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GENERALIZED STURM-LIOUVILLE EQUATIONS II

DANA FRAŇKOVÁ, ŠTEFAN SCHWABIK, Praha

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In this paper we deal with a system of generalized differential equations of the form

$$\mathrm{d}v = z\,\mathrm{d}R\,, \quad \mathrm{d}z = v\,\mathrm{d}P \tag{0.1}$$

which is closely connected to the classical Sturm-Liouville equation of the second order. For a more detailed description and motivation see [7] and [5]. In the case of the generalized differential equations of the form (0.1) the solutions in general exhibit discontinuities, and by (0.1) systems with strong impulses can be described in the sense of K. Kreith (see [5]).

The main goal in [7] was to derive some generalized version of the Sturmian comparison theorem. To this aim a certain identity was derived and we obtained a comparison theorem in which the distribution of "zeroes" of a solution was described for two equations of the form (0.1) with the same coefficient R and different coefficients P_1 and P_2. The method used in [7] was of "variational" nature and, moreover, the proof of the results was based on integration without mentioning anything which would correspond to differentiation.

Here we will derive a more general and complete result for systems of the form (0.1). First we derive a Sturm type comparison theorem for classical systems of the form

$$\dot{x} = r(s)\,y\,, \quad \dot{y} = p(s)\,x$$

with locally integrable coefficients r, p such that $r \geqq 0$ almost everywhere in the interval of definition of the system. The second part of the paper is devoted to the concept of prolongation of a function (of bounded variation) along a given increasing function. This concept then enables us to transfer results for classical systems of ordinary differential equations to systems of the form (0.1); this technique is also used for deriving the desired comparison theorem in the last part of the present paper.

The method of prolongation of a function along an increasing function seems to be useful for studying generalized differential equations in the sense of J. Kurzweil also from a general point of view. The corresponding results will be given in the paper [2] which is under preparation.

1. A COMPARISON RESULT FOR CLASSICAL STURM-LIOUVILLE SYSTEMS

1.1. Lemma. *Assume that functions* $r, p: (\alpha, \beta) \to \mathbb{R}$ *satisfy*

$$r, p \in L^1_{\text{loc}}(\alpha, \beta) \tag{1.1}$$

and

$$r(s) \geqq 0 \quad \text{for almost all} \quad s \in (\alpha, \beta) . \tag{1.2}$$

If the pair (x, y) *is a solution of the system of ordinary differential equations*

$$\dot{x} = r(s)\, y\,, \quad \dot{y} = p(s)\, x \tag{1.3}$$

on an interval $[\gamma, \delta] \subset (\alpha, \beta)$ *then the set*

$$N = \{s \in [\gamma, \delta];\ x(s) = 0\}$$

consists of a finite number of connected components in $[\gamma, \delta]$.

Proof. The solution (x, y) is assumed to be nontrivial, otherwise we have $N = [\gamma, \delta]$ and N consists of only one component. Since $x: [\gamma, \delta] \to R$ is continuous, every component of N is either a closed interval in $[\gamma, \delta]$ or an isolated point. Assume that we have a non-trivial solution (x, y) of (1.3) and that the number of components of N is infinite. Then a sequence $[s_k, s'_k]$, $s_k < s'_k$, $k = 1, 2, \ldots$ of components of N can be found such that $s'_k < s'_{k+1}$, $s'_k - s_k \to 0$ and $s'_k \to s_0 \in [\gamma, \delta]$ for $k \to \infty$. By the continuity of x in $[\gamma, \delta]$ we have $x(s_0) = 0$ because $x(s'_k) = 0$ and $y(s_0) \neq 0$ since the solution (x, y) of (1.3) is not trivial. Hence by the continuity of y at s_0 there is a $\delta > 0$ such that, e.g., $y(s) > 0$ for $s \in [s_0 - \delta, s_0 + \delta] \cap [\gamma, \delta]$ and there exists a $k_0 \in \mathbb{N}$ such that $[s_{k_0}, s'_{k_0}] \subset [s_0 - \delta, s_0 + \delta] \cap [\gamma, \delta]$. By the definition of a solution we have

$$\int_{s_{k_0}}^{s_0} r(\sigma)\, y(\sigma)\, d\sigma = x(s_0) - x(s_{k_0}) = 0$$

because $x(s_0) = x(s_{k_0}) = 0$. Hence necessarily $r(s) = 0$ for almost all $s \in [s_{k_0}, s_0]$ and, consequently,

$$x(s) = x(s_{k_0}) + \int_{s_{k_0}}^{s} r(\sigma)\, y(\sigma)\, d\sigma = x(s_{k_0}) = 0$$

for all $s \in [s_{k_0}, s_0]$. This means that the interval $[s_{k_0}, s_0]$ is a part of some component of N and this component contains all intervals $[s_k, s'_k]$ for $k \geqq k_0$. Hence we get a contradiction since $[s_k, s'_k]$ are components of the set N.

For the system (1.3) it is convenient to use the well-known Prüfer transform (see [4]), i.e.

$$\varrho = \sqrt{(x^2 + y^2)}\,, \quad \varphi = \operatorname{Arctg}(x/y)\,. \tag{1.4}$$

This means that ϱ and φ are such continuous functions that

$$x(s) = \varrho(s) \sin \varphi(s)\,, \quad y(s) = \varrho(s) \cos \varphi(s)\,, \quad s \in (\alpha, \beta)\,. \tag{1.5}$$

Under this transformation the system (1.3) has the form

$$\dot{\varphi} = r(s) \cos^2 \varphi - p(s) \sin^2 \varphi = r(s) - (r(s) + p(s)) \sin^2 \varphi \tag{1.6}$$

$$\dot{\varrho} = (r(s) + p(s))\, \varrho \sin \varphi \cos \varphi\,. \tag{1.7}$$

1.2. Lemma. *Suppose that* $r, p: (\alpha, \beta) \to \mathbb{R}$ *satisfy* (1.1) *and* (1.2). *Let* (x, y) *be a nontrivial solution of the system* (1.3) *on the interval* $[\gamma, \delta] \subset (\alpha, \beta)$ *such that*

$$N = \{s \in [\gamma, \delta];\ x(s) = 0\} = N^0 \cup N^1$$

where

$$N^1 = \bigcup_{j=1}^{k} [s_j, s'_j]$$

with $\gamma < s_1 \leqq s'_1 < s_2 \leqq s'_2 < \ldots < s_k \leqq s'_k \leqq \delta$ *and* $N^0 = [s_0, s'_0]$ *is the component of* N *containing* $\gamma = s_0$. (*If* $\gamma \notin N$ *then* $N^0 = \emptyset$ *and we set* $s'_0 = \gamma$ *in this case.*) *Further let us suppose that the functions* $\varphi, \varrho: [\gamma, \delta] \to \mathbb{R}$ *are given by the Prüfer transform* (1.4) *and that* $0 \leqq \varphi(\gamma) < \pi$. *Then*

$$\varphi(s) = j\pi \quad \text{for} \quad s \in [s_j, s'_j]\,, \quad j = 1, 2, \ldots, k \tag{1.8}$$

$$j\pi < \varphi(s) < (j + 1)\,\pi \quad \text{for} \quad s \in (s'_j, s_{j+1})\,, \quad j = 0, 1, \ldots, k \quad \text{where} \quad s_{k+1} = \delta\,. \tag{1.8}$$

Proof. Let us set $A = \int_\gamma^\delta |r(s) + p(s)|\, ds \geqq 0$. For $s \in [\gamma, \delta]$ we have $\varrho(s) > 0$ since the solution (x, y) is nontrivial. Since (1.5) implies $x(s) = \varrho(s) \sin \varphi(s)$ we obtain

$$x(s) = 0 \quad \text{if and only if} \quad \varphi(s) = 0 \pmod{\pi}\,. \tag{1.9}$$

Since φ is continuous and $\varphi(s) = 0 \pmod{\pi}$ for $s \in N$ we obtain that on every component $[s_j, s'_j]$ of N the function φ is constant and assumes a value which is a multiple of π, i.e., we have $\varphi(s) = l\pi$ for $s \in [s_j, s'_j]$, $l \in \mathbb{N}$. Assume that $\varphi(s) > l\pi$ for $s < s_j$. By (1.6) we have

$$\begin{aligned} 0 < \varphi(s) - \varphi(s_j) &= \varphi(s) - l\pi = \\ = \int_{s_j}^{s} r(\sigma)\, d\sigma &- \int_{s_j}^{s} (r(\sigma) + p(\sigma)) \sin^2 \varphi(\sigma)\, d\sigma \leqq \\ &\leqq \int_s^{s_j} |r(\sigma) + p(\sigma)|\, (\varphi(\sigma) - l\pi)^2\, d\sigma\,. \end{aligned}$$

Since φ is continuous there exists $\lambda > 0$ such that

$$A(\varphi(s) - l\pi) < 1 \tag{1.10}$$

for $s \in [\gamma, s_j] \cap [s_j - \lambda, s_j] = I$ and there is an $s' \in I$ such that $\varphi(s') = \max_{s \in I} \varphi(s)$ and consequently $(\varphi(\sigma) - l\pi)^2 \leqq (\varphi(s') - l\pi)^2$ for $\sigma \in I$. By (1.9) for every $s \in I$ we have

$$0 < \varphi(s) - l\pi \leqq \int_s^{s_j} |r(\sigma) + p(\sigma)|\, (\varphi(s') - l\pi)^2\, d\sigma \leqq (\varphi(s') - l\pi)^2 A\,.$$

This inequality holds also for $s = s' \in I$ and consequently, we get

$$\varphi(s') - l\pi \leqq (\varphi(s') - l\pi)^2 A$$

and also $1 \leqq (\varphi(s') - l\pi) A$ because $\varphi(s') - l\pi > 0$. But this evidently contradicts (1.10). Hence $\varphi(s) \leqq l\pi$ for $s < s_j$ and $\varphi(s) < l\pi$ for $s \in (s'_{j-1}, s_j)$.

Similarly we can show that $\varphi(s) > l\pi$ for $s \in (s'_j, s_{j+1})$ provided $\varphi(s'_j) = l\pi$. Assume that $\varphi(s) < l\pi$ for $s > s'_j$. By (1.6) we get for $s > s'_j$

$$(1.11) \qquad 0 > \varphi(s) - l\pi = \varphi(s) - \varphi(s'_j) =$$
$$= \int_{s_j'}^{s} r(\sigma)\, d\sigma - \int_{s_j'}^{s} (r(\sigma) + p(\sigma)) \sin^2 \varphi(\sigma)\, d\sigma \geqq$$
$$\geqq - \int_{s_j'}^{s} |r(\sigma) + p(\sigma)|\, (\varphi(\sigma) - l\pi)^2\, d\sigma\,.$$

The continuity of φ implies the existence of a $\lambda > 0$ such that

$$(1.12) \qquad 0 \leqq A(l\pi - \varphi(s)) < 1\,, \quad s \in [s'_j, \delta] \cap [s'_j, s'_j + \lambda] = I\,,$$

and also the existence of $s' \in I$ such that $\varphi(s') = \max_{s \in I} \varphi(s)$. Hence by (1.11)

$$0 > \varphi(s') - l\pi \geqq -A(\varphi(s') - l\pi)^2\,,$$

and also $A(l\pi - \varphi(s')) \geqq 1$ for $s' \in I$. This contradicts (1.12) and proves that for $s \in (s'_j, s_{j+1})$ we have $l\pi < \varphi(s)$.

Using the continuity of φ and the properties of the set N we obtain in this way the following statement:

(A) If $\varphi(s) = l\pi$ for $s \in [s_j, s'_j]$, $l \in \mathbb{N}$ then $(l - 1)\pi < \varphi(s) < l\pi$ for $s \in (s'_{j-1}, s_j)$ and $l\pi < \varphi(s) < (l + 1)\pi$ for $s \in (s'_j, s_{j+1})$.

By assumption we have $0 \leqq \varphi(\gamma) < \pi$; hence $\varphi(s) > 0$ on (s'_0, s_1). By the continuity of φ we have either $\varphi(s_1) = 0$ or $\varphi(s_1) = \pi$. By (A) the first possibility can be excluded because in this case we would have $\varphi(s) < 0$ for $s \in (s'_0, s_1)$ and also $\varphi(\gamma) < 0$. Hence $\varphi(s_1) = \pi$ and also $\varphi(s) = \pi$ for $s \in [s_1, s'_1]$. By (A) we get $\pi < \varphi(s) < 2\pi$ for $s \in (s'_1, s_2)$ and again the possibility $\varphi(s_2) = \pi$ can be excluded and we obtain $\varphi(s) = 2\pi$ for $s \in [s_2, s'_2]$. In this way it can be proved step by step that (1.8) holds.

Now we prove a statement which corresponds to the first Sturmian comparison theorem for second order linear differential equations in the form stated in the monograph [4]. Our result concerns systems of the form

$$(1.13)_j \qquad \dot{x} = r_j y\,, \quad \dot{y} = p_j x\,, \quad j = 1, 2\,.$$

1.3. Theorem. *Assume that the functions* $r_j, p_j: (\alpha, \beta) \to \mathbb{R}$, $j = 1, 2$ *satisfy* (1.1) *and* (1.2) *and that, moreover,*

$$(1.14) \qquad r_1(s) \leqq r_2(s)\,, \quad p_1(s) \geqq p_2(s) \quad \textit{for almost all} \quad s \in [\gamma, \delta] \subset (\alpha, \beta)\,.$$

Let (x_1, y_1) *be a nontrivial solution of the system* $(1.13)_1$ *on* $[\gamma, \delta]$ *such that for* $N = N^0 \cup N^1$ *given in Lemma* 1.2 *we have* $N^1 = \bigcup_{j=1}^{k} [s_j, s'_j]$ *with* $\gamma < s_1 \leqq s'_1 < s_2 \leqq s'_2 < \ldots < s_k \leqq s'_k \leqq \delta$. *(*$N^0$ *is the component of the set of zeroes of* x_1 *in* $[\gamma, \delta]$ *which contains the point* γ, *or* $N^0 = \emptyset$ *if* $\gamma \notin N$.*)*

Let (x_2, y_2) *be a solution of* $(1,13)_2$ *on* $[\gamma, \delta]$ *such that*

$$(1.15) \qquad \frac{y_1(\gamma)}{x_1(\gamma)} \geqq \frac{y_2(\gamma)}{x_2(\gamma)}$$

holds. (If $x_1(\gamma) = 0$, $x_2(\gamma) = 0$ *then we set* $y_1(\gamma)/x_1(\gamma) = \infty$, $y_2(\gamma)/x_2(\gamma) = \infty$, *respectively.)*

Then

$$M = \{s \in [\gamma, s_k];\ x_2(s) = 0\} = M^0 \cup M^1$$

where M^0 *is again the component of* M *containing* γ *or* $M^0 = \emptyset$ *and the set* M^1 *consists of at least* k *components.*

If in addition the inequality in (1.15) *is strict or there is a nondegenerate interval* $J \subset [\gamma, s_k]$ *such that either*

$$p_1(s) > p_2(s) \quad \textit{for almost all} \quad s \in J \quad \textit{and} \quad \int_J r_1(\sigma)\,\mathrm{d}\sigma > 0$$

or

$$r_1(s) < r_2(s) \quad \textit{for almost all} \quad s \in J \quad \textit{and} \quad \int_J p_1(\sigma)\,\mathrm{d}\sigma \neq 0$$

then

$$M = M^l \cup M^i \cup M^r$$

where M^l, M^r *are the components of* M *containing the points* γ, s_k, *respectively (or empty sets), and* M^i *consists of at least* k *components.*

Proof. Using the Prüfer transform for the system $(1.13)_j$, $j = 1, 2$ we obtain the equation for the polar angle

$$(1.16)_j \qquad \dot{\varphi} = r_j(s) \cos^2 \varphi - p_j(s) \sin^2 \varphi\,.$$

The functions $\varphi_j(s) = \operatorname{Arctg}(x_j(s)/y_j(s))$, $j = 1, 2$ are solutions of $(1.16)_j$ on $[\gamma, \delta]$. Assume that $\varphi_1(\gamma), \varphi_2(\gamma) \in [0, \pi)$. Since (1.15) is satisfied we get

$$(1.17) \qquad 0 \leqq \varphi_1(\gamma) \leqq \varphi_2(\gamma) < \pi\,.$$

Let us prove $\varphi_1(s) \leqq \varphi_2(s)$ for all $s \in [\gamma, \delta]$. To this aim we consider the sequence of initial value problems, $n \in \mathbb{N}$,

$$(1.18) \quad \dot{\psi} = r_2(s) \cos^2 \psi - p_2(s) \sin^2 \psi + (1/n)\,|r_2(s) + p_2(s)|\,, \quad \psi(\gamma) = \varphi_2(\gamma)\,.$$

If $\psi_n(s)$ is a solution of (1.18) defined for $s \geqq \gamma$ then the continuous dependence theorem for $n \to \infty$ can be used for obtaining that

$$(1.19) \qquad \lim_{n\to\infty} \psi_n(s) = \varphi_2(s) \quad \text{uniformly on} \quad [\gamma, \delta]\,.$$

Let $n \in \mathbb{N}$ be fixed. Assume that there is a value $s_1 \in (\gamma, \delta]$ such that $\psi_n(s_1) < \varphi_1(s_1)$. Let us set

$$s_2 = \inf\{\sigma \in [\gamma, s_1];\ \psi_n(s) < \varphi_1(s) \text{ for } s \in [\sigma, s_1]\}\,.$$

Since the functions ψ_n, φ_1 are continuous we have $\psi_n(s_2) = \varphi_1(s_2)$ and $\psi_n(s) < \varphi_1(s)$ for $s \in (s_2, s_1]$. Moreover, such a point $s_3 \in (s_2, s_1]$ can be found that

$$|\sin^2 \varphi_1(s) - \sin^2 \psi_n(s)| < (1/n) \quad \text{for} \quad s \in [s_2, s_3]\,.$$

Using this and (1.14) we obtain

$$0 < \varphi_1(s_3) - \psi_n(s_3) =$$
$$= \int_{s_2}^{s_3} [r_1(s) \cos^2 \varphi_1(s) - p_1(s) \sin^2 \varphi_1(s) - (r_2(s) \cos^2 \psi_n(s) - p_2(s) \sin^2 \psi_n(s)]\,\mathrm{d}s -$$

$$- (1/n) \int_{s_2}^{s_3} |r_2(s) + p_2(s)| \, ds =$$
$$= \int_{s_2}^{s_3} [(r_1(s) - r_2(s)) \cos^2 \varphi_1(s) - (p_1(s) - p_2(s)) \sin^2 \varphi_1(s)] \, ds +$$
$$+ \int_{s_2}^{s_3} [(r_2(s)(\cos^2 \varphi_1(s) - \cos^2 \psi_n(s)) + p_2(s)(\sin^2 \psi_n(s) - \sin^2 \varphi_1(s)) -$$
$$- (1/n) |r_2(s) + p_2(s)|] \, ds =$$
$$= \int_{s_2}^{s_3} [(r_1(s) - r_2(s)) \cos^2 \varphi_1(s) - (p_1(s) - p_2(s)) \sin^2 \varphi_1(s)] \, ds +$$
$$+ \int_{s_2}^{s_3} (r_2(s) + p_2(s))(\sin^2 \psi_n(s) - \sin^2 \varphi_1(s)) \, ds - (1/n) \int_{s_2}^{s_3} |r_2(s) + p_2(s)| \, ds \leqq$$
$$\leqq \int_{s_2}^{s_3} (r_2(s) + p_2(s))(\sin^2 \psi_n(s) - \sin^2 \varphi_1(s)) \, ds - (1/n) \int_{s_2}^{s_3} |r_2(s) + p_2(s)| \, ds \leqq$$
$$\leqq (1/n) \int_{s_2}^{s_3} |r_2(s) + p_2(s)| \, ds - (1/n) \int_{s_2}^{s_3} |r_2(s) + p_2(s)| \, ds = 0 .$$

This inequality is a contradiction which shows that the assumption of the existence of a value $s_1 \in (\gamma, \delta]$ with $\psi_n(s_1) < \varphi_1(s_1)$ is false. Hence

$$\varphi_1(s) \leqq \psi_n(s) \quad \text{for all} \quad s \in [\gamma, \delta] \quad \text{and} \quad n \in \mathbb{N} .$$

Consequently, by (1.19) we obtain also

$$\varphi_1(s) \leqq \varphi_2(s) \quad \text{for all} \quad s \in [\gamma, \delta] . \tag{1.20}$$

By Lemma 1.2 we have $\varphi_1(s_k) = k\pi$ and the above inequality yields $\varphi_2(s_k) \geqq k\pi$. Again by the results of Lemma 1.2 this inequality indicates that the set M^1 consists of at least k components because $0 \leqq \varphi_2(\gamma) < \pi$. In this way the first part of the theorem is proved.

Let us assume that the assumptions of the second part of the statement are satisfied. If the inequality in (1.15) is strict, then by (1.4) also $\varphi_1(\gamma) < \varphi_2(\gamma)$. Let $\psi(s)$ be a solution of the equation $(1.16)_2$ such that $\psi(\gamma) = \varphi_1(\gamma)$. Then using the first part of the theorem proved above we have $\varphi_1(s) \leqq \psi(s)$ for $s \in [\gamma, \delta]$ by (1.20). Since the solutions of $(1.16)_2$ are uniquely determined by the initial conditions and $\psi(\gamma) < \varphi_2(\gamma)$ we have $\psi(s) < \varphi_2(s)$ for $s \in [\gamma, \delta]$. Hence

$$\varphi_1(s) \leqq \psi(s) < \varphi_2(s) , \quad s \in [\gamma, \delta]$$

and consequently $k\pi = \varphi_1(s_k) < \varphi_2(s_k)$. The inequality $\varphi_2(s_k) > k\pi$ shows that the number of components of M^i is necessarily greater than k, and this proves the statement.

Let us now assume that (1.15) holds with the equality sign and that the assertion is not valid. In this case the results proved above can be used to state that $\varphi_1(s) = \varphi_2(s)$ for $s \in [\gamma, s_k]$. Then also

$$\dot{\varphi}_2(s) - \dot{\varphi}_1(s) = (r_2(s) - r_1(s)) \cos^2 \varphi_1(s) - (p_2(s) - p_1(s)) \sin^2 \varphi_1(s) = 0$$

for almost all $s \in [\gamma, s_k]$ and since (1.14) holds we conclude that

$$(r_2(s) - r_1(s)) \cos^2 \varphi_1(s) = 0 , \tag{1.21}$$

$$(p_2(s) - p_1(s)) \sin^2 \varphi_1(s) = 0 \tag{1.22}$$

for almost all $s \in [\gamma, s_k]$.

Let now $J \subset [\gamma, s_k]$ be a nondegenerate interval such that $p_1(s) > p_2(s)$ a.e. in J and $\int_J r_1(\sigma) \, d\sigma > 0$. Then by (1.22) we have $\sin \varphi_1(s) = 0$ for $s \in J$ and by (1.5) also $x_1(s) = 0$, $s \in J$. Hence $\dot{y}_1 = 0$ almost everywhere in J, and y_1, being absolutely continuous, is equal to a constant $c \neq 0$ on J. (The value of y_1 is nonzero in J, otherwise the solution (x_1, y_1) would be trivial.) Hence for every $s, s' \in J$ we have, by the definition of a solution,

$$0 = x_1(s') - x_1(s) = \int_s^{s'} r_1(\sigma) \, y_1(\sigma) \, d\sigma = c \int_s^{s'} r_1(\sigma) \, d\sigma$$

and consequently $\int_J r_1(\sigma) \, d\sigma = 0$ – a contradiction.

If, on the other hand, $J \subset [\gamma, s_k]$ is a nondegenerate interval such that $r_1(s) < r_2(s)$ for almost all $s \in J$ and $\int_J p_1(\sigma) \, d\sigma \neq 0$ then by (1.21) $\cos \varphi_1(s) = 0$ for $s \in J$ and by (1.5) we have $y_1(s) = 0$ for $s \in J$. Hence $\dot{x}_1(s) = 0$ for almost all $s \in J$ and $x_1(s) = c$ in J where $c \neq 0$ is a constant. This constant is nonzero since for $s \in J$ we have $\sin \varphi_1(s) \neq 0$. Hence by the definition of a solution we have

$$y_1(s') - y_1(s) = \int_s^{s'} p_1(\sigma) \, x_1(\sigma) \, d\sigma = c \int_s^{s'} p_1(\sigma) \, d\sigma = 0 \, .$$

Hence $\int_J p_1(\sigma) \, d\sigma = 0$, again a contradiction.

Since in both cases we reached contradictions, there is necessarily a value $\bar{s} \in (\gamma, s_k]$ such that $\varphi_1(\bar{s}) < \varphi_2(\bar{s})$ which yields also $\varphi_1(s_k) = k\pi < \varphi_2(s_k)$ and shows that the assertion of the second part of the theorem is true.

2. PROLONGATION OF A FUNCTION

2.1. Definition. A function $z\colon (a, b) \to \mathbb{R}^n$ is called *regulated* if for every $t \in (a, b)$ the onesided limits

$$\lim_{s \to t-} z(s) = z(t-) \, , \quad \lim_{s \to t+} z(s) = z(t+)$$

exist and are finite.

Remark. The class of regulated functions defined on an interval is well known and commonly used. For more details see e.g. [1], [3].

In our considerations of generalized differential equations the following concept of the prolongation of a function along an increasing function will be useful.

2.2. Definition. Given a regulated function $z\colon (a, b) \to \mathbb{R}^n$ and an increasing function $w\colon (a, b) \to \mathbb{R}$ we say that the function $\bar{z}\colon [w(c), w(d)] \to \mathbb{R}^n$ is the *prolongation of the function* $z\colon [c, d] \to \mathbb{R}^n$, $[c, d] \subset (a, b)$ *along* w if for $t \in [c, d]$ we have $\bar{z}(w(t)) = z(t)$ and on every interval of the type $[w(t-), w(t)]$, $[w(t), w(t+)]$, $t \in [c, d]$ the function $\bar{z}$ is linear.

More precisely, if $w(t+) > w(t)$ for some $t \in [c, d)$ then for $s \in [w(t), w(t+)]$ we set

$$\bar{z}(s) = z(t) + \frac{s - w(t)}{w(t+) - w(t)} \, (z(t+) - z(t))$$

and similarly, if $w(t-) < w(t)$ for some $t \in (c, d]$ then for $s \in [w(t-), w(t)]$ we set

$$\bar{z}(s) = z(t) + \frac{w(t) - s}{w(t) - w(t-)}(z(t-) - z(t)) = z(t-) + \frac{s - w(t-)}{w(t) - w(t-)} z(t) - z(t-)) .$$

Remark. It is clear that when an increasing $w: (a, b) \to \mathbb{R}$ is fixed then the prolongation of an arbitrary $z: (a, b) \to \mathbb{R}^n$ along w can be defined in the same manner provided at every point of the left- or right- discontinuity of w the corresponding onesided limits of z exist.

Further, since $w: (a, b) \to \mathbb{R}$ is increasing, for every closed interval $[c, d] \subset (a, b)$ the interval $[w(c), w(d)]$ is closed and bounded. Nevertheless, since the continuity of w is not required, the image of a closed interval $[c, d] \subset (a, b)$ need not be the whole interval $[w(c), w(d)]$. It should be also noted that the set of points of discontinuity of the function w in every interval $[c, d] \subset (a, b)$ is at most countable.

If an increasing function $w: (a, b) \to \mathbb{R}$ is given then for $[c, d] \subset (a, b)$ it is useful to have a function defined on the whole interval $[w(c), w(d)]$ which plays the role of the "inverse" function to $w: [c, d] \to \mathbb{R}$ also in the case when w has discontinuity points in $[c, d]$. Therefore we introduce the following definition.

2.3. Definition. Let $w: [c, d] \to \mathbb{R}$ be an increasing function. Define w_{-1}: $[w(c), w(d)] \to \mathbb{R}$ as follows:

if $s \in [w(c), w(d)]$ and $s = w(t)$ for some $t \in [c, d]$ then $w_{-1}(s) = t$, and if $s \in$ $\in [w(t-), w(t)) \cup (w(t), w(t+)]$ for some $t \in [c, d]$ then $w_{-1}(s) = t$.

Remark. Let us mention that if $w: [c, d] \to \mathbb{R}$ from the definition is continuous then the function $w_{-1}: [w(c), w(d)] \to \mathbb{R}$ coincides with the usual inverse function to w.

2.4. Lemma. *If $w: [c, d] \to \mathbb{R}$ is increasing then the function $w_{-1}: [w(c), w(d)] \to \to \mathbb{R}$ given by Definition 2.3 is nondecreasing and continuous on $[w(c), w(d)]$, and $w_{-1}(w(c)) = c$, $w_{-1}(w(d)) = d$.*

Proof. If $s_1, s_2 \in [w(c), w(d)]$, $s_1 < s_2$ then $s_i \in [w(t_i-), w(t_i+)]$, $i = 1, 2$ and evidently $t_1 \leqq t_2$. Hence $w_{-1}(s_1) = t_1 \leqq w_{-1}(s_2) = t_2$ and w_{-1} is nondecreasing. The continuity of w_{-1} easily follows from the fact that w_{-1} maps $[w(c), w(d)]$ onto the whole interval $[c, d]$ and that w_{-1} is monotone.

2.5. Lemma. *If $z: (a, b) \to \mathbb{R}^n$ is regulated and $w: (a, b) \to \mathbb{R}$ is increasing and if, moreover,*

$$\|z(t_2) - z(t_1)\| \leqq K(w(t_2) - w(t_1)), \quad K \geqq 0 \tag{2.1}$$

for $c \leqq t_1 \leqq t_2 \leqq d$ where $[c, d] \subset (a, b)$ then the prolongation $\bar{z}: [w(c), w(d)] \to \to \mathbb{R}^n$ of z along w satisfies

$$\|\bar{z}(s_2) - \bar{z}(s_1)\| \leqq K|s_2 - s_1| \tag{2.2}$$

for $s_1, s_2 \in [w(c), w(d)]$, *i.e., the function* $\bar{z}$ *is Lipschitzian with the constant* K *in* $[w(c), w(d)]$.

Proof. If $s_i = w(t_i)$ where $t_i \in [c, d]$, $i = 1, 2$, then by definition $\bar{z}(s_i) = z(t_i)$ and

$$\|\bar{z}(s_2) - \bar{z}(s_1)\| = \|z(t_2) - z(t_1)\| \leqq K|w(t_2) - w(t_1)| = D|s_2 - s_1| .$$

If e.g. $s_1 = w(t_1)$, $t_1 \in [c, d]$ and $s_2 \in [w(t_2), w(t_2+)]$, $t_2 \in [c, d)$, $t_1 < t_2$ then

$$\|\bar{z}(s_2) - \bar{z}(s_1)\| = z(t_2) + \frac{s_2 - w(t_2)}{w(t_2+) - w(t_2)} (z(t_2+) - z(t_2)) - z(t_1) \leqq$$

$$\leqq \|z(t_2) - z(t_1)\| + (s_2 - w(t_2)) \frac{\|z(t_2+) - z(t_2)\|}{w(t_2+) - w(t_2)} \leqq$$

$$\leqq K(w(t_2) - w(t_1)) + K(s_2 - w(t_2)) = K(s_2 - w(t_1)) = K(s_2 - s_1)$$

because

$$\|z(t_2 + \delta) - z(t_2)\| \leqq K(w(t_2 + \delta) - w(t_2))$$

for every sufficiently small $\delta > 0$.

For all the other possible cases the same reasoning can be used to show (2.2) for arbitrary $s_1, s_2 \in [w(c), w(d)]$.

Remark. Let us mention that if (2.1) is satisfied then every point $t \in [c, d]$ which is a point of discontinuity of the function z is necessarily also a point of discontinuity of the function w. Moreover, Lemma 2.5 yields that if (2.1) is satisfied then the prolongation $\bar{z}: [w(c), w(d)] \to R^n$ of z along w is an absolutely continuous function on $[w(c), w(d)]$.

In the sequel we pay attention to real valued functions which are locally of bounded variation on the interval (a, b). Such functions are of course regulated and the concept of the prolongation along an increasing function w can be used for them.

If $g: [c, d] \to R$ is of bounded variation on $[c, d]$ ($g \in \mathrm{BV}([c, d])$) then

$$g(t) = g_c(t) + g_b(t) , \quad t \in [c, d] \tag{2.3}$$

where g_c is continuous on $[c, d]$ and g_b is a break function on $[c, d]$ (the Jordan decomposition of g).

Using Definition 2.2 we can easily show that the prolongation of $g \in \mathrm{BV}([c, d])$ along an increasing function $w: [c, d] \to R$ satisfies

$$\bar{g}(s) = \bar{g}_c(s) + \bar{g}_b(s) , \quad s \in [w(c), w(d)] .$$

It is also obvious that if $g \in \mathrm{BV}([c, d])$ and $w: [c, d] \to R$ is increasing then the prolongation $\bar{g}: [w(c), w(d)] \to R$ of g along w fulfils $\bar{g} \in \mathrm{BV}([w(c), w(d)])$ and also $g(w_{-1}(s)) \in \mathrm{BV}([w(c), w(d)])$.

For a given increasing $w: [c, d] \to R$, $-\infty < c < d < +\infty$ we denote

$$D_w^- = \{t \in (c, d];\ w(t-) < w(t)\} \tag{2.4}$$

and

$$D_w^+ = \{t \in [c, d);\ w(t) < w(t+)\} . \tag{2.5}$$

The set $D_w^-(D_w^+)$ is the set of all points in $(c, d]$ ($[c, d)$) at which the function w has a discontinuity from the left (right). Evidently, D_w^- and D_w^+ are at most countable, i.e.

$$D_w^- = \{t_1^-, t_2^-, \ldots\}, \quad D_w^+ = \{t_1^+, t_2^+, \ldots\},$$

and so is also the set

$$D_w = D_w^- \cup D_w^+$$

of all points of discontinuity of the function w.

Given a function $g \in \mathrm{BV}([c, d])$ we denote for $\tau \in (c, d]$

$$(2.6) \qquad g_\tau^-(t) = 0 \quad \text{if} \quad t \in [c, \tau),$$
$$g_\tau^-(t) = g(\tau) - g(\tau-) \quad \text{if} \quad t \in [\tau, d],$$

and similarly for $\tau \in [c, d)$ we set

$$g_\tau^+(t) = 0 \quad \text{if} \quad t \in [c, \tau],$$
$$g_\tau^+(t) = g(\tau+) - g(\tau) \quad \text{if} \quad t \in (\tau, d].$$

The functions g_τ^-, g_τ^+ are the simple jump functions describing the discontinuity of g at the point $\tau \in [c, d]$ from the left or right, respectively. The break part of the function $g \in \mathrm{BV}([c, d])$ is given by the expression

$$g_b(t) = \sum_{\tau \in [c,d)} g_\tau^+(t) + \sum_{\tau \in (c,d]} g_\tau^-(t).$$

In connection with a given increasing function $w: [c, d] \to \mathbb{R}$ we define the following functions, $t \in [c, d]$:

$$(2.8) \qquad g^-(t) = \sum_{\tau \in D_w^-} g_\tau^-(t),$$
$$g^+(t) = \sum_{\tau \in D_w^+} g_\tau^+(t).$$

$g^+, g^- \in \mathrm{BV}([c, d])$ are evidently break functions describing all the simple jumps of g from the right and left, which are at the same time also discontinuities of the function w from the right and left, respectively.

Using the notation from (2.8) we set

$$g_1(t) = g(t) - g^-(t) - g^+(t).$$

Then we have a certain decomposition of g of the form

$$(2.9) \qquad g(t) = g_1(t) + g^-(t) + g^+(t)$$

where $g_1 \in \mathrm{BV}([c, d])$ is the part of g from which all the discontinuities of g occurring at the discontinuity points of the function w are eliminated.

Using Definition 2.2 and the properties of the function $w_{-1}: [w(c), w(d)] \to \mathbb{R}$ given by Definition 2.3 we can easily see that for $s \in [w(c), w(d)]$ the equality

$$(2.10) \qquad \bar{g}_1(s) = g_1(w_{-1}(s))$$

holds.

Let us mention that for $s \in [w(c), w(d)]$ we have

$$(2.11)\qquad g_\tau^-(w_{-1}(s)) = 0 \quad \text{if} \quad s < w(\tau-)\,,$$
$$g_\tau^-(w_{-1}(s)) = g(\tau) - g(\tau-) \quad \text{if} \quad s \geqq w(\tau-)\,,$$

and similarly also

$$(2.12)\qquad g_\tau^+(w_{-1}(s)) = 0 \quad \text{if} \quad s \leqq w(\tau+)\,,$$
$$g_\tau^+(w_{-1}(s)) = g(\tau+) - g(\tau) \quad \text{if} \quad s > w(\tau+)\,.$$

For the prolongations g_τ^- and g_τ^+ of the simple jump functions g_τ^- and g_τ^+ the following identities hold by definition.

If $\tau \in (c, d]$ then

$$(2.13)\qquad \bar{g}_\tau^-(s) = 0 \quad \text{if} \quad s \in [w(c), w(\tau-))\,,$$
$$\bar{g}_\tau^-(s) = \frac{s - w(\tau-)}{w(\tau) - w(\tau-)}\,(g(\tau) - g(\tau-)) \quad \text{if} \quad s \in [w(\tau-), w(\tau))\,,$$
$$\bar{g}_\tau^-(s) = g(\tau) - g(\tau-) \quad \text{if} \quad s \in [w(\tau), w(d)]\,,$$

and similarly, if $\tau \in [c, d)$ then

$$(2.14)\qquad \bar{g}_\tau^+(s) = 0 \quad \text{if} \quad s \in [w(c), w(\tau)]\,,$$
$$\bar{g}_\tau^+(s) = \frac{s - w(\tau)}{w(\tau+) - w(\tau)}\,(g(\tau+) - g(\tau)) \quad \text{if} \quad s \in (w(\tau), w(\tau+)]\,,$$
$$g_\tau^+(s) = g(\tau+) - g(\tau) \quad \text{if} \quad s \in (w\tau+), w(d)]\,.$$

Using (2.8) we have

$$(2.15)\qquad g^-(w_{-1}(s)) = \sum_{\tau \in D_w^-} g_\tau^-(w_{-1}(s))\,,$$
$$g^+(w_{-1}(s)) = \sum_{\tau \in D_w^+} g_\tau^+(w_{-1}(s))\,.$$

Similarly, for the prolongations we also have

$$(2.16)\qquad \bar{g}^-(s) = \sum_{\tau \in D_w^-} \bar{g}_\tau^-(s)$$
$$\bar{g}^+(s) = \sum_{\tau \in D_w^+} \bar{g}_\tau^+(s)$$

and

$$(2.17)\qquad \operatorname{var}_{w(c)}^{w(d)} g^-(w_{-1}(s)) = \operatorname{var}_{w(c)}^{w(d)} \bar{g}^-(s) = \sum_{\tau \in D_w^-} |g(\tau) - g(\tau-)| < \infty\,,$$

$$(2.18)\qquad \operatorname{var}_{w(c)}^{w(d)} g^+(w_{-1}(s)) = \operatorname{var}_{w(c)}^{w(d)} \bar{g}^+(s) = \sum_{\tau \in D_w^+} |g(\tau+) - g(\tau)| < \infty\,.$$

For the considerations concerning generalized differential equations the following "substitution" result for the Perron-Stieltjes integrals will be useful.

2.6. Proposition. *Assume that* $-\infty < c < d < +\infty, f, g \in \mathrm{BV}([c, d]), w: [c, d] \to \to \mathbb{R}$ *is increasing. Denote by* $\bar{f}, \bar{g}: [w(c), w(d)] \to \mathbb{R}$ *the prolongation of* f, g

along w, respectively. Then both integrals

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}(s)\,, \quad \int_c^d f(t)\, \mathrm{d}g(t)$$

exist and

(1.19)
$$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}(s) = \int_c^d f(t)\, \mathrm{d}g(t) + $$
$$+ \tfrac{1}{2} \sum_{\tau \in D_w^+} (f(\tau+) - f(\tau))\,(g(\tau+) - g(\tau)) - $$
$$- \tfrac{1}{2} \sum_{\tau \in D_w^-} (f(\tau) - f(\tau-))\,(g(\tau) - g(\tau-))\,.$$

Proof. Since all functions involved are of bounded variation, both integrals evidently exist (see e.g. 1.23 in [6]). Using 1.24 and 1.25 from [6] we have

(2.20)
$$\int_c^d f(t)\, \mathrm{d}g(t) = \int_{w_{-1}(w(c))}^{w_{-1}(w(d))} f(t)\, \mathrm{d}g(t) = \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g(w_{-1}(s))$$

where $w_{-1}: [w(c), w(d)] \to \mathbb{R}$ is the nondecreasing continuous function given by Definition 2.3.

To prove (2.19) we consider the difference

(2.21)
$$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}(s) - \int_c^d f(t)\, \mathrm{d}g(t) =$$
$$= \int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g(w_{-1}(s)) =$$
$$= \int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}_1(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g_1(w_{-1}(s)) +$$
$$+ \int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}^-(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g^-(w_{-1}(s)) +$$
$$+ \int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}^+(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g^+(w_{-1}(s))\,.$$

Using the properties of $\bar{g}_1: [w(c), w(d)] \to \mathbb{R}$ (see (2.9)) we obtain

(2.22)
$$\bar{g}_1(\alpha_2) - \bar{g}_1(\alpha_1) = 0 \quad \text{if} \quad \alpha_2, \alpha_1 \in [w(\tau-), w(\tau+)]\,,$$

and we have also by definition

(2.23)
$$\bar{f}(s) = f(w_{-1}(s)) \quad \text{for} \quad s \in [w(c), w(d)] \setminus \bigcup_{\tau \in D_w} [w(\tau-), w(\tau+)]$$

Assume $\varepsilon > 0$ is given. Let us define a function $\delta_1: [w(c), w(d)] \to (0, +\infty)$ as follows:

If $s \in (w(\tau-), w(\tau+))$, $\tau \in D_w$ then let $\delta_1(s) > 0$ be such that $[s - \delta_1(s), s + \delta_1(s)] \subset (w(\tau-), w(\tau+))$;

if $s = w(t_k^-)$, $t_k^- \in D_w^-$ $(s = w(t_k^+)$, $t_k^+ \in D_w^+)$ then let $\delta_1(s) > 0$ be such that

(2.24)
$$|\bar{g}_1(\alpha) - \bar{g}_1(s)| < \varepsilon/2^{k+1}(|\bar{f}(s) - f(w_{-1}(s))| + 1)$$
$$\text{if} \quad \alpha \in [s - \delta_1(s), s + \delta_1(s)]$$

and

$$\delta_1(s) = 1 \quad \text{if} \quad s \in [w(c), w(d)] \setminus \bigcup_{\tau \in D_w} [w(\tau-), w(\tau+)]\,.$$

The possibility of finding such a function δ_1 is an evident consequence of the fact that for $\tau \in D_w$ the interval $(w(\tau-), w(\tau+))$ is open and that the function $\bar{g}_1$ is continuous at the points $w(\tau-)$, $\tau \in D_w^-$ and $w(\tau+)$, $\tau \in D_w^+$.

From (2.10) we further obtain

$$\int_{w(c)}^{w(d)} \bar{f}(s) \, d\bar{g}_1(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s)) \, dg_1(w_{-1}(s)) = \\ = \int_{w(c)}^{w(d)} (\bar{f}(s) - f(w_{-1}(s))) \, d\bar{g}_1(s) . \tag{2.25}$$

All integrals in (2.25) exist since the functions occurring here are of bounded variation on $[w(c), w(d)]$.

Let now $\varepsilon > 0$ be given and let δ: $[w(c), w(d)] \to)0, +\infty)$ be a gauge such that

$$\delta(s) \leqq \delta_1(s) , \quad s \in [w(c), w(d)] . \tag{2.26}$$

If

$$A = \{\alpha_0, \sigma_1, \alpha_1, \ldots, \alpha_{m-1}, \sigma_m, \alpha_m\}$$

is an arbitrary δ-fine partition of $[w(c), w(d)]$, i.e.

$$w(c) = \alpha_0 < \alpha_1 < \ldots < \alpha_m = w(d) ,$$

$$\alpha_{j-1} \leqq \sigma_j \leqq \alpha_j , \quad [\alpha_{j-1}, \alpha_j] \subset [\sigma_j - \delta(\sigma_j), \sigma_j + \delta(\sigma_j)] , \quad j = 1, 2, \ldots, m ,$$

then the corresponding integral sum for the integral on the right hand side of (2.25) is

$$\sum_{j=1}^{m} (\bar{f}(\sigma_j) - f(w_{-1}(\sigma_j))) (\bar{g}_1(\alpha_j) - \bar{g}_1(\alpha_{j-1})) . \tag{2.27}$$

If $\sigma_j \in (w(\tau-), w(\tau+))$ for some $\tau \in D_w$ then $[\alpha_{j-1}, \alpha_j] \subset (w(\tau-), w(\tau+))$ and by (2.22) we have

$$(\bar{f}(\sigma_j) - f(w_{-1}(\sigma_j))) (\bar{g}_1(\alpha_j) - \bar{g}_1(\alpha_{j-1})) = 0 . \tag{2.28}$$

If $\sigma_j \in [w(c), w(d)] \setminus \bigcup_{\tau \in D_w} [w(\tau-), w(\tau+)]$ then by (2.23), the equality (2.28) again holds. Hence the integral sum (2.27) consists only of such terms for which either $\sigma_j = w(t_k^-)$, $t_k^- \in D_w$ or $\sigma_j = w(t_k^+)$, $t_k^+ \in D_w^+$. Since the partition A is δ-fine and (2.26) holds we can use (2.24) for the estimate

$$\Big|\sum_{j=1}^{m} (\bar{f}(\sigma_j) - f(w_{-1}(\sigma_j))) (\bar{g}_1(\alpha_j) - \bar{g}_1(\alpha_{j-1}))\Big| \leqq \\ \leqq \sum_{j=1}^{m} |\bar{f}(\sigma_j) - f(w_{-1}(\sigma_j))| \, (|\bar{g}_1(\alpha_j) - \bar{g}_1(\sigma_j)| + \\ + |\bar{g}_1(\sigma_j) - \bar{g}_1(\alpha_{j-1})|) < \sum_{j=1}^{m} \varepsilon/2^k = \varepsilon ,$$

which yields

$$\int_{w(c)}^{w(d)} (\bar{f}(s) - f(w_{-1}(s))) \, d\bar{g}_1(s) = 0 ,$$

and by (2.25) also

$$\int_{w(c)}^{w(d)} \bar{f}(s) \, d\bar{g}_1(s) = \int_{w(c)}^{w(d)} f(w_{-1}(s)) \, d\bar{g}_1(w_{-1}(s)) . \tag{2.29}$$

Since we have (see (2.11), (2.13))

$$\operatorname{var}_{w(c)}^{w(d)} g_\tau^-(w_{-1}(s)) = |g(\tau) - g(\tau-)|\,,$$
$$\operatorname{var}_{w(c)}^{w(d)} \bar{g}_\tau^-(s) = |g(\tau) - g(\tau-)|$$

for $\tau \in D_w^-$ and (2.17) holds, for any $\varepsilon > 0$ there exists a finite set $K^- \subset D_w^-$ such that

(2.30) $$\operatorname{var}_{w(c)}^{w(d)}\Big(\sum_{\tau\in D_w^-\setminus K^-} g_\tau^-(w_{-1}(s))\Big) < \varepsilon/(M+1)$$

and

(2.31) $$\operatorname{var}_{w(c)}^{w(d)}\Big(\sum_{\tau\in D_w^-\setminus K^-} \bar{g}_\tau^-(s)\Big) < \varepsilon/(M+1)$$

where $M > 0$ is such a constant that $|\bar{f}(s)| \leqq M$ and $|f(w_{-1}(s))| \leqq M$ for all $s \in [w(c), w(d)]$. Such a constant evidently exists because $\bar{f}(s)$ and $f(w_{-1}(s))$ are of bounded variation in $[w(c), w(d)]$. (In fact we can set $M = |f(c)| + \operatorname{var}_c^d f$.)

Hence using the obvious estimates for the Perron-Stieltjes integrals (see e.g. 1.19 in [6]) we have by (2.30)

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\Big(\sum_{\tau\in D_w^-\setminus K^-} \bar{g}_\tau^-(s)\Big) \leqq M \,.\, \operatorname{var}_{w(c)}^{w(d)}\Big(\sum_{\tau\in D_w^-\setminus K^-} \bar{g}_\tau^-(s)\Big)$$

and finally, also

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}^-(s) = \sum_{\tau\in D_w^-} \int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}_\tau^-(s)\,.$$

For similar reasons the equality

$$\int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g^-(w_{-1}(s)) = \sum_{d\tau\in D_w^-} \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g_\tau^-(w_{-1}(s))$$

holds. Hence

(2.32) $$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}^-(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}g^-(w_{-1}(s)) =$$
$$= \sum_{\tau\in D_w^-} \Big(\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}_\tau^-(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, \mathrm{d}\bar{g}_\tau^-(w_{-1}(s))\Big)\,.$$

Using the definition of the prolongation of a function along w we have for $\tau \in D_w^-$

(2.33) $$\int_{w(c)}^{w(d)} \bar{f}(s)\, \mathrm{d}\bar{g}_\tau^-(s) =$$
$$= \int_{w(\tau-)}^{w(\tau)} \left(f(\tau-) + \frac{s - w(\tau-)}{w(\tau) - w(\tau-)}(f(\tau) - f(\tau-))\, \mathrm{d}\, \frac{s - w(\tau-)}{w(\tau) - w(\tau-)}(g(\tau) - g(\tau-))\right) =$$
$$= \frac{g(\tau) - g(\tau-)}{w(\tau) - w(\tau-)} \int_{w(\tau-)}^{w(\tau)} \left(f(\tau-) + \frac{s - w(\tau-)}{w(\tau) - w(\tau-)}(f(\tau) - f(\tau-))\right) \mathrm{d}s =$$
$$= \frac{g(\tau) - g(\tau-)}{w(\tau) - w(\tau-)} \Big[f(\tau-)(w(\tau) - w(\tau-)) +$$
$$+ \frac{f(\tau) - f(\tau-)}{w(\tau) - w(\tau-)} \big(\tfrac{1}{2}(w(\tau))^2 - \tfrac{1}{2}(w(\tau-))^2 - w(\tau-)(w(\tau) - w(\tau-))\big)\Big] =$$
$$= (g(\tau) - g(\tau-))(f(\tau-) + \tfrac{1}{2}f(\tau) - \tfrac{1}{2}f(\tau-)) =$$
$$= \tfrac{1}{2}(f(\tau) - f(\tau-))(g(\tau) - g(\tau-))\,.$$

Further, from Definition 2.3 and (2.11) we obtain for every $\eta \in [w(c), w(\tau-))$

$$\int_{w(c)}^{w(d)} f(w_{-1}(s))\, d\bar{g}_\tau^-(w_{-1}(s)) = \int_\eta^{w(\tau-)} f(w_{-1}(s))\, dg_\tau^-(w_{-1}(s))$$

and consequently, (see e.g. 1.13 and 1.14 in [6])

$$\int_{w(c)}^{w(d)} f(w_{-1}(s))\, dg_\tau^-(w_{-1}(s)) = \lim_{\eta \to w(\tau-)-} \int_\eta^{w(\tau-)} f(w_{-1}(s))\, dg_\tau^-(w_{-1}(s)) =$$

$$= \lim_{\eta \to w(\tau-)-} f(w_{-1}(w(\tau-)))\,(g_\tau^-(w_{-1}(w(\tau-))) - g_\tau^-(w_{-1}(w(\tau-) - \eta))) =$$

$$= f(w_{-1}(w(\tau-)))\,(g_\tau^-(w_{-1}(w(\tau-))) - g_\tau^-(w_{-1}(w(\tau-) -))) =$$

$$= f(\tau)\,(g(\tau) - g(\tau-))\,.$$

Using this equality and (2.33) we have

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, d\bar{g}_\tau^-(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, dg_\tau^-(w_{-1}(s)) =$$

$$= [\tfrac{1}{2}(f(\tau) - f(\tau-) - f(\tau)]\,(g(\tau) - g(\tau-)) =$$

$$= -\tfrac{1}{2}(f(\tau) - f(\tau-))\,(g(\tau) - g(\tau-))$$

and by (2.32) also

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, d\bar{g}^-(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, dg^-(w_{-1}(s)) = \tag{2.34}$$

$$= -\tfrac{1}{2} \sum_{\tau \in D_w^-} (f(\tau) - f(\tau-))\,(g(\tau) - g(\tau-))\,.$$

A completely analogous computation leads to the equality

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, d\bar{g}^+(s) - \int_{w(c)}^{w(d)} f(w_{-1}(s))\, dg^+(w_{-1}(s)) =$$

$$= \tfrac{1}{2} \sum_{\tau \in D_w^+} (f(\tau+) - f(\tau))\,(g(\tau+) - g(\tau))\,,$$

which together with (2.34), (2.29) and (2.21) yields the equality (2.19).

2.7. Corollary. *If, in addition to the assumptions of Proposition* 2.6, *at every point* $\tau \in D_w$ *one of the functions* f *and* g *is continuous from the left and the other from the right, then*

$$\int_{w(c)}^{w(d)} \bar{f}(s)\, d\bar{g}(s) = \int_c^d f(\tau)\, dg(\tau)\,.$$

3. STURMIAN THEOREM FOR GENERALIZED STURM-LIOUVILLE SYSTEMS

Assume that $-\infty \leqq a < b \leqq +\infty$ and that $R, P\colon (a, b) \to \mathbb{R}$ satisfy the following assumptions:

$$R, P \in \mathrm{BV}_{\mathrm{loc}}(a, b)\,, \tag{3.1}$$

$$R \quad \text{is increasing in} \quad (a, b)\,, \tag{3.2}$$

$$R(t-) = R(t)\,, \quad P(t+) = P(t)\,, \quad t \in (a, b)\,. \tag{3.3}$$

Further we assume that $w\colon (a, b) \to \mathbb{R}$ is an increasing function such that

$$R(t_2) - R(t_1) \leqq w(t_2) - w(t_1)\,, \tag{3.4}$$

$$|P(t_2) - P(t_1)| \leqq w(t_2) - w(t_1) \quad \text{for} \quad a < t_1 \leqq t_2 < b\,.$$

It should be noticed that if (3.1) and (3.2) are satisfied then there always exists an increasing function $w: (a, b) \to \mathbb{R}$ for which (3.4) is satisfied. Indeed, it is sufficient in this case to take

$$w(t) = R(t) + \operatorname{var}_c^t P\,, \quad t \in (a, b)$$

for some fixed $c \in (a, b)$ with the usual convention that for $t < c$ we take $\operatorname{var}_c^t P = -\operatorname{var}_t^c P$.

Let $[c, d] \subset (a, b)$ be a given closed interval. According to Definition 2.2 we denote by $\bar{R}, \bar{P}: [w(c), w(d)] \to \mathbb{R}$ the prolongations of the functions R, P along the increasing function w.

Since (3.4) is assumed, by Lemma 2.5 the functions $\bar{R}, \bar{P}$ are absolutely continuous on $[w(c), w(d)]$. This yields also that the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}s}\bar{R}(s) = r(s)\,, \quad \frac{\mathrm{d}}{\mathrm{d}s}\bar{P}(s) = p(s) \tag{3.5}$$

exist almost everywhere in $[w(c), w(d)]$ and r, p are Lebesgue integrable in $[w(c), w(d)]$.

Since (3.2) holds, the function $\bar{R}: [w(c), w(d)] \to \mathbb{R}$ is evidently nondecreasing (compare Definition 2.2), and consequently the derivative r from (3.5) is nonnegative a.e. in $[w(c), w(d)]$.

Let us now consider the generalized differential equation

$$\mathrm{d}v = z\,\mathrm{d}R\,, \quad \mathrm{d}z = v\,\mathrm{d}P \tag{3.6}$$

and the generalized differential equation

$$\mathrm{d}x = y\,\mathrm{d}\bar{R}\,, \quad \mathrm{d}y = x\,\mathrm{d}\bar{P} \tag{3.7}$$

under the assumptions (3.1), (3.2), (3.3) and (3.4), where $w: (a, b) \to \mathbb{R}$ is a given increasing function. Since R and P are functions defined on (a, b), the prolongations $\bar{R}, \bar{P}$ can be defined on the open interval $(w(a+), w(b-))$ $(w(a+) = \lim_{t \to a+} w(t)$, $w(b-) = \lim_{t \to b-} w(t))$. Hence $(w(a+), w(b-))$ is the interval in which the coefficients of the system (3.7) are defined and in which this system can be considered.

From Definition 2.2 of the prolongation of a function along w and by the assumption (3.3) we have the following assertion:

if $t \in (a, b)$ then

$$\begin{aligned} \bar{R}(s) &= R(t) = R(t-) \quad \text{for} \quad s \in [w(t-), w(t)]\,, \\ \bar{P}(s) &= P(t) = P(t+) \quad \text{for} \quad s \in [w(t), w(t+)]\,. \end{aligned} \tag{3.8}$$

3.1. Proposition. *Let* (3.1), (3.2), (3.3) *and* (3.4) *be satisfied and let* $(v, z): [c, d] \to \mathbb{R}^2$ *be a solution of* (3.6) *on an interval* $[c, d] \subset (a, b)$. *Denote by* (x, y): $[w(c), w(d)] \to \mathbb{R}^2$ *the prolongation of* (v, z) *along the function* w.

Then (x, y) *is a solution of* (3.7) *on* $[w(c), w(d)]$.

Proof. By the definition of a solution of (3.6) we have

$$(3.9) \qquad v(t_2) - v(t_1) = \int_{t_1}^{t_2} z(\tau)\, dR(\tau)\,,$$
$$z(t_2) - z(t_1) = \int_{t_1}^{t_2} v(\tau)\, dP(\tau)$$

for $t_1, t_2 \in [c, d]$. The integral used here is the Perron-Stieltjes integral in the sense of J. Kurzweil, see [7]. For the functions v, z we have

$$(3.10-) \qquad v(t-) = v(t)\,,$$
$$z(t-) = z(t) - \Delta^- P(t)\, v(t) = z(t) - (P(t) - P(t-))\, v(t)\,,$$
$$t \in (c, d]$$

and

$$(3.10+) \qquad v(t+) = v(t) + \Delta^+ R(t)\, z(t) = v(t) + (R(t+) - R(t))\, z(t)\,,$$
$$z(t+) = z(t)\,,$$
$$t \in [c, d)\,.$$

Hence by Definition 2.2 the prolongations x, y satisfy

$$(3.11) \qquad x(s) = v(t) = v(t-)\,, \quad s \in [w(t-), w(t)]\,, \quad t \in (c, d]\,,$$
$$y(s) = z(t) = z(t+)\,, \quad s \in [w(t), w(t+)]\,, \quad t \in [c, d)\,.$$

Assume that $s_1, s_2 \in [w(c), w(d)]$ and that e.g. $s_1 < s_2$. Then there exist $t_1, t_2 \in [c, d]$ such that $t_1 \leqq t_2$ and $s_1 \in [w(t_1-), w(t_1+)]$, $s_2 \in [w(t_2-), w(t_2+)]$. Assume further that e.g. $s_1 \in [w(t_1-), w(t_1)]$ and $s_2 \in [w(t_2), w(t_2+)]$. Then

$$(3.12) \qquad x(s_2) - x(s_1) = x(s_2) - x(w(t_2)) + x(w(t_2)) - x(w(t_1)) + $$
$$+ x(w(t_1)) - x(s_1)\,.$$

By Definition 2.2 we have (cf. (2.10+))

$$(3.13) \quad x(s_2) - x(w(t_2)) = v(t_2) + \frac{s_2 - w(t_2)}{w(t_2+) - w(t_2)}\,(v(t_2+) - v(t_2)) - v(t_2) =$$
$$= \frac{s_2 - w(t_2)}{w(t_2+) - w(t_2)}\,(v(t_2+) - v(t_2)) = \frac{s_2 - w(t_2)}{w(t_2+) - w(t_2)}\,(R(t_2+) - R(t_2))\, z(t_2) =$$
$$= (\bar{R}(s_2) - \bar{R}(w(t_2)))\, z(t_2) = \int_{w(t_2)}^{s_2} d\bar{R}(\tau)\, z(t_2) = \int_{w(t_2)}^{s_2} y(\sigma)\, d\bar{R}(\sigma)$$

since by (3.11) we have $y(\sigma) = z(t_2)$ for every $\sigma \in [w(t_2), w(t_2+)]$. Using the first relation in (3.11) we have (cf. (3.8))

$$(3.14) \qquad x(w(t_1)) - x(s_1) = 0 = \int_{s_1}^{w(t_1)} y(\sigma)\, d\bar{R}(\sigma)$$

and Proposition 2.6 implies

$$(3.15) \qquad x(w(t_2)) - x(w(t_1)) = v(t_2) - v(t_1) =$$
$$= \int_{t_1}^{t_2} z(\tau)\, dR(\tau) = \int_{w(t_1)}^{w(t_2)} y(\sigma)\, d\bar{R}(\sigma)$$

because by (3.10) and (3.3) we have

$$(z(t+) - z(t))(R(t+) - R(t)) = 0\,, \quad t \in [c, d)\,,$$
$$(z(t) - z(t-))(R(t) - R(t-)) = 0\,, \quad t \in (c, d]\,.$$

Using (3.13)–(3.15) we obtain by (3.12)

(3.16) $$x(s_2) - x(s_1) = \int_{w(t_2)}^{s_2} y(\sigma)\,\mathrm{d}\bar{R}(\sigma) + \int_{w(t_1)}^{w(t_2)} y(\sigma)\,\mathrm{d}\bar{R}(\sigma) + \int_{s_1}^{w(t_1)} y(\sigma)\,\mathrm{d}\bar{R}(\sigma) = \int_{s_1}^{s_2} y(\sigma)\,\mathrm{d}\bar{R}(\sigma)\,.$$

Analogously it can be shown that also the relation

(3.17) $$y(s_2) - y(s_1) = \int_{s_1}^{s_2} x(\sigma)\,\mathrm{d}\bar{P}(\sigma)$$

holds. By a similar argument we can conclude that (3.16) and (3.17) hold for arbitrary positions of $s_1, s_2 \in [w(c), w(d)]$ and this is nothing else than the fact that the function $(x, y)\colon [w(c), w(d)] \to \mathbb{R}^2$ is a solution of the generalized differential equation (3.7).

3.2. Proposition. *Let* (3.1)–(3.4) *be satisfied and let* $(x, y)\colon [\alpha, \beta] \to \mathbb{R}^2$ *be a solution of* (3.7) *on the interval* $[\alpha, \beta] \subset (w(a+), w(b-))$. *Then there exists a solution* (v, z) *of* (3.6) *such that* (x, y) *is a prolongation of* (v, z) *along the function* w.

Proof. Since $w(a+) < \alpha < \beta < w(b-)$ there exist $c, d \in (a, b)$, $c \leqq d$ such that $\alpha \in [w(c-), w(c+)]$, $\beta \in [w(d-), w(d+)]$. Given $t \in [c, d]$ define

$$v(t) = x(w(t))\,, \quad z(t) = y(w(t))$$

and set $v(c-) = v(c)$, $z(c-) = z(c) - \Delta^- P(c)\, v(c)$, $v(d+) = v(d) + \Delta^+ R(d)\, z(d)$, $z(d+) = z(d)$.

By the definition of a solution we have

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} y(\sigma)\,\mathrm{d}\bar{R}(\sigma)\,, \qquad y(s_2) - y(s_1) = \int_{s_1}^{s_2} x(\sigma)\,\mathrm{d}\bar{P}(\sigma)$$

for every $s_1, s_2 \in [\alpha, \beta]$. Using these equations we can show that in the intervals of the form $[w(t-), w(t)]$, $[w(t), w(t+)]$ the functions x and y are linear. For example, if $s \in [w(t-), w(t)]$, $t \in [c, d]$ then $\bar{R}(s) = R(t)$. Consequently,

$$x(s) = x(w(t)) + \int_{w(t)}^{s} y(\sigma)\,\mathrm{d}\bar{R}(\sigma) = x(w(t)) = v(t)$$

for $s \in [w(t-), w(t)]$ and

$$y(s) = y(w(t)) + \int_{w(t)}^{s} x(\sigma)\,\mathrm{d}\bar{P}(\sigma) = y(w(t)) + x(w(t))\,(\bar{P}(s) - P(w(t))) =$$
$$= z(t) + x(w(t)) \frac{w(t) - s}{w(t) - w(t-)} (P(t-) - P(t)) =$$
$$= z(t) + \frac{w(t) - s}{w(t) - w(t-)} (\bar{P}(w(t-)) - \bar{P}(w(t))\, x\,(w(t)) =$$
$$= z(t) + \frac{w(t(- s}{w(t) - w(t-)} \int_{w(t)}^{w(t-)} x(\sigma)\,\mathrm{d}\bar{P}(\sigma) =$$
$$= z(t) + \frac{w(t) - s}{w(t) - w(t-)} (y(w(t-)) - y(w(t))) = z(t) + \frac{w(t) - s}{w(t) - w(t-)} (z(t-) - z(t))$$

since the onesided limit $w(t-)$ exists and therefore $y(w(t-)) = z(t-)$ and $y(w(t)) = z(t)$ by definition.

Similarly, for $s \in [w(t), w(t+)]$ it can be shown that

$$x(s) = v(t) + \frac{s - w(t)}{w(t+) - w(t)} (v(t+) - v(t))$$

and

$$y(s) = y(w(t)) = z(t) .$$

Hence the functions x, y are the prolongations of the functions v, z, respectively.

Taking arbitrary $t_1, t_2 \in [c, d]$ we have by Proposition 2.6

$$v(t_2) - v(t_1) = x(w(t_2)) - x(w(t_1)) = \int_{w(t_1)}^{w(t_2)} y(\sigma) \, \mathrm{d}\bar{R}(\sigma) = \int_{t_1}^{t_2} z(\tau) \, \mathrm{d}R(\tau)$$

since $R(t-) = R(t)$ by (3.3) and $z(t+) = y(w(t+)) = y(w(t)) = z(t)$. Similarly we have also

$$z(t_2) - z(t_1) = \int_{t_1}^{t_2} v(\tau) \, \mathrm{d}P(\tau)$$

and these equalities mean that (v, z) is a solution of (3.6) on $[c, d]$.

Let us now mention that the "prolongated" system (3.7) of generalized differential equations is closely connected with a certain system of classical differential equations. This connection follows from the general result on the equivalence of Carathéodory differential equations and generalized differential equations as was explained in [6], Theorem 4A.1.

If we take $\gamma \in (w(a+), w(b-))$ then (3.5) implies

$$\bar{R}(s) - \bar{R}(\gamma) = \int_\gamma^s r(\sigma) \, \mathrm{d}\sigma , \quad \bar{P}(s) - \bar{P}(\gamma) = \int_\gamma^s p(\sigma) \, \mathrm{d}\sigma ,$$
$$(w(a+), w(b-))$$

and these relations together with the results from [6] mentioned above yield the following assertion.

3.3. Proposition. *Le (3.1)–(3.4) be satisfied. The couple of functions (x, y): $[\alpha, \beta] \to \mathbb{R}^2$, $[\alpha, \beta] \subset (w(a+), w(b-))$ is a solution of the generalized differential equation (3.7) on $[\alpha, \beta]$ if and only if it is a solution of the classical Carathéodory system of differential equations*

$$\dot{x} = r(t) \, y , \quad \dot{y} = p(t) \, x \tag{3.18}$$

on $[\alpha, \beta]$.

3.4. Remark. The result of Proposition 3.3 can be used in connection with Proposition 3.1 to state that under the assumptions of Proposition 3.1 the prolongation of any solution (v, z): $[c, d] \to \mathbb{R}^2$ of the system (3.6) along the increasing function w is a solution of the system of equations (3.18) in the sense of Carathéodory.

Let us now recall the concept of a zero of a nontrivial solution of the system of generalized differential equations (3.6). This concept was introduced in [7].

3.5. Definition. If (v, z) is a maximal nontrivial solution of (3.6) defined on (a, b)

then we say that *the point $t^* \in (a, b)$ is a zero of the solution (v, z)* if 0 belongs to the interval with the endpoints $v(t^*)$ and $v(t^*+) = v(t^*) + \Delta^+ R(t^*)\, z(t^*)$.

3.6. Remark. A maximal solution of (3.6) can be defined quite analogously to the corresponding definition of a maximal solution of a system of ordinary differential equations. As in the classical case the linearity of (3.6) implies that any maximal solution is defined on the whole interval (a, b) on which the coefficients R and P are defined.

Assume that $t^* \in [c, d] \subset (a, b)$ and that the function $x: [w(c), w(d)] \to \mathbb{R}$ is the prolongation of v along w where v is the first component of a maximal nontrivial solution of the system (3.6). Then t^* is a zero of (v, z) if and only if there is a point $s^* \in [w(t^*), w(t^*+)]$ such that $x(s^*) = 0$. This statement is clear if the interval with the endpoints $v(t^*)$, $v(t^*+)$ is degenerate, i.e. if $v(t^*) = v(t^*+)$. If in this case $w(t^*) < w(t^*+)$ then the prolongation x of v along w is zero on the whole interval $[w(t^*), w(t^*+)]$. If the interval with the endpoints $v(t^*)$, $v(t^*+)$ is nondegenerate, then $\Delta^+ R(t^*) > 0$ and by (3.4) the interval $[w(t^*), w(t^*+)]$ is also nondegenerate; the statement then easily follows from the linearity of the prolongation x on the interval $[w(t^*), w(t^*+)]$, and from $x(w(t^*)) = v(t^*)$, $x(w(t^*+)) = v(t^*+)$.

Further, the following assertion holds: If $t^* \in (a, b)$ is a zero of the nontrivial solution (v, z) of (3.6) then $z(t^*) \neq 0$.

Indeed, if $z(t^*) = 0$ then also

$$v(t^*+) - v(t^*) = \Delta^+ R(t^*)\, z(t^*) = 0$$

and consequently, by the definition of a zero, $v(t^*) = v(t^*+) = 0$. Hence we have $v(t^*) = z(t^*) = 0$ and consequently the solution (v, z) would be trivial.

Now we turn our attention to two systems of the form (3.6), i.e. to

$$\mathrm{d}v = z\, \mathrm{d}R_i\,, \quad \mathrm{d}z = v\, \mathrm{d}P_i\,, \quad i = 1, 2 \tag{3.19$_i$}$$

with the aim of deriving a comparison theorem of Sturm type. The corresponding result reads as follows.

3.7. Theorem. *Assume that $-\infty \leqq a < b \leqq +\infty$ and that $R_j, P_j: (a, b) \to \mathbb{R}$, $j = 1, 2$ are given such that for $j = 1, 2$ we have*

$$R_j, P_j \in \mathrm{BV}_{\mathrm{loc}}(a, b)\,, \tag{3.20}$$

$$R_j \textit{ is increasing}\,, \tag{3.21}$$

$$R_j(t-) = R_j(t)\,, \quad P_j(t+) = P_j(t)\,, \quad t \in (a, b)\,. \tag{3.22}$$

Suppose that

$$R_2 - R_1 \textit{ and } P_1 - P_2 \textit{ are nondecreasing on } [c, d] \subset (a, b) \tag{3.23}$$

and that the couple (v_1, z_1) is a nontrivial solution of the equation $(3.19)_1$ *on* $[c, d]$ *which has exactly k zeroes*

$$c < t_1 < t_2 \ldots < t_k \leqq d$$

in the interval $(c, d]$.

Further assume that (v_2, z_2) *is a solution of* $(3.19)_2$ *on* $[c, d]$ *and that*

$$\frac{z_1(c)}{v_1(c)} \geqq \frac{z_2(c)}{v_2(c)} \tag{3.24}$$

(if $v_1(c) = 0$ *or* $v_2(c) = 0$ *then we set* $z_1(c)/v_1(c) = \infty$, $z_2(c)/v_2(c) = \infty$, *respectively).*

Then the solution (v_2, z_2) *has at least* k *zeros in* $(c, t_k]$.

Proof. Let us take

$$w(t) = R_1(t) + R_2(t) + \operatorname{var}_c^t P_1 + \operatorname{var}_c^t P_2\,, \quad t \in (a, b)$$

for some $c \in (a, b)$. Then by (3.21), $w: (a, b) \to \mathbb{R}$ is evidently increasing and

$$\begin{aligned} R_i(t_2) - R_i(t_1) &\leqq w(t_2) - w(t_1)\,, \\ |P_i(t_2) - P_i(t_1)| &\leqq w(t_2) - w(t_1) \end{aligned} \tag{3.25}$$

for $a < t_1 \leqq t_2 < b$, $i = 1, 2$.

Let $\bar{R}_i$, $\bar{P}_i$ be the prolongations of R_i, P_i along w for $i = 1, 2$. Since (3.25) is satisfied, by Lemma 2.5 the functions $\bar{R}_i$, $\bar{P}_i$ are Lipschitzian (with the constant 1) on $(w(a+), w(b))$ and therefore they are also locally absolutely continuous in this interval.

By (3.21) and (3.23) we can easily see that the functions $\bar{R}_i$, $i = 1, 2$, $\bar{R}_2 - \bar{R}_1$ and $\bar{P}_1 - \bar{P}_2$ are nondecreasing on $[w(c), w(d)]$. Hence in the same way as in (3.5) the derivatives

$$r_i(s) = \frac{\mathrm{d}}{\mathrm{d}s} \bar{R}_i(s)\,, \quad p_i(s) = \frac{\mathrm{d}}{\mathrm{d}s} \bar{P}_i(s)\,, \quad i = 1, 2$$

exist a.e. in $[w(c), w(d)]$ and we have

$$r_i(s) \geqq 0\,, \quad i = 1, 2 \quad \text{a.e. in} \quad [w(c), w(d)] \tag{3.26}$$

and

$$r_1(s) \leqq r_2(s)\,, \quad p_1(s) \geqq p_2(s) \quad \text{a.e. in} \quad [w(c), w(d)]\,. \tag{3.27}$$

Denote further by (x_i, y_i), $i = 1, 2$ the prolongation of (v_i, z_i) along w. As we have shown (see Propositions 3.1, 3.3 and Remark 3.4) the couple of functions (x_i, y_i) is a nontrivial solution of the classical system of ordinary differential equations

$$\dot{x} = r_i(t)\, y\,, \quad \dot{y} = p_i(t)\, x \tag{3.28_i}$$

for $i = 1, 2$ on the interval $[w(c), w(d)]$.

Since the solution (v_1, z_1) has exactly k zeroes $t_1, \ldots, t_k$ in $(c, d]$ the set of zeroes

$$N = \{s \in [w(c), w(d)];\ x_1(s) = 0\}$$

has exactly k components and possibly one component more, namely the component N^0 of N which contains the point $w(c)$ in the case when $w(c) \in N$. Using

(3.24) and the definition of the prolongation (Def. 2.2) we obtain

$$\frac{y_1(w(c))}{x_1(w(c))} = \frac{z_1(c)}{v_1(c)} \geqq \frac{z_2(c)}{v_2(c)} = \frac{y_2(w(c))}{x_2(w(c))}$$

and this inequality corresponds to (1.15) from Theorem 1.3.

Let us denote by $s_k \in (w(c), w(d)]$ the right endpoint of the component of N which contains the zero of x_1 located in $[w(t_k), w(t_k+)]$. Since all the assumptions of Theorem 1.3 are satisfied we obtain by this theorem that the set

$$M = \{s \in [w(c), s_k];\ x_2(s) = 0\}$$

is the union of at least k components and possibly of the component containing $w(c)$. This means that at least k components of the set of zeroes of the function x_2 lie in the interval $(w(c), s_k]$. From Remark 3.6 it is now apparent that for every component of M which is contained in $(w(c), s_k)$ there is a corresponding zero of the solution (v_2, z_2) of $(3.19)_2$ in $(c, t_k]$. It is also evident that to different components of M there correspond different zeroes of (v_2, z_2), and this concludes the proof of the theorem.

Remark. Theorem 3.7 is a Sturm type comparison theorem for the generalized linear differential equations of the form $(3.19)_i$, $i = 1, 2$. If we set $R_1 = R_2$ then this theorem can be used for deriving the existence of at least one zero of the solution of the equation $(3.19)_2$ between two consecutive zeroes of the equation $(3.19)_1$ in the form given by Theorem 1 in [7].

It should be also noted that the methods used in [7] and in the present paper are different. In [7] the approach was based purely on integration without mentioning the classical differential equations and even any derivative. Here we use the concept of the prolongation of solutions of $(3.19)_i$ along a function w satisfying (3.25) in order to connect $(3.19)_i$ with the classical system $(3.28)_i$. We obtained a comparison theorem which is more general than the result given in [7]. The result of the first part of Theorem 1.3 was used. It should be mentioned that also the second statement of Theorem 1.3 can be used for the systems $(3.19)_i$, $i = 1, 2$ if finer results are needed. This is only a question of translating the assumptions of Theorem 1.3 back into the assumptions on the coefficients of $(3.19)_i$. For example, the existence of a nondegenerate interval $[\alpha, \beta] \subset [w(c), s_k]$ such that $p_1(s) > p_2(s)$ a.e. in $[\alpha, \beta]$ and $\int_\alpha^\beta r_1(\sigma)\, d\sigma > 0$, which is one of the requirements of Theorem 1.3 for the system $(3.28)_i$, can be "translated" to function R_i, P_i, $i = 1, 2$ so that we require that there be a nondegenerate interval $[\gamma, \delta] \subset [c, t_k]$ such that $P_1 - P_2$ strictly increases in $[\gamma, \delta]$, and R_j is continuous in $[\gamma, \delta]$. Then for the interval $[\alpha, \beta] = [(w(\gamma), w(\delta)]$ and for the derivatives r_i, p_i of the prolongations $\bar{R}_i$, $\bar{P}_i$ on $[\alpha, \beta]$ the corresponding requirement of Theorem 1.3 is satisfied.

The method used here provides also a certain instructive model showing the way how to transfer generalized differential equations with discontinuous solutions of bounded variation back to classical differential equations with absolutely continuous solutions. In a more general setting these questions will be dealt with in the paper [2].

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Authors' address: 115 67 Praha 1, Žitná 25, Czechoslovakia (Matematický ústav ČSAV).