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ON INTEGRATION IN BANACH SPACES, IX (INTEGRATION WITH RESPECT TO POLYMEASURES)

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INTRODUCTION

In this part we start our integration theory of d-tuples of functions with respect to an operator valued d-polymeasure which is separately countably additive in the strong operator topology, see part VIII = [13]. More precisely, we give a generalization of the main results of part I = [5] (except Theorem I.17, whose generalization will be included in the forthcoming part X), and of Theorem II.17, i.e., of the Lebesgue dominated convergence theorem in $L_1(m)$.

Since the problem on existence of control d-polymeasures for a vector d-polymeasure, see the Problem in Section 3 in part VIII, has not yet been solved, we are forced in general to define all integrable d-tuples of functions by transfinite induction, see the paragraph before Definition 2 below.

Our approach to integration with respect to polymeasures differs from that of K. Ylinen [22], I. Kluvánek [20], M. M. Rao [21], Chang and Rao [3], and of A. K. Katsaras [19]. In the subsequent parts we will clarify the connections.

We will use the notation and concepts from the previous part VIII. (Previous parts are treated as chapters when referred to.) In particular, we will use the abbreviated symbols $(A_i) = (A_1, \ldots, A_d)$, $\mathsf{X}\mathscr{P}_i = \mathscr{P}_1 \times \ldots \times \mathscr{P}_d$, and $L^{(d)}(X_i; Y) = L^{(d)}(X_1, \ldots, X_d: Y)$.

INTEGRABLE d-TUPLES OF FUNCTIONS

In what follows we assume that $\Gamma: \mathsf{X}\mathscr{P}_i \to L^{(d)}(X_i; Y)$ is an operator valued d-polymeasure separately countably additive in the strong operator topology, see Definition 1 in part VIII = [13]. According to Definition VIII.5 we say that the semivariation $\hat{\Gamma}: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$, see Definition VIII.3, is locally σ-finite on $\mathsf{X}\sigma(\mathscr{P}_i)$ if for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ there are δ-rings $\mathscr{P}_{i,(A_i)} \subset \sigma(\mathscr{P}_i) \cap A_i$, $i = 1, 2, \ldots, d$, such that $\sigma(\mathscr{P}_{i,(A_i)}) = \sigma(\mathscr{P}_i) \cap A_i$ for each $i = 1, \ldots, d$ and the semivariation $\hat{\Gamma}$ is finite valued on $\mathsf{X}\mathscr{P}_{i,(A_i)}$. Similarly we introduce the local σ-finiteness of the semivarion $\hat{\Gamma}$ on $\mathsf{X}\mathscr{P}_i$. Unfortunately, the author has been unable to find an example in which $\hat{\Gamma}$ is locally σ-finite on $\mathsf{X}\mathscr{P}_i$, but not locally σ-finite on $\mathsf{X}\sigma(\mathscr{P}_i)$.

The following simple facts are important:

Lemma 1. *The following conditions are equivalent*:

(i) *The semivariation* $\hat{\Gamma}: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ *is locally σ-finite on* $\mathsf{X}\sigma(\mathscr{P}_i)$,

(ii) *For each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *there are δ-rings* $\mathscr{P}_{i,(A_i)} \subset \mathscr{P}_i \cap A_i$, $i = 1, \ldots, d$, *such that* $\sigma(\mathscr{P}_{i,(A_i)}) = \sigma(\mathscr{P}_i) \cap A_i$ *for each* $i = 1, \ldots, d$, *and* $\hat{\Gamma}$ *is finite valued on* $\mathsf{X}\mathscr{P}_{i,(A_i)}$.

(iii) *For each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *there are* $(A_{i,n}) \in \mathsf{X}\mathscr{P}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$, *such that* $A_{i,n} \nearrow A_i$ *for each* $i = 1, \ldots, d$, *and* $\hat{\Gamma}(A_{i,n}) < +\infty$ *for each* $n = 1, 2, \ldots$.

Proof. (i) $\Rightarrow$ (ii). Let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Take δ-rings $\mathscr{P}'_{i,(A_i)} \subset \sigma(\mathscr{P}_i) \cap A_i$, $i = 1, \ldots, d$, such that $\sigma(\mathscr{P}'_{i,(A_i)}) = \sigma(\mathscr{P}_i) \cap A_i$ for each $i = 1, \ldots, d$ and the semivariation $\hat{\Gamma}$ is finite valued on $\mathsf{X}\mathscr{P}'_{i,(A_i)}$. Clearly, the δ-rings $\mathscr{P}_{i,(A_i)} = \bigcup_{P_i \in \mathscr{P}_i} (\mathscr{P}'_{i,(A_i)} \cap P_i)$, $i = 1, \ldots, d$, satisfy the requirements in (ii).

Obviously (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). Let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Take $(A_{i,n}) \in \mathsf{X}\mathscr{P}_i$, $n = 1, 2, \ldots$ according to (iii), and put $\mathscr{P}_{i,(A_i)} = \bigcup_{n=1}^{\infty} A_{i,n} \cap \mathscr{P}_i$.

If not otherwise specified, in what follows we assume that the semivation $\hat{\Gamma}$ is locally σ-finite on $\mathsf{X}\sigma(\mathscr{P}_i)$.

Lemma 2. *Let* $f_i: T_i \to X_i$ (*or* $f_i: T_i \to [0, +\infty)$) *be* $\mathscr{P}_i$*-measurable for each* $i = 1, \ldots, d$. *Then*

1) $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ *is σ-finite*;

2) *without assuming the local σ-finiteness of the semivariation* $\hat{\Gamma}$ *on* $\mathsf{X}\sigma(\mathscr{P}_i)$, *if* $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ *is locally σ-finite, then there are δ-rings* $\mathscr{P}_{i,(f_i)} \subset \mathscr{P}_i$, $i = 1, \ldots, d$, *such that* $\hat{\Gamma}$ *is finite valued on* $\mathsf{X}\mathscr{P}_{i,(f_i)}$, *and for any* $i = 1, \ldots, d$ *each* $\mathscr{P}_i$*-measurable function* $g_i: T_i \to X_i$ (*or* $g_i: T_i \to [0, +\infty)$) *with* $\{t_i \in T_i, g_i(t_i) \neq 0\} \subset \{t_i \in T_i, f_i(t_i) \neq 0\}$ *is* $\mathscr{P}_{i,(f_i)}$*-measurable*; *and*

3) *if* $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ *is separately continuous* (*hence bounded by Theorem* VIII.6), *then there are δ-rings* $\widetilde{\mathscr{P}}_{i,(f_i)} \subset \mathscr{P}_i$, $i = 1, \ldots, d$, *such that* $\hat{\Gamma}$ *is separately continuous on* $\mathsf{X}\widetilde{\mathscr{P}}_{i,(f_i)}$, *and for any* $i = 1, \ldots, d$ *each* $\mathscr{P}_i$*-measurable function* $g_i: T_i \to X_i$ *or* ($g_i: T_i \to [0, +\infty)$) *with* $\{t_i \in T_i, g_i(t_i) \neq 0\} \subset \{t_i \in T_i, f_i(t_i) \neq 0\}$ *is* $\widetilde{\mathscr{P}}_{i,(f_i)}$*-measurable.*

Proof. For $i = 1, \ldots, d$ put $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$.

1) Since $\hat{\Gamma}: \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty]$ is σ-finite by assumption, by (ii) of Lemma 1 there are σ-rings $\mathscr{P}_{i,(F_i)} \subset \mathscr{P}_i$, $i = 1, \ldots, d$, such that $\sigma(\mathscr{P}_{i,(F_i)}) = \sigma(\mathscr{P}_i) \cap F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}$ is finite valued on $\mathsf{X}\mathscr{P}_{i,(F_i)}$. Put $\mathscr{P}'_i = \bigcup_{k=1}^{\infty} \{t_i \in T_i, 0 < |f_i(t_i)| < k\} \cap \mathscr{P}_i$, and let $\mathscr{P}^*_i = \mathscr{P}_{i,(F_i)} \cap \mathscr{P}'_i$, $i = 1, \ldots, d$. Then clearly $\sigma(\mathscr{P}^*_i) = \sigma(\mathscr{P}_i) \cap F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}[(f_i), (\cdot)]$ is finite valued on $\mathsf{X}\mathscr{P}^*_i$.

2) Since $\hat{\Gamma}[(f_i), (\cdot)]$: $\mathsf{X}(F_i \cap (\mathscr{P}_i)) \to [0, +\infty]$ is σ-finite, there are δ-rings $\mathscr{P}'_i \subset \sigma(\mathscr{P}_i) \cap F_i$, $i = 1, \dots, d$, such that $\sigma(\mathscr{P}'_i) = \sigma(\mathscr{P}_i) \cap F_i$ for each i and $\hat{\Gamma}[(f_i), (\cdot)]$ is finite valued on $\mathsf{X}\mathscr{P}'_i$. Clearly f_i is $\mathscr{P}'_i$-measurable for each $i = 1, \dots, d$. Put $\mathscr{P}''_i = \bigcup_{k=1}^{\infty} \{t_i \in T_i, |f_i(t_i)| > 1/k\} \cap \mathscr{P}'_i$, $i = 1, \dots, d$. Then obviously $\mathscr{P}''_i$ is a δ-ring such that $\sigma(\mathscr{P}''_i) = \sigma(\mathscr{P}'_i)$, and the semivariation $\hat{\Gamma}$ is finite valued on $\mathsf{X}\mathscr{P}''_i$ by the Tschebyscheff inequality, see Theorem VIII.3–7). Since $A_i \cap A'_i \in \mathscr{P}'_i$ and $A_i \cap A''_i \in \mathscr{P}''_i$ for any $A_i \in \mathscr{P}_i$, $A'_i \in \mathscr{P}'_i$ and $A''_i \in \mathscr{P}''_i$, $i = 1, \dots, d$, $\mathscr{P}''_i$-measurable functions g_i: $T_i \to X_i$ (or to $[0, +\infty)$) are exactly those $\mathscr{P}_i$-measurable functions g_i for which $\{t_i \in T_i, g_i(t_i) \neq 0\} \subset F_i$.

3) may be proved similarly as 2).

Theorem 1. *Let* $(f_{i,n}) \in \mathscr{S}_0 = \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \dots$, *and let* $f_{i,n}(t_i) \to f_i(t_i) \in X_i$ *for each* $i = 1, \dots, d$ *and each* $t_i \in T_i$. *Then the following conditions are equivalent*:

a) *the vector d-polymeasures* γ_n, $\gamma_n(A_i) = \int_{(A_i)} (f_{i,n})\, \mathrm{d}\Gamma$, $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, $n = 1, 2, \dots$ *are separately uniformly countably additive on* $\mathsf{X}\sigma(\mathscr{P}_i)$, *and*

b) $\lim_{n\to\infty} \int_{(A_i)} (f_{i,n})\, \mathrm{d}\Gamma \in Y$ *exists for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. b) $\Rightarrow$ a) by the (VHSN)-theorem for polymeasures, see the beginning of part VIII = [13].

a) $\Rightarrow$ b). Let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and let $\varepsilon > 0$. For $B_1 \in \sigma(\mathscr{P}_1) \cap A_1$ put

$$\mu_1(B_1) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup \{|\gamma_n(C_1 \cap A_1, A_2, \dots, A_d)|, C_1 \in B_1 \cap \sigma(\mathscr{P}_1)\}}{1 + \bar{\gamma}_n(A_i)},$$

where $\bar{\gamma}_n$ is the supremation of the vector d-polymeasure γ_n, see Definition VIII.2 in [13]. Then clearly $\mu_1: A_1 \cap \sigma(\mathscr{P}_1) \to [0, 1)$ is a subadditive submeasure in the sense of Definition 1 in [16]. (Of course, instead of the above suprema we may alternatively use control measures for the vector measures $\gamma_n(\cdot, A_2, \dots, A_d)$: $A_1 \cap \sigma(\mathscr{P}_1) \to Y$, $n = 1, 2, \dots$.) Applying the Egoroff-Lusin theorem, see Section 1.4 in part I = [6], to the convergence $f_{1,n} \cdot \chi(A_1) \to f_1 \cdot \chi(A_1)$ and the submeasure $\mu_1: A_1 \cap \sigma(\mathscr{P}_1) \to [0, 1)$, we obtain a set $N_1 \in A_1 \cap \sigma(\mathscr{P}_1)$ and a sequence $F_{1,k} \in A_1 \cap \mathscr{P}_1$, $k = 1, 2, \dots$ such that $\mu_1(N_1) = 0$, $F_{1,k} \nearrow A_1 \cap (F_1 - N_1)$, where $F_1 = \bigcup_{n=1}^{\infty} \{t_1 \in T_1, f_{1,n}(t_1) \neq 0\} \in \sigma(\mathscr{P}_1)$, and on each $F_{1,k}$, $k = 1, 2, \dots$ the sequence $f_{1,n}$, $n = 1, 2, \dots$ converges uniformly to the function f_1. Since the semivariation $\hat{\Gamma}$: $\mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ is locally σ-finite by assumption and $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, by Lemma 1 (iii) there are $F''_{i,k} \in \mathscr{P}_i$, $i = 1, \dots, d$ and $k = 1, 2, \dots$, such that $F''_{i,k} \nearrow A_i$ for each $i = 1, \dots, d$, and $\hat{\Gamma}(F''_{i,k}) < +\infty$ for each $k = 1, 2, \dots$. For $k = 1, 2, \dots$ put $F'_{1,k} = F''_{1,k} \cap \{t_1 \in T_1, |f_1(t_1)| \leqq k\}$. Then obviously $F'_{1,k} \in A_1 \cap \sigma(\mathscr{P}_1)$ and $F'_{1,k} \nearrow A_1 \cap (F_1 - N_1)$. Now by the assumed separate uniform countable additivity of the vector d-polymeasures γ_n: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$, $n = 1, 2, \dots$ there is a k_1 such that

$$|\gamma_n(A_1, \dots, A_d) - \gamma_n(F'_{1,k_1}, A_2, \dots, A_d)| < \varepsilon/4d$$

for each $n = 1, 2, \dots$.

Repeating the above consideration for the second coordinate and starting with $(F'_{1,k_1}, A_2, \ldots, A_d)$, we obtain a set $F'_{2,k_2} \in A_2 \cap \sigma(\mathscr{P}_2)$ such that the sequence $f_{2,n}$, $n = 1, 2, \ldots$ converges uniformly to f_2 on F'_{2,k_2}, $|f_2(t_2)| \leqq k_2$ for each $t_2 \in F'_{2,k_2}$, and

$$|\gamma_n(F'_{1,k_1}, A_2, A_3, \ldots, A_d) - \gamma_n(F'_{1,k_1}, F'_{2,k_2}, A_3, \ldots, A_d)| < \varepsilon/4d$$

for each $n = 1, 2, \ldots$.

Continuing in this way we obtain successively $F'_{3,k_3}, \ldots, F'_{d,k_d}$ with the corresponding properties. Hence

$$a_n = |\gamma_n(A_i) - \gamma_n(F'_{i,k_i})| < \varepsilon/4$$

for each $n = 1, 2, \ldots$.

By separate linearity of the integral $\int_{(B_i)} (\cdot) \, d\Gamma \colon \mathsf{X}S(\mathscr{P}_i, X_i) \to Y$ for each $(B_i) \in \in \mathsf{X}\sigma(\mathscr{P}_i)$ we have the following inequalities for any $n, j = 1, 2, \ldots$:

$$\begin{aligned} b_{n,j} = |\gamma_n(F'_{i,k_i}) - \gamma_j(F'_{i,k_i})| &= |\textstyle\int_{(F'_{i,ki})} (f_{i,n}) \, d\Gamma - \int_{(F'_{i,ki})} (f_{i,j}) \, d\Gamma| \leqq \\ &\leqq |\textstyle\int_{(F'_{i,ki})} ((f_{i,n} - f_{1,j}), f_{2,n}, \ldots, f_{d,n}) \, d\Gamma| + \\ &+ |\textstyle\int_{(F'_{i,ki})} (f_{1,j}, (f_{2,n} - f_{2,j}), f_{3,n}, \ldots, f_{d,n}) \, d\Gamma| + \ldots \\ &\ldots + |\textstyle\int_{(F'_{i,ki})} (f_{1,j}, \ldots, f_{d-1,j}, (f_{d,n} - f_{d,j})) \, d\Gamma| \leqq \\ &\leqq \|f_{1,n} - f_{1,j}\|_{F'_{1,k1}} \cdot \|f_{2,n}\|_{F'_{2,k2}} \cdot \ldots \cdot \|f_{d,n}\|_{F'_{d,kd}} \cdot \hat{\Gamma}(F'_{i,k_i}) + \\ &+ \|f_{1,j}\|_{F'_{1,k1}} \cdot \|f_{2,n} - f_{2,j}\|_{F'_{2,k2}} \cdot \|f_{3,n}\|_{F'_{3,k3}} \cdot \ldots \cdot \|f_{d,n}\|_{F'_{d,kd}} \cdot \hat{\Gamma}(F'_{i,k_i}) + \ldots \\ &\ldots + \|f_{1,j}\|_{F'_{1,k1}} \cdot \ldots \cdot \|f_{d-1,j}\|_{F'_{d-1,kd-1}} \cdot \|f_{d,n} - f_{d,j}\|_{F'_{d,kd}} \cdot \hat{\Gamma}(F'_{i,k_i}) . \end{aligned}$$

Since on each F'_{i,k_i}, $i = 1, \ldots, d$, the sequence $f_{i,n}$, $n = 1, 2, \ldots$, converges uniformly to the function f_i, and $\|f_i\|_{F'_{i,ki}} \leqq k_i$, there is an n_0 such that $n \geqq n_0$ implies $\|f_{i,n}\|_{F'_{i,ki}} \leqq \leqq k_i + 1$ for each $i = 1, \ldots, d$, and at the same time $n, j \geqq n_0$ implies

$$\|f_{i,n} - f_{i,j}\|_{F'_{i,ki}} \prod_{i=1}^{d} (k_i + 1) \, \hat{\Gamma}(F'_{i,k_i}) < \varepsilon/2d$$

for each $i = 1, \ldots, d$. Hence $b_{n,j} < \varepsilon/2$ for $n, j \geqq n_0$. Thus for $n, j \geqq n_0$ we have the inequality

$$|\gamma_n(A_i) - \gamma_j(A_i)| \leqq a_n + a_j + b_{n,j} < \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon .$$

Since $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and $\varepsilon > 0$ were arbitrary, the implication a) $\Rightarrow$ b) holds, and the theorem is proved.

If now $(f'_{i,n}) \in \mathscr{I}_0 = \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ is another sequence such that $f'_{i,n} \to f_i$ for each $i = 1, \ldots, d$, and the vector d-polymeasures $\gamma'_n(\cdot) = \int_{(\cdot)} (f'_{i,n}) \, d\Gamma$, $n = 1, 2, \ldots$ are separately uniformly countably additive on $\mathsf{X}\sigma(\mathscr{P}_i)$, then the same is true for the sequence $(f''_{i,n})$, $n = 1, 2, \ldots$, where

$$f''_{i,n} = \begin{cases} f_{i,n} & \text{for } n \text{ odd} , \\ f'_{i,n} & \text{for } n \text{ even} . \end{cases}$$

Hence the integral in the next definition is unambigously defined.

Definition 1. Let $f_i \colon T_i \to X_i$, $i = 1, \ldots, d$, be $\mathscr{P}_i$-measurable functions. We say that the d-tuple (f_i) belongs to the *first integrable class* $\mathscr{I}_1(\Gamma)$, i.e., $(f_i) \in \mathscr{I}_1(\Gamma)$, if there is a sequence of d-tuples $(f_{i,n}) \in \mathscr{I}_0 = \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \dots$ such that $f_{i,n}(t_i) \to f_i(t_i)$ for each $i = 1, \dots, d$ and each $t_i \in T_i$, and the vector d-polymeasures γ_n, $\gamma_n(A_i) = \int_{(A_i)} (f_{i,n}) \,\mathrm{d}\Gamma$, $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, $n = 1, 2, \dots$ are separately uniformly countably additive on $\mathsf{X}\sigma(\mathscr{P}_i)$. In this case we define

$$\int_{(A_i)} (f_i) \,\mathrm{d}\Gamma = \lim_{n \to \infty} \int_{(A_i)} (f_{i,n}) \,\mathrm{d}\Gamma$$

for $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

We easily obtain

Theorem 2. 1) *Let* $(f_i) \in \mathsf{X}\bar{S}(\mathscr{P}_i, X_i)$, *let* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *and let* $\hat{\Gamma}(A_i) < +\infty$. *Then* $(f_i\, \chi(A_i)) \in \mathscr{I}_1(\Gamma) = \mathscr{I}_1$.

2) *Let* f_i: $T_i \to X_i$, $i = 1, \dots, d$, *be* $\mathscr{P}_i$*-measurable functions and let* $\hat{\Gamma}[(f_i), (T_i)] = 0$. *Then* $(f_i) \in \mathscr{I}_1(\Gamma) = \mathscr{I}_1$. *In particular, if* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *and* $\hat{\Gamma}(A_i) = 0$, *then* $(f_i\, \chi(A_i)) \in \mathscr{I}_1$ *for any* $\mathscr{P}_i$*-measurable functions* f_i: $T_i \to X_i$, $i = 1, \dots, d$. (Of course, the integral is identically equal to 0.)

Proof. 1) follows immediately via the implication b) $\Rightarrow$ a) of Theorem 1.

2) Take $f_{i,n} \in S(\mathscr{P}_i, X_i)$, $i = 1, \dots, d$, $n = 1, 2, \dots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$ for each $i = 1, \dots, d$, and apply Corollary 3 of Theorem VIII.4.

In the next theorem we summarize the basic properties of the elements of $\mathscr{I}_1(\Gamma)$ and of the integral on them.

Theorem 3. 1) *If* $(f_1, f_2, \dots, f_d)$, $(g_1, f_2, \dots, f_d) \in \mathscr{I}_1(\Gamma)$ *and* $a_1, b_1 \in K$ = *scalars, then* $(a_1 f_1 + b_1 g_1, f_2, \dots, f_d) \in \mathscr{I}_1(\Gamma)$ *and*

$$\int_{(A_i)} (a_1 f_1 + b_1 g_1, f_2, \dots, f_d) \,\mathrm{d}\Gamma = a_1 \int_{(A_i)} (f_i) \,\mathrm{d}\Gamma + b_1 \int_{(A_i)} (g_1, f_2, \dots, f_d) \,\mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. *Analogous identities hold in the coordinates* $i = 2, \dots, d$.

2) *If* $c_i \in K$, $i = 1, \dots, d$, *and* $(f_i) \in \mathscr{I}_1(\Gamma)$, *then* $(c_i f_i) \in \mathscr{I}_1(\Gamma)$ *and*

$$\int_{(A_i)} (c_i f_i) \,\mathrm{d}\Gamma = \prod_{i=1}^{d} c_i \int_{(A_i)} (f_i) \,\mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

3) *If* $(f_i) \in \mathscr{I}_1(\Gamma)$ *and* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *then* $(f_i\, \chi(A_i)) \in \mathscr{I}_1(\Gamma)$ *and*

$$\int_{(B_i)} (f_i\, \chi(A_i)) \,\mathrm{d}\Gamma = \int_{(A_i \cap B_i)} (f_i) \,\mathrm{d}\Gamma$$

for each $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

4) *For a fixed* $(f_i) \in \mathscr{I}_1(\Gamma)$ *the indefinite integral* $\int_{(\cdot)} (f_i) \,\mathrm{d}\Gamma$: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$ *is a vector d-polymeasure* $(0 - 0)$ *absolutely continuous with respect to* $\hat{\Gamma}$, *see Definitions* VIII.2.

5) *If each* T_i, $i = 1, \dots, d$, *is locally compact Hausdorff topological space and* $\Gamma(\cdot)(x_i)$: $\mathsf{X}\delta(\mathscr{C}_i) \to Y$ *is a separately regular vector Borel d-polymeasure for each* $(x_i) \in \mathsf{X}X_i$, *then for each* $(f_i) \in \mathscr{I}_1(\Gamma)$ *the indefinite integral* $\int_{(\cdot)} (f_i) \,\mathrm{d}\Gamma$: $\mathsf{X}\sigma(\mathscr{C}_i) \to Y$ *is a separately regular vector Borel d-polymeasure.* (*Note that each vector Baire d-polymeasure is separately regular, see also Theorem 7 in* [15].)

6) $\hat{\Gamma}(A_i) = \sup \{|\int_{(A_i)} (f_i) \,\mathrm{d}\Gamma|;\ (f_i) \in \mathscr{I}_1(\Gamma),\ \|f_i\|_{A_i} \leqq 1,\ i = 1, \dots, d\} = \hat{\Gamma}^1(A_i)$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *and similarly* $\hat{\Gamma}[(f_i),(A_i)] = \hat{\Gamma}^1[(f_i),(A_i)]$ *for any* $\mathscr{P}_i$-*measurable functions* $f_i\colon T_i \to X_i$ (*or* $f_i\colon T_i \to [0,+\infty)$), $i = 1,\dots,d$ *and each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Assertions 1)–4) are immediate consequences of the definitions.

5) follows from the following assertion.

Supplement to (VHSN)-Theorem for polymeasures, see the beginning of part VIII. Let each T_i, $i = 1,\dots,d$, be a locally compact Hausdorff topological space, let $\gamma_n\colon \mathsf{X}\delta(\mathscr{C}_i) \to L^{(d)}(X_i; Y)$, $n = 1,2,\dots$ be operator valued Borel d-polymeasures separately regular in the strong operator topology, and let $\lim_{n\to\infty} \gamma_n(A_i)(x_i) = \gamma(A_i)(x_i) \in Y$ exist for each $(A_i) \in \mathsf{X}\delta(\mathscr{C}_i)$ and each $(x_i) \in \mathsf{X}X_i$. Then γ_n, $n = 1,2,\dots$ are separately uniformly (or equi-) regular in the strong operator topology, and $\gamma\colon \mathsf{X}\delta(\mathscr{C}_i) \to L^{(d)}(X_i; Y)$ is separately regular in the strong operator topology. The same is true if $\delta(\mathscr{C}_i)$ is replaced by $\sigma(\mathscr{C}_i)$ or $\sigma(\mathscr{U}_i)$, where $\mathscr{U}_i$ denotes the lattice of all open subsets of T_i, $i = 1,\dots,d$.

This supplement is an easy standard consequence of the (VHSN) – theorem.

6) We prove the first equality, since the second can be proved similarly.

Let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, let $(f_i) \in \mathscr{I}_1(\Gamma)$ and $\|f_i\|_{A_i} \leqq 1$, $i = 1,\dots,d$, and let $\varepsilon > 0$. Take a defining sequence $(f_{i,n}) \in \mathscr{I}_0 = \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1,2,\dots$ and use the notation from the proof of Theorem 1. Since clearly $\hat{\Gamma}(A_i) \leqq \hat{\Gamma}^1(A_i)$, it is enough to suppose that $\hat{\Gamma}(A_i) < +\infty$.

Take n_0 such that

$$\left|\int_{(A_i)} (f_i)\,\mathrm{d}\Gamma\right| \leqq \varepsilon + \left|\int_{(A_i)} (f_{i,n})\,\mathrm{d}\Gamma\right|$$

for $n \geqq n_0$. According to Theorem VIII.1 there is a k_0 such that

$$\left|\int_{(A_i)} (f_{i,n})\,\mathrm{d}\Gamma\right| \leqq \varepsilon + \left|\int_{A_i \cap F_{i,k})} (f_{i,n})\,\mathrm{d}\Gamma\right| \leqq$$
$$\leqq \varepsilon + \|f_{1,n}\|_{A_1\cap F_{1,k}} \cdot \ldots \cdot \|f_{d,n}\|_{A_d \cap F_{d,k}} \hat{\Gamma}(A_i)$$

for each $k \geqq k_0$ and each $n = 1,2,\dots$.

Let $k \geqq k_0$ be fixed. Since on each $F_{i,k}$, $i = 1,\dots,d$,the sequence $f_{i,n}$, $n = 1,2,\dots$ converges uniformly to the function f_i and $\|f_i\|_{A_i} \leqq 1$, $i = 1,\dots,d$, there is an $n_1 \geqq n_0$ such that $\|f_{i,n}\|_{A_i \cap F_{i,k}} \leqq 1 + \varepsilon$ for each $i = 1,\dots,d$ and each $n \geqq n_1$. Thus

$$\left|\int_{(A_i)} (f_i)\,\mathrm{d}\Gamma\right| \leqq 2\varepsilon + (1+\varepsilon)^d\, \hat{\Gamma}(A_i)\,.$$

Since $(f_i) \in \mathscr{I}_1(\Gamma)$ with $\|f_i\|_{A_i} \leqq 1$, $i = 1,\dots,d$, and $\varepsilon > 0$ were arbitrary, $\hat{\Gamma}^1(A_i) \leqq \leqq \hat{\Gamma}(A_i)$. This implies (6) and the theorem is proved.

Owing to the theorem just proved we may apply the extension procedure of Theorem 1 and Definition 1, starting with $\mathscr{I}_1(\Gamma)$ instead of $\mathscr{I}_0$, thus obtaining the second integrable class $\mathscr{I}_2(\Gamma)$ for which the analog of Theorem 3 holds. Unfortunately, the author has been able neither to prove the equality $\mathscr{I}_2(\Gamma) = \mathscr{I}_1(\Gamma)$ in general, nor to construct an example with $\mathscr{I}_2(\Gamma) \neq \mathscr{I}_1(\Gamma)$. In Theorem X.3 we will prove the equality $\mathscr{I}_2(\Gamma) = \mathscr{I}_1(\Gamma)$ under the assumption that the vector d-

polymeasure $\Gamma(\cdot)(x_i): \mathsf{X}\mathscr{P}_i \to Y$ has a locally control d-polymeasure for each $(x_i) \in \mathsf{X}X_i$; this occurs, for example, if each $\mathscr{P}_i$, $i = 1, \ldots, d$, is generated by a countable family of sets, see Section 3 in part VIII = [13]. Hence in general we are forced to use transfinite induction for the definition of the class of all integrable d-tuples of functions. Namely, we introduce

Definition 2. Let α be a countable ordinal and let $f_i: T_i \to X_i$, $i = 1, \ldots, d$, be $\mathscr{P}_i$-measurable functions. We say that the d-tuple (f_i) belongs to the *α-integrable class $\mathscr{I}_\alpha(\Gamma)$*, shortly to $\mathscr{I}_\alpha$, if there is a sequence of countable ordinals $\alpha_n < \alpha$, $n = 1, 2, \ldots$, and a sequence of d-tuples of functions $(f_{i,n}) \in \mathscr{I}_{\alpha_n}$, $n = 1, 2, \ldots$ such that $f_{i,n}(t_i) \to f_i(t_i)$ for each $i = 1, \ldots, d$ and each $t_i \in T_i$, and the indefinite integrals $\int_{(\cdot)} (f_{i,n}) \, \mathrm{d}\Gamma: \mathsf{X}\sigma\mathscr{P}_i) \to Y$, $n = 1, 2, \ldots$ are separately uniformly countably additive. In this case we define

$$\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma = \lim_{n \to \infty} \int_{(A_i)} (f_{i,n}) \, \mathrm{d}\Gamma$$

for $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Clearly $\mathscr{I}_{\alpha_1} \subset \mathscr{I}_{\alpha_2}$ for $\alpha_1 \leqq \alpha_2$. We put $\mathscr{I} = \mathscr{I}(\Gamma) = \bigcup_{\alpha < \Omega} \mathscr{I}_\alpha(\Gamma)$, where Ω is the first uncountable ordinal number. The elements $(f_i) \in \mathscr{I}$ are called *d-tuples of functions integrable with respect to the operator valued d-polymeasure Γ*, or simply d-tuples of integrable functions.

In fact, by transfinite induction we also have

Theorem 4. 1) *The assertions of Theorem* 3 *remain valid if $\mathscr{I}_1$ is replaced by any $\mathscr{I}_\alpha$, $0 \leqq \alpha < \Omega$, hence also by $\mathscr{I}$.*

2) *Let α_n, $n = 1, 2, \ldots$ be countable ordinals, let $(f_{i,n}) \in \mathscr{I}_{\alpha_n}$, $n = 1, 2, \ldots$, and let $f_{i,n}(t_i) \to f_i(t_i) \in X_i$ for each $i = 1, \ldots, d$ and each $t_i \in T_i$. Then the analogs of* a) *and* b) *of Theorem* 1 *in this setting are equivalent, and if they hold, then $(f_i) \in \mathscr{I}_\alpha$ for any α satisfying $\alpha > \alpha_n$ for all $n = 1, 2, \ldots$, and*

$$\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma = \lim_{n \to \infty} \int_{(A_i)} (f_{i,n}) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

3) *Let $U \in L(Y, Z)$. Then $U\Gamma: \mathsf{X}\mathscr{P}_i \to L^{(d)}(X_i; Z)$ is separately countably additive in the strong operator topology, $U\,\Gamma(A_i) \leqq |U|\, \hat{\Gamma}(A_i)$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and $(f_i) \in \mathscr{I}_\alpha(U\Gamma)$ for each $(f_i) \in \mathscr{I}_\alpha$ for any countable ordinal α.*

Using Theorem VIII.1 and the assertions 1) and 2) of Theorem 4 we immediately have

Corollary 1. *For each $i = 1, \ldots, d$ let $\mathscr{P}'_i$ be a δ-ring such that $\mathscr{P}_i \subset \mathscr{P}'_i \subset \sigma(\mathscr{P}_i)$, and let Γ be the restriction to $\mathsf{X}\mathscr{P}_i$ of an operator valued d-polymeasure $\Gamma': \mathsf{X}\mathscr{P}'_i \to L^{(d)}(X_i; Y)$ separately countably additive in the strong operator topology with necessarily locally σ-finite semivariation $\hat{\Gamma}'$ on $\mathsf{X}\sigma(\mathscr{P}'_i) = \mathsf{X}\sigma(\mathscr{P}_i)$. Then $\mathsf{X}S(\mathscr{P}'_i, X_i) \subset \mathscr{I}_1(\Gamma)$, hence $\mathscr{I}_\alpha(\Gamma') \subset \mathscr{I}_{\alpha+1}(\Gamma)$ for each countable ordinal α, and thus $\mathscr{I}(\Gamma') = \mathscr{I}(\Gamma)$. Further, $\hat{\Gamma}' = \hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}_i)$, and similar equalities hold for $\hat{\Gamma}'[(\cdot), (\cdot)]$, $\tilde{\Gamma}'$, and $\|\Gamma'\|$.*

Similarly we have the following useful

Corollary 2. *Let f_i: $T_i \to X_i$, $i = 1, \ldots, d$, be $\mathscr{P}_i$-measurable functions, and put $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$. Let α be countable ordinal and suppose there are sets $F_{i,k} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ such that $F_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$, and $(f_i \chi_{F_{i,k}}) \in \mathscr{I}_\alpha$ for each $k = 1, 2, \ldots$. Then $(f_i) \in \mathscr{I}_{\alpha+1}$ if and only if there is a vector d-polymeasure γ: $\mathsf{X}\sigma(\mathscr{P}_i) \to Y$ such that*

$$\gamma(A_i \cap F_{i,k}) = \int_{(A_i)} (f_i \chi_{F_{i,k}}) \, d\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $k = 1, 2, \ldots$. If this is the case, then

$$\gamma(A_i) = \int_{(A_i)} (f_i) \, d\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

We easily deduce also the following two corollaries:

Corollary 3. *Let α, α_n, $n = 1, 2, \ldots$ be countable ordinals such that $\alpha > \alpha_n$ for each $n = 1, 2, \ldots$, let $(f_{i,n}) \in \mathscr{I}_{\alpha_n}$, $n = 1, 2, \ldots$, and for each $i = 1, \ldots, d$ let the sequence $f_{i,n}$, $n = 1, 2, \ldots$ converge uniformly to a function f_i: $T_i \to X_i$. Then $(f_i \, \chi(A_i)) \in \mathscr{I}_\alpha$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ with finite semivariation $\hat{\Gamma}(A_i)$.*

Corollary 4. *Let α be a countable ordinal and let d_1 be a positive integer such that $1 \leqq d_1 < d$. Suppose that f_i: $T_i \to X_i$, $i = 1, \ldots, d_1$ are $\mathscr{P}_i$-measurable functions such that $(f_1, \ldots, f_{d_1}, x_{d_1+1} \, \chi(A_{d_1} + 1), \ldots, x_d \, \chi(A_d)) \in \mathscr{I}_\alpha$ for each $(x_{d_1+1}, \ldots$ $\ldots, x_d) \in X_{d_1+1} \times \ldots \times X_d$ and each $(A_{d_1+1}, \ldots, A_d) \in \mathscr{P}_{d_1+1} \times \ldots \times \mathscr{P}_d$, let $f_i \in$ $\in \bar{S}(\mathscr{P}_i, X_i)$ for $i = d_1 + 1, \ldots, d$, let $(B_{d_1+1}, \ldots, B_d) \in \sigma(\mathscr{P}_{d_1+1}) \times \ldots \times \sigma(\mathscr{P}_d)$ and $\hat{\Gamma}[(f_1, \ldots, f_d, \chi(A_{d_1+1}), \ldots, \chi(A_d)), (T_i)] < +\infty$. Then $(f_1, \ldots, f_{d_1}, f_{d_1+1} \, \chi(A_{d_1+1}), \ldots$ $\ldots, f_d \, \chi(A_d)) \in \mathscr{I}_{\alpha+1}$.*

Now we can prove

Theorem 5. *Let α be a countable ordinal, let $(f_i) \in \mathscr{I}_\alpha$ and let φ_i: $T_i \to K$, $i =$ $= 1, \ldots, d$, be bounded $\mathscr{P}_i$-measurable scalar valued functions. Then $(\varphi_i f_i) \in \mathscr{I}_{\alpha+2}$.*

Proof. For $i = 1, \ldots, d$ put $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$. Since by assumption the semivariation $\hat{\Gamma}$: $\mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty]$ is σ-finite, there are $F_{i,k} \in \mathscr{P}_i$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ such that $F_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(F_{i,k}) <$ $< +\infty$ for each $k = 1, 2, \ldots$. The sequences $F'_{i,k} = F_{i,k} \cap \{t_i \in T_i, |f_i(t_i)| \leqq k\}$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ possess the same properties. Since φ_i: $T_i \to K$, $i = 1, \ldots, d$, are bounded scalar $\mathscr{P}_i$-measurable functions, it is well known, see Thoerem B in § 20 in [18], that there are sequences $\varphi_{i,n} \in S(\sigma(\mathscr{P}_i), K)$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ such that the sequence $\varphi_{i,n}$, $n = 1, 2, \ldots$ converges uniformly on T_i to the function φ_i for each $i = 1, \ldots, d$. Let k be fixed. Since on each $F_{i,k}$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ the sequence $\varphi_{i,n} f_i$, $n = 1, 2, \ldots$ converges uniformly to the function $\varphi_i f_i$, and since $(\varphi_{i,n} f_i \, \chi(F_{i,k})) \in \mathscr{I}_\alpha$ for each $k, n = 1, 2, \ldots$ by Theorem 4, we have $(\varphi_i f_i \, \chi(F_{i,k})) \in$

$\in \mathscr{I}_{\alpha+1}$ by Corollary 3 of Theorem 4 for each $k = 1, 2, \ldots$. Hence $(f_i) \in \mathscr{I}_{\alpha+2}$ by Corollary 2 of Theorem 4. The theorem is proved.

The next theorem is obvious.

Theorem 6. *Let f_i: $T_i \to X_i$, $i = 1, \ldots, d$, be $\mathscr{P}_i$-measurable functions and put $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$. Let the variation $v(\Gamma, (\cdot))$: $\mathsf{X}(F_i \cap \mathscr{P}_i) \to [0, +\infty]$, see Definition* VIII.3-c) *in* [13], *be σ-finite, and let $(|f_i|) \in \mathscr{I}(v(\Gamma, (\cdot))) = \mathscr{I}_1(v(\Gamma, (\cdot)))$. Then $(f_i) \in \mathscr{I}_1(\Gamma)$ and*

$$v(\textstyle\int_{(\cdot)} (f_i) \, d\Gamma, (A_i)) \leqq \int_{(A_i)} (|f_i|) \, dv(\Gamma, (\cdot))$$

for each $(A_i) \in \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i))$.

The following important result is a generalization of the Lebesgue dominated convergence theorem in $\mathscr{L}_1(m)$, see Theorem II.17 in [7].

Theorem 7. *Let g_i: $T_i \to X_i$, $i = 1, \ldots, d$, be $\mathscr{P}_i$-measurable functions, and let the multiple L_1-gauge $\hat{\Gamma}[(g_i), (\cdot)]$: $\mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ be separately continuous (then $\hat{\Gamma}[(g_i), (T_i)] < +\infty$ by Theorem* VIII.6). *Further, let $f_{i,n}$: $T_i \to X_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ be $\mathscr{P}_i$-measurable functions, let $f_{i,n}(t_i) \to f_i(t_i) \in X_i$ for each $i = 1, \ldots, d$ and each $t_i \in T_i$, and let $|f_{i,n}| \leqq |g_i|$ for each $i = 1, \ldots, d$ and each $n = 1, 2, \ldots$. Then*

1) $\lim_{n\to\infty} (\hat{\Gamma}[(f_1 - f_{1,n}, g_2, \ldots, g_d), (T_i)] + \hat{\Gamma}[(g_1, f_2 - f_{2,n}, g_3, \ldots, g_d), (T_i)] + \ldots$
$\ldots + \hat{\Gamma}[(g_1, \ldots, g_{d-1}, f_d - f_{d,n}), (T_i)]) = 0$;

2) $(g_i) \in \mathscr{I}_1(\Gamma)$, *and for any $g_{i,n} \in S(\mathscr{P}_i, X_i)$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ such that $g_{i,n} \to g_i$ and $|g_{i,n}| \nearrow |g_i|$, $i = 1, \ldots, d$, we have*

$$\lim_{n_1, \ldots, n_d \to \infty} \int_{(A_i)} (g_{i,n_i}) \, d\Gamma = \int_{(A_i)} (g_i) \, d\Gamma$$

uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and

$$\lim_{n_1, \ldots, n_d \to \infty} \hat{\Gamma}[g_{i,n_i}), (A_i)] = \hat{\Gamma}[(g_i), (A_i)]$$

uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and

3) $(f_i), (f_{i,n}) \in \mathscr{I}_1(\Gamma)$, $n = 1, 2, \ldots$, *and*

$$\lim_{n_1, \ldots, n_d \to \infty} \int_{(A_i)} (f_{i,n_i}) \, d\Gamma = \int_{(A_i)} (f_i) \, d\Gamma ,$$

$$\lim_{n_1, \ldots, n_d \to \infty} \hat{\Gamma}[(f_{i,n_i}), (A_i)] = \hat{\Gamma}[(f_i, (A_i)]$$

both uniformly with respect to $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. 1) In view of the symmetry in the coordinates $i = 1, \ldots, d$ it is sufficient to prove

(i) $\lim_{n\to\infty} \hat{\Gamma}[(f_1 - f_{1,n}, g_2, \ldots, g_d), (T_i)] = \lim_{n\to\infty} \hat{\Gamma}[(f_1 - f_{1,n}, g_2, \ldots, g_d), (S_{g_i})] = 0$,

where $S_{g_i} = \{t_i \in T_i, g_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$.

Let $\mathscr{P}'_1 = \bigcup_{k=1}^{\infty} \mathscr{P}_1 \cap \{t_1 \in T_1, |g_1(t_1)| > 1/k\}$, and for $A_1 \in \sigma(\mathscr{P}'_1) = \sigma(\mathscr{P}_1) \cap S_{g_1}$

put $\mu_1(A_1) = \hat{\Gamma}[(\chi(A_1), g_2, \ldots, g_d), (S_{g_i})]$. Then by the separate continuity of $\hat{\Gamma}[(g_i), (\cdot)]: X\sigma(\mathscr{P}_i) \to [0, +\infty)$ and other properties of the multiple L_1-gauge, see the assertions of Theorems VIII.3 and VIII.4, we immediately see that $\mu_1: \sigma(\mathscr{P}'_1) \to \to [0, +\infty)$ is a subadditive submeasure in the sense of Definition 1 in [16].Hence by the Egoroff-Lusin theorem, see Section 1.4 in part I = [6], which remains valid for such μ_1, there are $N_1 \in \sigma(\mathscr{P}'_1)$ and $F_{k_1} \in \mathscr{P}'_1$, $k_1 = 1, 2, \ldots$ such that $\mu_1(N_1) = 0$, $F_{k_1} \nearrow S_{g_1} - N_1$, and on each F_{k_1}, $k_1 = 1, 2, \ldots$ the sequence $f_{1,n}$, $n = 1, 2, \ldots$ converges uniformly to the function f_1.

Let $\varepsilon > 0$. Since obviously $\hat{\Gamma}[(2g_1, g_2, \ldots, g_d), (S_{g_i})]: X\sigma(\mathscr{P}_i) \to [0, +\infty)$ is separately continuous, there is a positive integer $k_{1,0}$ such that

$$\hat{\Gamma}[(2g_1, g_2, \ldots, g_d), (S_{g_1} - N_1 - F_{k_{1,0}}, S_{g_2}, \ldots, S_{g_d})] < \varepsilon/2 .$$

Since $F_{k_{1,0}} \in \mathscr{P}'_1$, there is a positive integer k_0 such that $F_{k_{1,0}} \in \mathscr{P}_1 \cap \{t_1 \in T_1, |g_1(t_1)| > 1/k_0\}$. Hence $(1/k_0)\, \chi(F_{k_{1,0}})(t_1) < |g_1(t_1)|$ for each $t_1 \in T_1$. But then $\hat{\Gamma}[(\chi(F_{k_{1,0}}), g_2, \ldots, g_d), (F_{k_{1,0}}, S_{g_2}, \ldots, S_{g_d})] < k_0\, \hat{\Gamma}[(g_i), (T_i)] < +\infty$. Since on $F_{k_{1,0}}$ the sequence $f_{1,n}$, $n = 1, 2, \ldots$ converges uniformly to the function f_1, there is a positive integer n_0 such that

$$b_n = \|f_1 - f_{1,n}\|\, F_{k_{1,0}}\, \hat{\Gamma}[(\chi(F_{k_{1,0}}), g_2, \ldots, g_d), (F_{k_{1,0}}, S_{g_2}, \ldots, S_{g_d})] \leqq$$
$$\leqq \|f_1 - f_{1,n}\|\, F_{k_{1,0}}\, k_0\, \hat{\Gamma}[(g_i), (T_i)] < \varepsilon/2$$

for $n \geqq n_0$.

Hence for $n \geqq n_0$,

$$\hat{\Gamma}[(f_1 - f_{1,n}, g_2, \ldots, g_d), (S_{g_i})] \leqq$$
$$\leqq \hat{\Gamma}[(f_1 - f_{1,n}, g_2, \ldots, g_d), (S_{g_1} - N_1 - F_{k_{1,0}}, S_{g_2}, \ldots, S_{g_d})] +$$
$$+ \hat{\Gamma}[(f_1 - f_{1,n}, g_2, \ldots, g_d), (F_{k_{1,0}}, S_{g_2}, \ldots, S_{g_d})] < \varepsilon/2 + b_n < \varepsilon .$$

Since $\varepsilon > 0$ was arbitrary, (i) and hence also 1) is proved.

2) Let $(g_{i,n}) \in XS(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ be such that $g_{i,n} \to g_i$ and $|g_{i,n}| \nearrow |g_i|$ for each $i = 1, \ldots, d$. Then we have the inequalities

$$\left|\int_{(A_i)} (g_{i,k_{i,1}})\, \mathrm{d}\Gamma - \int_{(A_i)} (g_{i,k_{i,2}})\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq \left|\int_{(A_i)} (g_{1,k_{1,1}} - g_{1,k_{1,2}}, g_{2,k_{2,1}}, \ldots, g_{d,k_{d,1}})\, \mathrm{d}\Gamma\right| +$$
$$+ \left|\int_{(A_i)} (g_{1,k_{1,2}}, (g_{2,k_{2,1}} - g_{2,k_{2,2}}), g_{3,k_{3,1}}, \ldots, g_{d,k_{d,1}})\, \mathrm{d}\Gamma\right| + \ldots$$
$$\ldots + \left|\int_{(A_i)} (g_{1,k_{1,2}}, \ldots, g_{d-1,k_{d-1,2}}, (g_{d,k_{d,1}} - g_{d,k_{d,2}}))\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq \hat{\Gamma}[(g_{1,k_{1,1}} - g_{1,k_{1,2}}), g_2, \ldots, g_d), (T_i)] + \ldots$$
$$\ldots + \hat{\Gamma}[(g_1, \ldots, g_{d-1}, (g_{d,k_{d,1}} - g_{d,k_{d,2}})), (T_i)] \leqq$$
$$\leqq \hat{\Gamma}[((g_1 - g_{1,k_{1,1}}), g_2, \ldots, g_d), (T_i)] +$$
$$+ \hat{\Gamma}[((g_1 - g_{1,k_{1,2}}), g_2, \ldots, g_d), (T_i)] + \ldots$$
$$\ldots + \hat{\Gamma}[(g_1, \ldots, g_{d-1}, (g_d - g_{d,k_{d,1}}), (T_i)] +$$
$$+ \hat{\Gamma}[(g_1, \ldots, g_{d-1}, (g_d - g_{d,k_{d,2}}), (T_i)]$$

for any multiindices $(k_{i,1})$ and $(k_{i,2})$ and any $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Hence 1) $\Rightarrow$ 2). It is easy to see that similarly 1) $\Rightarrow$ 3). The theorem is proved.

For the next theorem we need

Lemma 3. *Let* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *let* $N_i \in A_i \cap \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, *and let* $\hat{\Gamma}(N_1, A_2, \ldots, A_d) + \ldots + \hat{\Gamma}(A_1, \ldots, A_{d-1}, N_d) = 0$. *Further let* $f_i\colon T_i \to X_i$, $i = 1, \ldots, d$, *be* $\mathscr{P}_i$*-measurable functions and let* α *be a countable ordinal number. Then* $(f_i \chi_{A_i}) \in \mathscr{I}_\alpha$ *if and only if* $(f_i\, \chi(A_i - N_i)) \in \mathscr{I}_\alpha$. *If* $(f_i\, \chi(A_i)) \in \mathscr{I}_\alpha$, *then*

$$\int_{(B_i)} (f_i \chi_{A_i})\, \mathrm{d}\Gamma = \int_{(B_i)} (f_i\, \chi(A_i - N_i))\, \mathrm{d}\Gamma \tag{1}$$

for each $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Suppose that $(f_i\, \chi(A_i - N_i)) \in \mathscr{I}_\alpha$. Since $(f_1\, \chi(N_1), f_2\, \chi(A_2 - N_2, \ldots$ $\ldots, f_d\, \chi(A_d - N_d) \in \mathscr{I}_1 \subset \mathscr{I}_\alpha$ and $\int_{(B_i)} (f_1\, \chi(N_1), f_2\, \chi(A_2 - N_2), \ldots$ $\ldots, f_d\, \chi(A_d - N_d))\, \mathrm{d}\Gamma = 0$ for each $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ by the assertion 2) of Theorem 2, $(f_1\, \chi(A_1), f_2\, \chi(A_2 - N_2), \ldots, f_d\, \chi(A_d - N_d)) \in I_\alpha$ and $\int_{(B_i)} (f_1\, \chi(A_1), f_2\, \chi(A_2 - N_2), \ldots, f_d\, \chi(A_d - N_d))\, \mathrm{d}\Gamma = \int_{(B_i)} (f_i\, \chi(A_i - N_i))\, \mathrm{d}\Gamma$ for each $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Repeating the above consideration with the coordinates $i = 2, \ldots, d$ we obtain that $(f_i\, \chi(A_i)) \in \mathscr{I}_\alpha$ and that equality (1) holds for each $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. The converse assertion is clear.

Using this lemma we now prove the following generalization of Theorem I.11.

Theorem 8. *Let* $f_i\colon T_i \to X_i$, $i = 1, \ldots, d$, *be bounded* $\mathscr{P}_i$*-measurable functions, let* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *and* $\hat{\Gamma}(A_i) < +\infty$. *Further, let* α, α_n, $n = 1, 2, \ldots$ *be countable ordinals and let* $\alpha > \alpha_n$ *for each* $n = 1, 2, \ldots$. *Finally, suppose that there are* $(f_{i,n}) \in \mathscr{I}_{\alpha_n}$, $n = 1, 2, \ldots$ *such that*

$$(*) \qquad a_n(\delta) = \hat{\Gamma}(\{t_1 \in A_1, |f_1(t_1) - f_{1,n}(t_1)| > \delta\}, A_2, \ldots, A_d) + \ldots$$
$$\ldots + \hat{\Gamma}(A_1, \ldots, A_{d-1}, \{t_d \in T_d, |f_d(t_d) - f_{d,n}(t_d)| > \delta\}) \to 0 \quad \text{as} \quad n \to \infty$$

for each $\delta > 0$. *Then* $(f_i \chi_{A_i}) \in \mathscr{I}_\alpha$. *If, moreover,* $b = \sup_{i,n} \|f_{i,n}\|_{A_i} < +\infty$, *then*

$$\lim_{n\to\infty} \int_{(B_i)} (f_{i,n}\, \chi(A_i))\, \mathrm{d}\Gamma = \int_{(B_i)} (f_i\, \chi(A_i))\, \mathrm{d}\Gamma$$

uniformly with respect to $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. First, suppose $b < +\infty$. Using the monotonocity and separate countable subadditivity of the semivariation $\hat{\Gamma}\colon \mathsf{X}(A_i \cap \sigma(\mathscr{P}_i)) \to [0, +\infty)$, we deduce from $(*)$ in the usual way, see § 22 in [18], that there are sets $N_i \in A_i \cap \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, and a subsequence $\{n_k\} \subset \{n\}$ such that $\hat{\Gamma}(N_1, A_2, \ldots, A_d) + \ldots$ $\ldots + \hat{\Gamma}(A_1, \ldots, A_{d-1}, N_d) = 0$ and $f'_{i,k} = f_{i,n_k}\, \chi(A'_i) \to f_i\, \chi(A'_i)$ for each $i = 1, \ldots, d$, where $A'_i = A_i - N_i$. According to Lemma 3 we may replace each A_i, $i = 1, \ldots, d$, by A'_i. Obviously,

$$\left|\int_{(B_i)} (f'_{i,k})\, \mathrm{d}\Gamma - \int_{(B_i)} (f'_{i,j})\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq \left|\int_{(B_i)} ((f'_{1,k} - f'_{1,j}), f'_{2,k}, \ldots, f'_{d,k})\, \mathrm{d}\Gamma\right| + \ldots$$

$$\dots + \left|\int_{(B_i)} (f'_{1,j}, \dots, f'_{d-1,j}, (f'_{d,k} - f'_{d,j}))\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq \hat{\Gamma}[((f'_{1,k} - f'_{1,j}), b\chi_{A_2'}, \dots, b\chi_{A_d'}), (A'_i)] + \dots$$
$$\dots + \hat{\Gamma}[(b\,\chi(A'_1), \dots, b\,\chi(A'_{d-1}), (f'_{d,k} - f_{d,j})), (A'_i)]$$

for each $j, k = 1, 2, \dots$ and each $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

By the symmetry in the coordinates it is sufficient to estimate the first term on the right hand side of the above inequality. Clearly

$$\hat{\Gamma}[((f_{1,k} - f_{1,j}), b\,\chi(A'_2), \dots, b\,\chi(A'_d)), (A'_i)] \leqq$$
$$\leqq \hat{\Gamma}[((f_{1,k} - f_1), b\,\chi(A'_2), \dots, b\,\chi(A'_d)), (A'_i)] +$$
$$+ \hat{\Gamma}[((f_1 - f_{1,j}), b\,\chi(A'_2), \dots, b\,\chi(A'_d)), (A'_i)]\,, \quad \text{and}$$
$$\hat{\Gamma}[((f_1 - f_{1,r}), b\,\chi(A'_2), \dots, b\,\chi(A'_d)), (A'_i)] \leqq$$
$$\leqq 2b^d\, \hat{\Gamma}(\{t_1 \in A'_1, |f_1(t_1) - f_{1,r}(t_1)| > \delta\}, A'_2, \dots, A'_d) + \delta b^{d-1}\, \hat{\Gamma}(A'_i)$$

for any $r = 1, 2, \dots$ and any $\delta > 0$. Hence we immediately see that $(f_i\, \chi(A'_i)) \in \mathscr{I}_\alpha$ and $\lim_{k\to\infty} \int_{(B_i)} (f_{i,n_k}\, \chi(A'_i))\, \mathrm{d}\Gamma = \int_{(B_i)} (f_i\, \chi(A'_i))\, \mathrm{d}\Gamma$ uniformly with respect to $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

If it were not true that $\lim_{n\to\infty} \int_{(B_i)} (f_{i,n}\, \chi(A'_i))\, \mathrm{d}\Gamma = \int_{(B_i)} (f_i\, \chi(A'_i))\, \mathrm{d}\Gamma$ uniformly with respect to $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ (i.e., either the above limit does not exist for some $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, or it is not uniform with respect to $(B_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$), then there would be an $\varepsilon > 0$, a subsequence $\{n'_j\} \subset \{n\}$, and d-tuples of sets $(B_{i,j}) \in \mathsf{X}(\mathscr{P}_i)$, $j = 1, 2, \dots$ such that

$$\left|\int_{(B_{i,j})} (f_{i,n_j'}\, \chi(A'_i))\, \mathrm{d}\Gamma - \int_{(B_{i,j})} (f_i\, \chi(A'_i))\, \mathrm{d}\Gamma\right| > \varepsilon$$

for each $j = 1, 2, \dots$. However, by the first part of the proof there is a subsequence $\{j_k\} \subset \{j\}$ for which this is not true. Hence the theorem is proved in the case $b < +\infty$.

Now let $b = +\infty$. By $(*)$ there is a subsequence $\{n_k\} \subset \{n\}$ such that

$$\hat{\Gamma}(\{t_1 \in A_1, |f_1(t_1) - f_{1,n_k}(t_1)| > 1/2^k\}, A_2, \dots, A_d) + \dots$$
$$\dots + \hat{\Gamma}(A_1, \dots, A_{d-1}, \{t_d \in A_d, |f_d(t_d) - f_{d,n_k}(t_d)| > 1/2^k\}) < 1/2^k\,.$$

Put

$$A_{i,k} = \{t_i \in A_i, |f_i(t_i) - f_{i,n_k}(t_i)| > 1/2^k\}\,,$$

let $N_{i,k} = \bigcup_{j=k}^{\infty} A_{i,j}$ and $N_i = \bigcap_{k=1}^{\infty} N_{i,k}$, $i = 1, \dots, d$, $k = 1, 2, \dots$. Then clearly $\hat{\Gamma}(N_1, A_2, \dots, A_d) + \dots + \hat{\Gamma}(A_1, \dots, A_{d-1}, N_d) = 0$. For i and k as above put $f'_{i,k} = f_{i,n_k}\, \chi(A_i - N_{i,k})$. Then obviously $(f'_{i,k}) \in \mathscr{I}_{\alpha_{nk}}$ for each $k = 1, 2, \dots$, $f'_{i,k} \to f_i\, \chi(A_i - N_i)$ for each $i = 1, \dots, d$, and $\|f'_{i,k}\|_{T_i} \leqq \|f_i\|_{A_i} + 1$ for all i and k considered. Hence $(f_i\, \chi(A_i - N_i)) \in \mathscr{I}_\alpha$ by the first part of the proof. Thus $(f_i\, \chi(A_i)) \in \mathscr{I}_\alpha$ by Lemma 3. The theorem is proved.

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