}^{s-\varepsilon}_{p,q}(\Omega)\|\,. \tag{13}$$

An imbedding theorem and (12) yield

$$\|u_k^{(\pm)} \mid \tilde{F}^{-\varepsilon}_{p,q}(\Omega)\| \leqq C\,. \tag{14}$$

(13), (14), and the well-known inequality $\|f \mid \tilde{F}^{s-\varepsilon}_{p,q}\| \leqq \delta\|f \mid \tilde{F}^s_{p,q}\| + C_\delta\|f \mid \tilde{F}^{-\varepsilon}_{p,q}\|$ show

$$\|u_{k+1}^{(\pm)} \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq D + 1/2\|u_k^{(\pm)} \mid \tilde{F}^s_{p,q}(\Omega)\|\,. \tag{15}$$

If $M > \max(2D, \|u_+ \mid \tilde{F}^s_{p,q}(\Omega)\|, \|u_- \mid \tilde{F}^s_{p,q}(\Omega)\|)$, then induction yields

$$\|u_k^{(\pm)} \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq M\,. \tag{16}$$

There exist $u^{(\pm)} \in L_\infty$ with

$$u_k^{(\pm)} \to u^{(\pm)} \quad \text{if} \quad k \to \infty \quad \text{pointwise}\,. \tag{17}$$

Let S be the coretraction constructed in [3, 4.1]. We may suppose that supp Sf is uniformly bounded for all f. The construction of S yields

$$Su_k^{(\pm)} \to Su^{(\pm)} \quad \text{if} \quad k \to \infty \tag{18}$$

pointwise. Lebesgue's theorem proves that (18) holds also for the weak $\sigma(S(R_n),$

$S'(R_n)$)-topology. The Fatou-property [3, 2.6] yields

$$u^{(\pm)} \in \tilde{F}^s_{p,q}(\Omega) . \tag{19}$$

The $u^{(\pm)}$ are continuous, for $\tilde{F}^s_{p,q} \subset \tilde{C}$, and Dini's theorem proves that (17) holds in $\tilde{C}(\bar{\Omega})$. Now it is not hard to check that $u^{(\pm)}$ are solutions of (6).

Proof of Lemma 1. Let $w \in \tilde{B}^{\varepsilon-2}_{\infty,\infty}(\Omega)$, $\varepsilon > 0$, be non-negative. If $\psi \in \tilde{C}^\infty(\bar{\Omega})$, $\psi \mid \partial\Omega = 0$, then $\psi \in \tilde{B}^{2-\varepsilon}_{1,0}(\Omega)$. If ψ is non-negative, then the proof of Triebel [16, 3.4.3] proves that ψ can be approximated in $\tilde{B}^{2-\varepsilon}_{1,0}(\Omega)$ by non-negative $\tilde{C}^\infty_0$-functions. Thus $\psi(w)$ is well-defined (for the dual space of $\tilde{B}^{2-\varepsilon}_{1,0}(\Omega)$ is $\tilde{B}^{\varepsilon-2}_{\infty,\infty}(\Omega)$) and non-negative. Let

$$A^* = - \sum_{1 \leqq |\alpha| \leqq 2} (-1)^{|\alpha|} D^\alpha a_\alpha .$$

If $\varphi \in \tilde{C}^\infty_0(\Omega)$, then the unique solution ψ of $A^*\psi + (\omega - \lambda_1)\psi = \varphi$, $\psi \mid \partial\Omega = 0$ is non-negative if φ is. For, if $a \in \tilde{C}^\infty_0(\Omega)$ is non-negative, then there exists a non-negative $b \in \tilde{C}^\infty(\bar{\Omega})$ with $b \mid \partial\Omega = 0$, $Ab + (\omega - \lambda_1) b = a$ (cf. Fučík [5, 34.2]). Now

$$\int_\Omega \psi(x)\, a(x)\, dx = \int_\Omega \varphi(x)\, b(x)\, dx \geqq 0$$

proves the above statement.

Let v be the same as in the formulation of Lemma 1. Let φ be non-negative, $\varphi \in \tilde{C}^\infty_0(\Omega)$, $\varphi = A^*\psi + (\omega - \lambda_1)\psi$, $\psi \in \tilde{C}^\infty(\bar{\Omega})$, $\psi \mid \partial\Omega = 0$, and ψ non-negative. Then an easy limiting argument proves

$$\int_\Omega \varphi(x)\, v(x)\, dx = \int_\Omega \{(A^* + \omega - \lambda_1)\psi\}\, v(x)\, dx = \\ = \int_\Omega \psi(Av + (\omega - \lambda_1) v)\, dx \geqq 0 ,$$

which completes the proof of Lemma 1.

Proof of Theorem 1. Without loosing generality we suppose $s_- \leqq s_+$. We prove the existence of a subsolution u_- and of a supersolution u_+ of (6) with $u_- \leqq \leqq s_-\varphi^*$ and $u_+ \geqq s_+\varphi^*$ in Ω. Then an application of Theorem 2 completes the proof.

We prove the existence of u_+, the other part of the proof being almost the same. Let T be the operator assigning to $u \in \tilde{C}(\bar{\Omega})$ the unique solution v of

$$Av - \lambda_1 v = f - h_+(x, u(x)) - \varphi^* \langle \varphi^* ,\ f - h_+(x, u(x)) \rangle_{L_2} ,$$
$$\langle v, \varphi^* \rangle_{L_2} = 0 ,$$
$$v \mid \partial\Omega = 0 .$$

Then $\|Tu \mid \tilde{C}^1(\bar{\Omega})\| \leqq M$ (for h_+ is bounded and $f \in \tilde{B}^{-1}_{\infty,1}(\Omega)$), where M is independent of u. Therefore there exists a c with $Tu + c\varphi^* \geqq 0$ for all u (Fučík [5, 34.12]). We put

$$T_\lambda u = (s_+ + 1)\varphi^* + \lambda(Tu + c\varphi^*) , \quad 0 \leqq \lambda \leqq 1 . \tag{20}$$

Since h_+ is a bounded C^∞-function, there exist constants with

$$\|T_\lambda u \mid \tilde{C}(\bar{\Omega})\| \leqq A \tag{21}$$

if $0 \leqq \lambda \leqq 1$ and

$$\|T_\lambda u \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq B\|u \mid \tilde{F}^{s-\varepsilon}_{p,q}(\Omega)\| \tag{22}$$

if $u \in \tilde{F}^s_{p,q}(\Omega)$ and $\|u \mid \tilde{C}(\bar{\Omega})\| \leqq 2A$. Similar as in the proof of Theorem 2 we deduce

$$\|T_\lambda u \mid \tilde{F}^s_{p,q}(\Omega)\| \leqq D + 1/2\|u \mid \tilde{F}^s_{p,q}(\Omega)\| . \tag{23}$$

Let $G = \{u \in \tilde{F}^s_{p,q}(\Omega) \mid \|u \mid \tilde{F}^s_{p,q}(\bar{\Omega})\| < 3D,\ \|u \mid \tilde{C}(\bar{\Omega})\| < 2A,\ u > s_+\varphi^* \text{ in } \Omega\}$.

G is an open bounded subset of $\tilde{F}^s_{p,q}(\Omega)$, and T_λ is a compact homotopy. If $u \in G$ and $0 \leqq \lambda \leqq 1$, then

$$T_\lambda u \geqq (s_+ + 1)\,\varphi^* \quad \text{in} \quad \Omega\,,$$

$$\|T_\lambda u \mid \tilde{C}(\bar{\Omega})\| \leqq A\,,$$

and

$$\|T_\lambda u \mid \tilde{F}^s_{p,q}(\Omega)\| < 2{,}5D\,.$$

Thus $T_\lambda u \in G$, T_λ has not fixed point on ∂G.

Now an elementary Leray-Schauder argument (see 3.2) shows that T_1 has a fixed point u on G. Hence u is a supersolution of (6) and it holds $u \geqq s_+\varphi^*$ in Ω. The proof is complete.

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