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ON THE PICK INVARIANT

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In [4], I have shown that the bounded elliptic surfaces M in the equiaffine 3-space satisfying the relation $\Phi(H, K) = 0$ with $\Phi_H^2 + 4H\Phi_H\Phi_K + 4K\Phi_K^2 > 0$ and umbilical boundary are affine spheres. Here, I am going to study analogous problems.

1. Let $M \subset A^3$ be an elliptic surface in the equiaffine 3-dimensional space. With each point $m \in M$, let us associate a frame $\{m; v_1, v_2, v_3\}$; we have the fundamental equations

(1) $$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,, \quad \mathrm{d}v_i = \omega_i^j v_j$$

with $\omega_1^1 + \omega_2^2 + \omega_3^3 = 0$ and the usual integrability conditions. It is possible, see [4], to choose the frames in such a way that

(2) $$\omega_1^3 = \omega^1\,, \quad \omega_2^3 = \omega^2\,;$$

(3) $$\omega_1^1 = -\tfrac{1}{2}c\omega^1 + \tfrac{1}{2}b\omega^2\,, \quad \omega_2^2 = \tfrac{1}{2}c\omega^1 - \tfrac{1}{2}b\omega^2\,, \quad \omega_3^3 = 0\,,$$
$$\omega_1^2 + \omega_2^1 = b\omega^1 + c\omega^2\,.$$

The form ω being defined by

(4) $$\omega := \tfrac{1}{2}(\omega_1^2 - \omega_2^1)\,,$$

we have

(5) $$\omega_1^2 = \tfrac{1}{2}b\omega^1 + \tfrac{1}{2}c\omega^2 + \omega\,, \quad \omega_2^1 = \tfrac{1}{2}b\omega^1 + \tfrac{1}{2}c\omega^2 - \omega$$

and

(6) $$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega\,, \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega\,.$$

The integrability conditions of (3) are

(7) $$(\mathrm{d}c + 3b\omega - 2\omega_3^1) \wedge \omega^1 - (\mathrm{d}b - 3c\omega) \wedge \omega^2 = 0\,,$$
$$(\mathrm{d}c + 3b\omega) \wedge \omega^1 - (\mathrm{d}b - 3c\omega - 2\omega_3^2) \wedge \omega^2 = 0\,,$$
$$(\mathrm{d}b - 3c\omega + \omega_3^2) \wedge \omega^1 + (\mathrm{d}c + 3b\omega + \omega_3^1) \wedge \omega^2 = 0\,.$$

From $(7_{1,2})$, $\omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0$ follows, and we get the existence of functions α, β, γ such that

(8) $$\omega_3^1 = \alpha\omega^1 + \beta\omega^2\,, \quad \omega_3^2 = \beta\omega^1 + \gamma\omega^2\,.$$

We get the following invariant forms (see [3]): the *metric form*

(9) $$\mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2\,,$$

and the *form of the lines of the affine curvature*

(10) $$\mathscr{C} = \beta(\omega^1)^2 + (\alpha - \gamma)\,\omega^1\omega^2 - \beta(\omega^2)^2 .$$

Further, we consider the following invariants

(11) $$J = \tfrac{1}{2}(b^2 + c^2)\,, \quad H = -\tfrac{1}{2}(\alpha + \gamma)\,, \quad K = \alpha\gamma - \beta^2\,, \quad \varkappa = J + H\,,$$

i.e., the *Pick invariant*, the *mean* and *affine curvature* and the *Gauss curvature* resp.

Theorem 1. *Let $M \subset A^3$ be an elliptic real analytic surface which is a bounded simply connected domain. Let $D \subset \mathbb{R}^2$ be a domain such that, for each $m \in M$, $(H(m), K(m)) \in D$. On D, let a real analytic function $F(u, v)$ be given such that*

(12) $$\left(\frac{\partial F}{\partial u}\right)^2 + 4u\,\frac{\partial F}{\partial u}\,\frac{\partial F}{\partial v} + 4v\left(\frac{\partial F}{\partial v}\right)^2 > 0 \quad \text{in} \quad D .$$

Let us suppose: (i) *We have*

(13) $$J(m) = F(H(m), K(m)) \quad \text{for each} \quad m \in M .$$

(ii) *There is an arc $\gamma \subset M$ such that*

(14) $$J(m) = 0\,, \quad \mathscr{C}(m) \equiv 0 \quad \text{for each} \quad m \in \gamma .$$

Then M is a part of an elliptic quadratic surface.

Proof. We may substitute (8) into (7) and differentiate (8); we get

(15) $$\begin{aligned}(\mathrm{d}c + 3b\omega) \wedge \omega^1 - (\mathrm{d}b - 3c\omega) \wedge \omega^2 &= -2\beta\omega^1 \wedge \omega^2\,,\\ (\mathrm{d}b - 3c\omega) \wedge \omega^1 + (\mathrm{d}c + 3b\omega) \wedge \omega^2 &= (\gamma - \alpha)\,\omega^1 \wedge \omega^2\,,\\ (\mathrm{d}\alpha - 2\beta\omega) \wedge \omega^1 + (\mathrm{d}\beta + (\alpha - \gamma)\,\omega) \wedge \omega^2 &= \{\tfrac{1}{2}b(\alpha - \gamma) + c\beta\}\,\omega^1 \wedge \omega^2\,,\\ (\mathrm{d}\beta + (\alpha - \gamma)\,\omega) \wedge \omega^1 + (\mathrm{d}\gamma + 2\beta\omega) \wedge \omega^2 &= \{\tfrac{1}{2}c(\alpha - \gamma) - b\beta\}\,\omega^1 \wedge \omega^2 .\end{aligned}$$

Thus there are functions $b_1, \ldots, \gamma_2$ such that

(16) $$\begin{gathered}\mathrm{d}b - 3c\omega = b_1\omega^1 + b_2\omega^2\,, \quad \mathrm{d}c + 3b\omega = c_1\omega^1 + c_2\omega^2\,;\\ \mathrm{d}\alpha - 2\beta\omega = \alpha_1\omega^1 + \alpha_2\omega^2\,, \quad \mathrm{d}\beta + (\alpha - \gamma)\,\omega = \beta_1\omega^1 + \beta_2\omega^2\,,\\ \mathrm{d}\gamma + 2\beta\omega = \gamma_1\omega^1 + \gamma_2\omega^2\end{gathered}$$

with

(17) $$\begin{gathered}b_1 + c_2 = 2\beta\,, \quad c_1 - b_2 = \gamma - \alpha\,;\\ \beta_1 - \alpha_2 = \tfrac{1}{2}b(\alpha - \gamma) + c\beta\,, \quad \gamma_1 - \beta_2 = \tfrac{1}{2}c(\alpha - \gamma) - b\beta .\end{gathered}$$

On M, introduce coordinates (x, y) such that

(18) $$\omega^1 = r\,\mathrm{d}x\,, \quad \omega^2 = r\,\mathrm{d}y\,; \quad r = r(x, y) > 0 .$$

Then, from (6),

(19) $$\omega = -r^{-1}r_y\,\mathrm{d}x + r^{-1}r_x\,\mathrm{d}y\,;$$

here, $r_x = \partial r(x, y)/\partial x$, etc. Substituting into (16), we get

$$(20) \qquad rb_1 = b_x + 3r^{-1}r_y c\,, \quad rb_2 = b_y - 3r^{-1}r_x c\,, \quad rc_1 = c_x - 3r^{-1}r_y b\,,$$
$$rc_2 = c_y + 3r^{-1}r_x b\,;$$
$$r\alpha_1 = \alpha_x + 2r^{-1}r_y\beta\,, \quad r\alpha_2 = \alpha_y - 2r^{-1}r_x\beta\,,$$
$$r\beta_1 = \beta_x - r^{-1}r_y(\alpha - \gamma)\,, \quad r\beta_2 = \beta_y + r^{-1}r_x(\alpha - \gamma)\,,$$
$$r\gamma_1 = \gamma_x - 2r^{-1}r_y\beta\,, \quad r\gamma_2 = \gamma_y + 2r^{-1}r_x\beta\,.$$

The conditions (17) may be rewritten as

$$(21) \qquad b_x + c_y + 3r^{-1}r_x b + 3r^{-1}r_y c = 2r\beta\,,$$
$$c_x - b_y - 3r^{-1}r_y b + 3r^{-1}r_x c = r(\gamma - \alpha)\,;$$

$$(22) \qquad \beta_x - \alpha_y - r^{-1}r_y(\alpha - \gamma) + 2r^{-1}r_x\beta = \tfrac{1}{2}rb(\alpha - \gamma) + rc\beta\,,$$
$$\gamma_x - \beta_y - 2r^{-1}r_y\beta - r^{-1}r_x(\alpha - \gamma) = \tfrac{1}{2}rc(\alpha - \gamma) - rb\beta\,.$$

The supposition (13) reads

$$(23) \qquad \tfrac{1}{2}(b^2 + c^2) = F(-\tfrac{1}{2}(\alpha + \gamma),\ \alpha\gamma - \beta^2)$$

with the differential consequences

$$(24) \qquad 2bb_x + 2cc_x + (F_u - 2\gamma F_v)\,\alpha_x + (F_u - 2\alpha F_v)\,\gamma_x + 4\beta F_v\beta_x = 0\,,$$
$$2bb_y + 2cc_y + (F_u - 2\gamma F_v)\,\alpha_y + (F_u - 2\alpha F_v)\,\gamma_y + 4\beta F_v\beta_y = 0\,.$$

Let us use the following notation: $A \equiv B$ means $A = B +$ linear combination of $b, c, \alpha - \gamma, \beta$. Then, using (22), (24) may be rewritten as

$$(25) \qquad 2bb_x + 2cc_x + (F_u - 2\gamma F_v)\,(\alpha - \gamma)_x +$$
$$+ 2(F_u - (\alpha + \gamma)\,F_v)\,\beta_y + 4\beta F_v\beta_x \equiv 0\,,$$
$$2bb_y + 2cc_y - (F_u - 2\alpha F_v)\,(\alpha - \gamma)_y +$$
$$+ 2(F_u - (\alpha + \gamma)\,F_v)\,\beta_x + 4\beta F_v\beta_y \equiv 0\,.$$

Consider the $\mathbb{R}^4$-valued function

$$(26) \qquad u = (b, c, \alpha - \gamma, \beta)^T\,;$$

we get, from (21) and (25),

$$(27) \qquad Au_x + Bu_y + Cu = 0$$

with

$$(28) \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2b & 2c & F_u - 2\gamma F_v & 4\beta F_v \\ 0 & 0 & 0 & 2(F_u - (\alpha + \gamma)\,F_v) \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(F_u - (\alpha + \gamma)\,F_v) \\ 2b & 2c & -(F_u - 2\alpha F_v) & 4\beta F_v \end{pmatrix}.$$

The symbol of the system (27) is defined as

$$\sigma(\xi, \eta) = A\xi + B\eta\,; \quad (\xi, \eta) \in \mathbb{R}^2\,. \tag{29}$$

It is easy to see that

$$\det \sigma(\xi, \eta) = 2(F_u + 2HF_v)(\xi^2 + \eta^2)\,\Phi(\xi, \eta) \quad \text{with} \tag{30}$$

$$\Phi(\xi, \eta) = (F_u - 2\gamma F_v)\,\xi^2 + 4\beta F_v \xi\eta + (F_u - 2\alpha F_v)\,\eta^2\,.$$

The discriminant Δ of the form $\Phi(\xi, \eta)$ being equal to $F_n^2 + 4HF_uF_v + 4KF_v^2$, the supposition (12) implies that $\Phi(\xi, \eta)$ is definite. Further, $F_u + 2HF_v \neq 0$. Indeed, $F_u = -2HF_v$ implies $\Delta = -4(H^2 - K)\,F_v^2 \leqq 0$, a contradiction to (12). Thus the symbol $\sigma(\xi, \eta)$ is an invertible matrix for each real pair $(\xi, \eta) \neq (0, 0)$, and the system (27) is elliptic; see [6], p. 76. Now, let us apply Theorem 5.4.1 of [6] which claims that the zeros of any non-trivial solution of an elliptic system with real analytic coefficients are isolated and of finite order. This provides the uniqueness of the Cauchy problem, and we have $u = 0$ on M. But this means $J = 0$ on M, and we are finished.

Remark. For $F(u, v) = -u +$ const., the condition (13) turns out to be $\varkappa =$ $=$ const., and we get a version of the $\varkappa$-Theorem; compare with [2] and [3].

2. The affine normals of a surface in the equiaffine 3-space A^3 coincide with its projective normals if and only if the Pick invariant J of the surface is constant; see [1], p. 111. In this connection, I am going to prove the following

Theorem 2. *The only compact elliptic surfaces $M \subset A^3$ without boundary with equal affine and projective normals are the ellipsoids.*

Proof. Let $(M, \mathrm{d}s^2 = \delta_{ij}\omega^i\omega^j)$ be a Riemannian manifold, F_{ijk} a tensor on M, $F_{ijk;l}$ its covariant derivatives with respect to the coframes $\{\omega^i\}$. Let the 1-form φ on M be defined by

$$\varphi = \delta^{i_1 j_1}\delta^{i_2 j_2}\delta^{kl}(F_{i_1 i_2 i}F_{j_1 j_2 k;l} - F_{i_1 i_2 k}F_{j_1 j_2 i;l})\,\omega^i\,. \tag{31}$$

Theorem 1.1 of [5] says that the form $\mathrm{d} * \varphi$ does not contain the second covariant derivatives of F_{ijk}.

Let us apply this result to the invariant form (see [4])

$$F \equiv F_{ijk}\omega^i\omega^j\omega^k := -2\mathscr{A} = c(\omega^1)^3 - 3b(\omega^1)^2\,\omega^2 - 3c\omega^1(\omega^2)^2 + b(\omega^2)^3\,. \tag{32}$$

The covariant derivatives $F_{ijk;l}$ being defined by

$$\mathrm{d}F_{ijk} - F_{rjk}\varphi_i^r - F_{irk}\varphi_j^r - F_{ijr}\varphi_k^r = F_{ijk;l}\omega^l\,; \tag{33}$$

$$\varphi_1^2 = -\varphi_2^1 := \omega\,, \quad \varphi_1^1 = \varphi_2^2 = 0\,;$$

we get, from $(16_{1,2})$,

$$\begin{aligned} F_{111;1} &= c_1\,, & F_{111;2} &= c_2\,, & F_{112;1} &= -b_1\,, & F_{112;2} &= -b_2\,, \\ F_{122;1} &= -c_1\,, & F_{122;2} &= -c_2\,, & F_{222;1} &= b_1\,, & F_{222;2} &= b_2\,. \end{aligned} \tag{34}$$

The exterior differentiation of $(16_{1,2})$ yields

(35) $\{db_1 - (b_2 + 3c_1)\omega\} \wedge \omega^1 + \{db_2 + (b_1 - 3c_2)\omega\} \wedge \omega^2 = 3\varkappa c\omega^1 \wedge \omega^2$,
$\{dc_1 - (c_2 - 3b_1)\omega\} \wedge \omega^1 + \{dc_2 + (c_1 + 3b_2)\omega\} \wedge \omega^2 = -3\varkappa b\omega^1 \wedge \omega^2$,

and we get the existence of functions $b_{11}, \ldots, c_{22}$ such that

(36)
$$db_1 - (b_2 + 3c_1)\omega = b_{11}\omega^1 + b_{12}\omega^2,$$
$$db_2 + (b_1 - 3c_2)\omega = b_{21}\omega^1 + b_{22}\omega^2,$$
$$dc_1 - (c_2 - 3b_1)\omega = c_{11}\omega^1 + c_{12}\omega^2,$$
$$dc_2 + (c_1 + 3b_2)\omega = c_{21}\omega^1 + c_{22}\omega^2;$$

(37)
$$b_{21} - b_{12} = 3\varkappa c, \quad c_{21} - c_{12} = -3\varkappa b.$$

Considering the form (31), we have

(38)
$$\psi := -\tfrac{1}{4} * \varphi = (cb_1 - bc_1)\omega^1 + (cb_2 - bc_2)\omega^2$$

and

(39)
$$d\psi = 2(b_2c_1 - b_1c_2 + 3\varkappa J)\omega^1 \wedge \omega^2.$$

Now, let us suppose $J = \text{const.} > 0$. From (11_1) and $(16_{1,2})$, $bb_1 + cc_1 = bb_2 + cc_2 = 0$, which implies $b_2c_1 - b_1c_2 = 0$. Thus

(40)
$$0 = \int_{\partial M} \psi = \int_M d\psi = 6J \int_M \varkappa\omega^1 \wedge \omega^2,$$

i.e., $\int_M \varkappa\omega^1 \wedge \omega^2 = 0$. This means $\chi(M) = 0$, and M should be a torus. But this is impossible because of the ellipticity of M. Thus $J = 0$ on M, and we are finished.

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