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OPERATORS WHOSE TENSOR POWERS ARE ε-π-CONTINUOUS

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1. Introduction and preliminaries. Following [5, 6] we define $\mathcal{T}en_r$ as the class of all operatos S such that the r^{th} tensor power of S is continuous as an operator from ε- to π-tensor products.

In [7] we observed that the product of 13 operators from the class $\mathcal{T}en_3$ has absolutely summable eigenvalues. By a result of Pietsch [9] we then have

$$(\mathcal{T}en_3)^{13} \subset \mathcal{P}_2 , \tag{1}$$

where $\mathcal{P}_p$ denotes the ideal of absolutely p-summing operators equipped with the norm P_p. Nevertheless, it is possible to establish the relation of the class $\mathcal{T}en_3$ to the ideal of absolutely (p, q)-summing operators without passing through the eigenvalue estimates. In this paper we show that

$$(\mathcal{T}en_3)^l \subset \mathcal{L}^{(x)}_{6/(l-1),\infty} , \tag{2}$$

where the Schatten class $\mathcal{L}^{(x)}_{p,\infty}$ based on Weyl numbers is intimately connected with the ideal of absolutely $(p, 2)$-summing operators. In particular, from (2) we get

$$(\mathcal{T}en_3)^5 \subset \mathcal{P}_2 \tag{3}$$

which is better then (1).

The proof of (2) uses an estimate of Weyl numbers by Hilbert numbers due to B. Carl

$$x_n(S) \leqq n^{1/2}\Big(\prod_{i=1}^{n} h_i(S)\Big)^{1/n}$$

(cf. also Pietsch [10]). We are grateful to Bernd Carl for allowing us to bring here his proof and a remark on a certain optimality of this estimate.

Parallelly we also give the corresponding results for $\mathcal{T}en_4$, but we have not been able to get non-trivial results for $\mathcal{T}en_r$ where $r > 4$ (cf. Remark 16).

Definition. Let $S: E \to F$ be an operator between Banach spaces E, F. Let the operator

$$\otimes^r T = \underbrace{T \otimes T \otimes \ldots \otimes T}_{r-\text{times}} : \underbrace{E \otimes_\varepsilon E \otimes_\varepsilon \ldots \otimes_\varepsilon E}_{r-\text{times}} \to \underbrace{F \otimes_\pi F \otimes_\pi \ldots \otimes_\pi F}_{r-\text{times}}$$

be continuous. We will denote

$$\mathbf{Ten}_r(T) = (\|\otimes^r T\|)^{1/r}.$$

Furthermore, let us denote by $\mathcal{T}en_r(\boldsymbol{E}, \boldsymbol{F})$ the class of all operators T such that $\otimes^r T$ is continuous. As usual we put

$$\mathcal{T}en_r = \bigcup_{\boldsymbol{E},\boldsymbol{F}} \mathcal{T}en_r(\boldsymbol{E}, \boldsymbol{F}).$$

If $\mathcal{A}$, $\mathcal{B}$ are classes of operators then the class $\mathcal{A} \circ \mathcal{B}$ is formed by all possible compositions $A \circ B$, where $A \in B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, we put $\mathcal{A}^2 = \mathcal{A} \circ \mathcal{A}$ and $\mathcal{A}^{n+1} = \mathcal{A}^n \circ \mathcal{A}$ for all integers n.

If $p, q > 0$ and $a = (a_i)$ is a sequence of numbers then the quasi-norms $l_{p,q}$ are defined as usual by

$$l_{p,q}(a) = l_{p,q}(a_i) = \left(\sum_{i=1}^{\infty} i^{q/p-1} |a_i|^{*q}\right)^{1/q} \quad \text{if} \quad q < \infty$$

and

$$l_{p,\infty}(a) = l_{p,\infty}(a_i) = \sup_i i^{1/p} |a_i|^*,$$

where $|a_i|^*$ is the nonincreasing rearrangement of $|a_i|$. Further, for a sequence (x_i) of elements of a Banach space $\boldsymbol{E}$ we put

$$w_{p,q}(x_i) = \sup \{l_{p,q}(\langle a, x_i \rangle);\ a \in \boldsymbol{E}',\ \|a\| \leqq 1\}.$$

If $T \in \mathcal{L}(\boldsymbol{E}, \boldsymbol{F})$ is an arbitrary (continuous) operator, p, q, r positive numbers and n a natural number then we put

$$P^{(n)}_{(p,q,r)}(T) = \inf C,$$

where the infimum is over all constants C such that for all $(x_1, \ldots, x_n) \subset \boldsymbol{E}$ and all $(y'_1, \ldots, y'_n) \in \boldsymbol{F}'$ we have

$$l_p(\langle Tx_i, y'_i \rangle_{i=1}^n) \leqq C w_q((x_i)_{i=1}^n)\, w_r((y'_i)_{i=1}^n).$$

Evidently we have the absolutely (p, q, r)-summing quasi-norm $P_{(p,q,r)} = \sup_n P^{(n)}_{(p,q,r)}$.

By a_n, c_n and x_n we denote respectively the approximation, Gelfand and Weyl numbers of an operator $S: E \to F$ (cf. e.g. [10]). Thus $c_n(S) = \inf \{\|S/M\|;\ M \subset E,\ \operatorname{codim} M < n\}$ and

$$x_n(S) = \sup \{a_n(SX);\ X \in \mathcal{L}(l_2, E);\ \|X\| \leqq 1\}.$$

If x is the Weyl s-function then, as usual, we denote by $\mathcal{L}^{(x)}_{p,q}$ the class of operators which are of x-type $l_{p,q}$, i.e., the operators S such that $(x_n(S)) \in l_{p,q}$.

The following lemma is implicitly contained in [7].

Lemma 1. *Let $S_i \in \mathcal{L}(E_i, F_i)$, $i = 1, \ldots, l$ be continuous operators, let $Y_i = X_{i+1}$ for $i = 1, \ldots, l - 1$ so that the composition $S = S_1 \circ \ldots \circ S_l$ exists. Then for any $p > 0$, $q > 0$ and every natural number n we have*

a) $\max_{k \leqq n} (k^{1/p} x_k(S)) \leqq a_{l,p} \prod_{i=1}^{l} \max_{k \leqq n} (k^{1/lp} x_k(S_i)) \leqq a_{l,p} \prod_{i=1} P^{(n)}_{(lp,2)}(S_i)$,

b) $\max_{k \leq n} (k^{1/q-(1/2)} x_k(S)) \leq b_{l,q} \prod_{i=1}^{l} \max_{k \leq n} (k^{1/lq} x_k(S_i')) \leq b_{l,q} \prod_{i=1}^{l} P^{(n)}_{(lq,2)}(S_i')$,

where $a_{l,p}$, $b_{l,q}$ are numerical constants depending only on l, p and l, q, respectively $(a_{l,p} \leq (l+1)^{1/p},\ b_{l,q} \leq a_{l,q} \mathrm{e}^{1/q})$.

Proof. a) The multiplicativity of the Weyl numbers gives

$$x_{kl-l+1}(S) \leq x_k(S_1)\, x_{k(l-1)-l+2}(S_{l-1} \circ \ldots \circ S_1) \leq \ldots \leq \prod_{i=1}^{l} x_k(S_i)\,.$$

This in turn implies (for l fixed):

$$\max_{k \leq n} (k^{1/p} x_k(S)) \leq \max_{kl-l+1 \leq n} (kl+1)^{1/p} x_{kl-l+1}(S) \leq$$

$$\leq (l+1)^{1/p} \max_{kl-l+1 \leq n} k^{1/p} \prod_{i=1}^{l} x_k(S_i) \leq$$

$$\leq (l+1)^{1/p} \prod_{i=1}^{l} \max_{k \leq [(n-1)/l]+1} k^{1/lp} x_k(S_i) \leq$$

$$\leq (l+1)^{1/p} \prod_{i=1}^{l} \max_{k \leq n} k^{1/lp} x_k(S_i)\,.$$

The second inequality in a) is an immediate consequence of the general inequality holding for all $p > 0$ and for an arbitrary continuous operator S:

$$\max_{k \leq n} (k^{1/p} x_k(S)) \leq P^{(n)}_{(p,2)}(S) \quad \text{for all integers} \quad n \geq 1\,.$$

(Cf. [7, Prop. 3], which is in fact [10, Lemma 8].)

b) From (17) and the complete symmetry of the Hilbert numbers we have

$$\max_{k \leq n} (k^{1/q-(1/2)} x_k(S)) \leq \mathrm{e}^{1/q} \max_{k \leq n} (k^{1/q} h_k(S)) =$$

$$= \mathrm{e}^{1/q} \max_{k \leq n} (k^{1/q} h_k(S')) \leq \mathrm{e}^{1/q} \max_{k \leq n} (k^{1/q} x_k(S_1' \ldots S_1'))\,.$$

Application of a) now yields b).

As in [7] we use a result from [6]:

Lemma 2. *Let $r \geq 2$ be an integer. Then for every operator S and every natural number n we have*

$$\prod_{i=1}^{r} P^{(n)}_{(1,q_i,r_i)}(S) \leq n^{r-1}(\mathbf{Ten}_r(S)^r)\,, \quad \textit{where} \quad 0 < p_i\,, \quad q_i \leq \infty$$

and
$$\sum_{i=1}^{r} 1/p_i = \sum_{i=1}^{r} 1/q_i = 1\,.$$

2. The Weyl numbers of operators from $(\mathscr{T}\!en_r)^l$.

Theorem 1.

a) $(\mathscr{T}\!en_3)^l \subset \mathscr{L}^{(x)}_{6/(l-1),\infty}$,

b) $(\mathscr{T}\!en_4)^l \subset \mathscr{L}^{(x)}_{4/(l-1),\infty}$.

Proof. a) We will use Lemma 1 in the form

$$(P^{(n)}_{(1,2,\infty)}(S))^2\, P^{(n)}_{(1,\infty,1)}(S) \leqq n^3\, \mathbf{Ten}_3^3(S)$$

i.e., because $P_2 \leqq P_1$,

$$(1) \qquad (P^{(n)}_{(1,2)}(S))^2\, P^{(n)}_{(2,2)}(S') \leqq n^2\, \mathbf{Ten}_3^3(S) \quad \text{for all} \quad S \in \mathscr{T}en_3\,.$$

By Lemma 1 a), b) we have for $S = S_1 \circ \ldots \circ S_l$

$$(2) \qquad (n^{1/p}\, x_n(S))^2\, n^{1/q-(1/2)}\, x_n(S) \leqq c^3_{p,q,l} \prod_{i=1}^{l} (P^{(n)}_{(lp,2)}(S_i))^2\, P^{(n)}_{(lq,2)}(S_i')\,.$$

Now if $S_i \in \mathscr{T}en_3$ and if

$$(3) \qquad lp = 1\,, \quad lq = 2$$

then (1) yields

$$\prod_{i=1}^{l} (P^{(n)}_{(lp,2)}(S_i))^2\, P^{(n)}_{(lq,2)}(S_i') \leqq n^{2l} \prod_{i=1}^{l} \mathbf{Ten}_3^3(S_i)\,.$$

Together with (2) this implies that for every natural n we have

$$(x_n(S))^3\, n^{2/p+1/q-1/2-2l} \leqq c^3_{p,q,l} \prod_{i=1}^{l} \mathbf{Ten}_3^3(S_i)\,.$$

Evidently $2/p + 1/q - 2l = 1/2\, l$ and thus finally

$$x_n(S)\, n^{l-1/6} \leqq c_{p,q,l} \prod_{i=1}^{l} \mathbf{Ten}_3(S_i)$$

b) is proved similarly; (1), (2) and (3) are substituted by

$$(1') \qquad (P^{(n)}_{(1,2)}(S)\, P^{(n)}_{(1,2)}(S'))^2 \leqq n^3\, \mathbf{Ten}_4^4(S) \quad \text{for all} \quad S \in \mathscr{T}en_4\,,$$

$$(2') \qquad n^{1/p}\, x_n(S)\, n^{1/q-(1/2)}\, x_n(S) \leqq d^2 \prod_{i=1}^{l} P^{(n)}_{(lp,2)}(S_i)\, P^{(n)}_{(lq,2)}(S_i')\,,$$

$$(3') \qquad lp = lq = 1\,.$$

This finally yields

$$(x_n(S))^4\, n^{2/p+2/q-1-3l} \leqq d^4 \prod_{i=1}^{l} \mathbf{Ten}_4^4(S_i)$$

or

$$x_n(S)\, n^{l-1/4} \leqq d \prod_{i=1}^{l} \mathbf{Ten}_4(S_i)\,.$$

Remark 1. a) The multicativity of the Weyl numbers implies (cf. [10]) that

$$(\mathscr{L}^{(x)}_{6,\infty})^l \subset \mathscr{L}^{(x)}_{6/l,\infty}$$

and thus the special case of a) (for $l = 2$) gives

$$(\mathscr{T}en_3)^{2l} \subset (\mathscr{L}^{(x)}_{6,\infty})^l \subset \mathscr{L}^{(x)}_{6/l,\infty}\,,$$

which is worse than the result a).

b) $\mathscr{L}^{(x)}_{4/(l-1),\infty}$ is (strictly) contained in $\mathscr{L}^{(x)}_{6/(l-1),\infty}$. This is in accordance with the inclusion $\mathscr{T}en_4 \subset \mathscr{T}en_3$. Generally we have $\mathscr{T}en_{r+1} \subset \mathscr{T}en_r$ and we may expect better results with increasing r. Nevertheless, the method does not yield better estimates for $r>4$. (The reason is that (1) must be replaced by an inequality where absolutely $P^{(n)}_{(1,q)}$-summing norms ($q > 2$) occur. Weyl numbers of such operators are not better then those of absolutely $P^{(n)}_{(1,2)}$-summing ones.) Thus we have only

$$(\mathscr{T}en_r)^l \subset (\mathscr{T}en_4)^l \subset \mathscr{L}^{(x)}_{4/(l-1),\infty} \quad \text{for} \quad r \geqq 4 .$$

c) The method applied for $r = 2$ yields only that the Weyl numbers of operators from $(\mathscr{T}en_2)^l$ are bounded, which is automatically always the case. This is in accordance with the example of the Pisier's space E such that $\mathrm{Id}_E \in \mathscr{T}en_2$.

3. The eigenvalues of operators from $(\mathscr{T}en_r)^l$. By a result of Pietsch [10] the eigenvalues of operators from $\mathscr{L}^{(x)}_{p,q}$ are of the same order, i.e. belong to $l_{p,q}$. Thus Theorem 1 immediately implies that the eigenvalues of an operator $S \in (\mathscr{T}en_3)^l$ belong to $l_{6/l-1,\infty}$. Similarly for $(\mathscr{T}en_4)^l$. Here we prove a slightly better result. The reason is that the eigenvalues of Riesz operators S and S' coincide and we need not use Lemma 1 b) where $1/2$ in the exponent was lost.

Theorem 2. a) *The eigenvalues of operators* $S \in (\mathscr{T}en_3)^l$ *belong to* $l_{6/l,\infty}$ *if* $l \geqq 2$. b) *The eigenvalues of operators* $S \in (\mathscr{T}en_4)^l$ *belong to* $l_{4/l,\infty}$ *if* $l \geqq 1$.

Proof. The proof of a) is almost the same as in [7], only instead of starting from the inequality (2) in [7] we start now from our inequality (1). We show that any operator $S \in (\mathscr{T}en_3)^l$ is of Weyl type $l_{6,l,\infty}$ (cf. [9]). This means that if $S \in (\mathscr{T}en_3)^l$ (E_1, E_{l+1}) and $L \in \mathscr{L}(E_{l+1}, E_1)$ then LS is a Riesz operator and its eigenvalues λ_i belong to the Lorentz sequence space $l_{6/l,\infty}$.

Thus let $S_i \in \mathscr{T}en_3(E_i, E_{i+1})$ for $i = 1, \ldots, l$, let $L \in \mathscr{L}(E_{l+1}, E_1)$ and put $S = S_l \circ \ldots \circ S_1$. If $l \geqq 2$ then by [7, Proposition 2] $T = LS$ and $T' = S'L'$ are Riesz operators. Let us denote by $\{\lambda_k(T)\}$ the sequence of all non-zero eigenvalues of T, every eigenvalue counted according to its algebraic multiplicity. Moreover, we may suppose that $|\lambda_1(T)| \geqq |\lambda_2(T)| \geqq \ldots$. We know (cf. [8]) that

$$\lambda_k = |\lambda_k(T)| = |\lambda_k(T')| . \tag{4}$$

Now let us choose a natural number n. We may suppose that $\lambda_n \neq 0$. According to [10, Lemma 12] there exists an n-dimensional T-invariant subspace $L_n \subset E_1$ such that the operator $T_n \in \mathscr{L}(L_n, L_n)$ induced in L_n by the operator T has the eigenvalues $\lambda_1(T), \ldots \lambda_n(T)$, i.e.

$$|\lambda_k(T_n)| = |\lambda_k(T)| = \lambda_k \quad \text{for} \quad k = 1, \ldots, n . \tag{5}$$

Let us write $\lambda_m(T_n) = 0$ for all $m > n$. By [10, Lemmas 1 and 13] we know that

$$l_{p,\infty}(\lambda_k(S)) \leqq c_p l_{p,\infty}(x_k(S)) \tag{6}$$

for any Riesz operator S, where the numerical constant c_p depends only on p. Combining (6) with Lemma 1 a) for the operator T_n in L_n, we obtain

$$\max_{k \leqq n} (k^{1/p} \lambda_k(T_n)) \leqq c_p \max_{k \leqq n} (k^{1/p} x_k(T_n)) \leqq \tag{7}$$

$$\leqq c_p \max_{k \leqq n} (k^{1/p} x_k(T)) \leqq c_p a_{l,p} \|L\| \prod_{i=1}^{l} P^{(n)}_{(lp,2)}(S_i) .$$

Here we used the fact that by injectivity of the Weyl numbers

$$x_k(T_n\colon L_n \to L_n) = x_k(T_n\colon L_n \to E_1) = x_k(TI_{L_n}) \leqq x_k(T)\,,$$

(7) and (5) now imply

$$(8)\qquad n^{1/p}\lambda_n \leqq c_p a_{l,p}\|L\| \prod_{i=1}^{l} P^{(n)}_{(lp,2)}(S_i)\,.$$

Similarly, using (4) we get

$$(9)\qquad n^{1/q}\lambda_n \leqq c_q a_{l,q}\|L\| \prod_{i=1}^{l} P^{(n)}_{(lq,2)}(S'_i)\,.$$

If we choose p, q as in (3) then (8), (9) and (1) yield

$$(10)\qquad (n^{1/p}\lambda_n)^2\,(n^{1/q}\lambda_n) \leqq e^3_{p,q,l}\|L\|^3 \prod_{i=1}^{l} (P^{(n)}_{(lp,2)}(S_i))^2\, P^{(n)}_{(lq,2)}(S'_i) \leqq$$

$$\leqq e^3_{p,q,l}\|L\|^3\, n^{2l} \prod_{i=1}^{l} \mathbf{Ten}^3_3(S_i)\,,$$

i.e.

$$\lambda_n^3 n^{2/p+1/q-2l} \leqq e^3_{p,q,l}\|L\|^3 \prod_{i=1}^{l} \mathbf{Ten}^3_3(S_i)$$

or

$$\lambda_n n^{l/6} \leqq e_{p,q,l}\|L\| \prod_{i=1}^{l} \mathbf{Ten}_3(S_i)\,.$$

Thus $\lambda_n \in l_{6/l,\infty}$ if $S_i \in \mathcal{T}en_3$.

To show b) we again replace (1) by (1′), (3) by (3′) and thus (10) is replaced by

$$(n^{1/p}\lambda_n)^2\,(n^{1/q}\lambda_n)^2 \leqq f^4_{p,q,l} n^{3l}\|L\|^4 \prod_{i=1}^{l} \mathbf{Ten}^4_4(S_i)$$

i.e.

$$\lambda_n^4 n^{4/p-3l} \leqq f^4_{p,q,l}\|L\|^4 \prod_{i=1}^{l} \mathbf{Ten}^4_4(S_i)$$

or

$$\lambda_n n^{l/4} \leqq f_{p,q,l}\|L\| \prod_{i=1}^{l} \mathbf{Ten}_4(S_i)\,.$$

Remark 2. Note that by [7, Remark 1] every $T \in \mathcal{T}en_4$ is a Riesz operator and thus b) holds even for $l = 1$, i.e. we have that every operator $T \in \mathcal{T}en_4$ is of Weyl type $l_{4,\infty}$. In the case a) we do not know whether the operators $T \in \mathcal{T}en_3$ are Riesz operators. The eigenvalues in the statement a) in the case $l = 1$ should then be understood as eigenvalues $\{\lambda_n(T)\}$ lying in the Riesz part of the spectra of T (cf. [12]).

4. An inequality between Weyl and Hilbert numbers. A. Pietsch has shown that for $S \in \mathcal{L}(E, F)$,

$$x_{2n-1}(S) \leqq n^{1/2}\Big(\prod_{k=1}^{n} h_k(S)\Big)^{1/n}\,.$$

This is sufficient to deduce (17), which we used in the proof of Lemma 1 b). The following unpublished result of B. Carl is slightly better:

Theorem 3. (Carl, unpublished). *Let* $S \in \mathscr{L}(E, F)$. *Then*

$$x_n(S) \leqq n^{1/2}\Big(\prod_{k=1}^{n} h_k(S)\Big)^{1/n}. \tag{11}$$

Proof. First we show the inequality (14) below, contained in [10, Lemma 10]. We follow the proof of [10, Lemma 10]. Given $\varepsilon > 0$, we inductively choose $x_1, x_2, \ldots \in E$ and $b_1, b_2, \ldots \in F'$ such that $\|x_i\| \leqq 1$, $\|b_j\| \leqq 1$, $\langle Sx_i, b_j\rangle = 0$ for $i > j$, and $(1 + \varepsilon)\,|\langle Sx_k, b_k\rangle| \geqq c_k(S)$. (See also [1, Lemma 6 (ii)] for explicit formulation.)

Now we define operators $X_n \in \mathscr{L}(l_2^n, E)$ and $B_n \in \mathscr{L}(F, l_2^n)$ by

$$X_n(\xi_i) = \sum_{i=1}^{n} \xi_i x_i \quad \text{and} \quad B_n y = (\langle y, b_j\rangle).$$

Factoring the operator B_n through l_∞^n and using the fact that $P_2(I\colon l_\infty^n \to l_2^n) = \sqrt{n}$ (cf. [8, 22.4.9]) we get

$$\|B_n\| \leqq P_2(B_n) \leqq n^{1/2}. \tag{12}$$

Similarly we have

$$P_2(X_n') \leqq n^{1/2}. \tag{13}$$

Since $B_n S X_n\colon l_2^n \to l_2^n$ is generated by the triangle matrix $(\langle Sx_i, b_j\rangle)$, it follows from [10, Lemma 4] that

$$\prod_{k=1}^{n} |\langle Sx_k, b_k\rangle| = |\det(B_n S X_n)| \leqq \prod_{k=1}^{n} h_k(B_n S X_n).$$

Thus we have

$$\prod_{k=1}^{n} c_k(S\colon E \to F) \leqq (1 + \varepsilon)^n \prod_{k=1}^{n} h_k(B_n, S X_n). \tag{14}$$

For the next step we will suppose that $S \in \mathscr{L}(l_2, F)$. Since then X_n, $B_n S$, $B_n S X_n$ are operators between Hilbert spaces, we have (cf. [3])

$$\prod_{k=1}^{n} h_k(B_n S X_n) \leqq \prod_{k=1}^{n} h_k(B_n S) \prod_{k=1}^{n} h_k(X_n). \tag{15}$$

Now the inequality between the geometric and algebraic mean and (13) yield

$$\prod_{k=1}^{n} h_k(X_n) \leqq \left(\frac{\sum_{k=1}^{n} h_k^2(X_n)}{n}\right)^{n/2} \leqq \left(\frac{P_2^2(X_n)}{n}\right)^{n/2} \leqq 1. \tag{16}$$

Here we use the well known fact that for an operator X_n in a Hilbert space we have (cf. e.g. [8])

$$\sum_k h_k^2(X_n) = P_2^2(X_n) = P_2^2(X_n').$$

From (12), (14), (15) and (16) we get

$$\prod_{k=1}^{n} c_k(S: l_2 \to F) \leqq (1 + \varepsilon)^n \, n^{n/2} \prod_{k=1}^{n} h_k(S: l_2 \to F)\,.$$

Letting $\varepsilon \to 0$ we obtain

$$c_n(S: l_2 \to F) \leqq \Big(\prod_{k=1}^{n} c_k(S: l_2 \to F)\Big)^{1/n} \leqq n^{1/2}\Big(\prod_{k=1}^{n} h_k(S: l_2 \to F)\Big)^{1/n}\,.$$

Finally, let us consider a general operator $S \in \mathscr{L}(E, F)$ and $X \in \mathscr{L}(l_2, E)$, $\|X\| \leqq 1$. Then by what we have just shown we have

$$a_n(SX) = c_n(SX) \leqq n^{1/2}\Big(\prod_{k=1}^{n} h_k(SX)\Big)^{1/n} \leqq n^{1/2}\Big(\prod_{k=1}^{n} h_k(S)\Big)^{1/n}\,.$$

The definition of the Weyl numbers now yields (11).

Corollary (cf. [10]). *Let $S \in \mathscr{L}(E, F)$ and let $\alpha > 0$. Then*

$$\sup_{1 \leqq k \leqq n} k^{\alpha - 1/2}\, x_k(S) \leqq \mathrm{e}^{\alpha} \sup_{1 \leqq k \leqq n} k^{\alpha}\, h_k(S) \tag{17}$$

for all natural numbers n.

The proof follows that of [10, Lemma 1, Case (3)]. We will supply it here for the convenience of the reader. (11) implies that

$$k^{\alpha - 1/2}\, x_k(S) \leqq k^{\alpha}\Big(\prod_{i=1}^{k} h_i(S)\Big)^{1/k} \quad \text{for all} \quad k\,.$$

The inequality $\mathrm{e}^k \geqq k^k/k!$ gives $k \leqq \mathrm{e}(k!)^{1/k}$ and thus if $k \leqq n$ we obtain

$$k^{\alpha - 1/2}\, x_k(S) \leqq \mathrm{e}^{\alpha}\Big(\prod_{i=1}^{k} i^{\alpha}\, h_i(S)\Big)^{1/k} \leqq \mathrm{e}^{\alpha} \sup_{1 \leqq i \leqq k} i^{\alpha}\, h_i(S) \leqq \mathrm{e}^{\alpha} \sup_{1 \leqq i \leqq n} i^{\alpha}\, h_i(S)\,,$$

which proves (17).

The estimate (11) is optimal in the following sense.

Remark 3 (Carl, unpublished). Let $\alpha > 0$ and $\varrho > 0$ be such that

$$x_n(S) \leqq \varrho n^{\alpha}\Big(\prod_{k=1}^{n} h_k(S)\Big)^{1/n} \tag{18}$$

for all n, all $S \in \mathscr{L}(E, F)$ and all Banach spaces E and F. Then $\alpha \geqq 1/2$.

Indeed, consider $E = l_2$, $F = c_0$ and $S =$ the identity imbedding $I: l_2 \to c_0$. Then we show below that

$$a_n(I: l_2 \to c_0) = 1\,, \tag{19}$$

$$h_n(I: l_2 \to c_0) = n^{-1/2}\,. \tag{20}$$

Substitution into (18) implies

$$1 = x_n(I) \leqq \varrho n^{\alpha}\left(\frac{1}{n!}\right)^{1/2n} \leqq \varrho \mathrm{e}^{1/2} n^{\alpha - 1/2}\,,$$

because $n^n/n! \leqq \mathrm{e}^n$. This is possible for all n only if $\alpha - 1/2 \geqq 0$.

To show (19) we observe that using [8, 11.11.3] we have

$$a_n(I: l_2 \to c_0) \leqq \|I: l_2 \to c_0\| \, a_n(I: l_2 \to l_2) \leqq 1 .$$

On the other hand, evidently $a_n(I) \geqq c_n(I) = 1$. To show (20) we observe that by [10, Lemma 8] we have $n^{1/2} x_n(S) \leqq P_2(S)$ and thus

$$x_n(I: l_1 \to l_2) \leqq n^{-1/2} P_2(I: l_1 \to l_2) \leqq n^{-1/2} \tag{21}$$

because $P_2(I: l_1 \to l_2) = 1$. ([8]).

The complete symmetry of the Hilbert numbers and (21) yield

$$h_n(I: l_2 \to c_0) = h_n(I: l_1 \to l_2) \leqq x_n(I: l_1 \to l_2) \leqq n^{-1/2} . \tag{22}$$

Let us now consider the following canonical factorization of the identity $I_n: l_2^n \to l_2^n$:

$$I_n = (Q_n: l_2 \to l_2^n)(I: l_1 \to l_2)(J_n: l_1^n \to l_1)(I: l_2^n \to l_1^n) ,$$

where I is the identity and Q_n the canonical projection. We have

$$1 = h_n(I_n) \leqq \|I: l_2^n \to l_1^n\| \, \|J_n: l_1^n \to l_1\| \, \|Q_n: l_2 \to l_2^n\| \, h_n(I: l_1 \to l_2) .$$

However,

$$\|I: l_2^n \to l_1^n\| \leqq \sqrt{n} \quad \text{and} \quad \|J_n\| = \|Q_n\| = 1 .$$

Thus $h_n(I: l_1 \to l_2) \geqq n^{-1/2}$. This together with (22) yields (20).

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