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INFINITESIMAL RIGIDITY OF SURFACES IN A^3

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In what follows, I am going to prove the infinitesimal rigidity of bounded simply connected domains on elliptic surfaces of the equiaffine 3-space with respect to the induced equiaffine metric. In the proof, I show that the problem may be transformed into the Cauchy problem for the system (49) or (53) resp.; for such systems, I simply use the results as presented in [4]. For more detailed exposition of the theory of surfaces, see [2] and [3]. Theorem 1 stands in close relation to the result of § 90 in [1].

Let $M \subset A^3$ be a surface in the equiaffine 3-space A^3. With each of its points $m \in M$, associate a frame $\{m; v_1, v_2, v_3\}$ such that $v_1, v_2 \in T_m(M)$; we may write

(1) $$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,, \quad \mathrm{d}v_i = \omega_i^j v_j \quad (i, j = 1, 2, 3)$$

with

(2) $$\omega_1^1 + \omega_2^2 + \omega_3^3 = 0$$

and the integrability conditions

(3) $$\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i\,, \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j\,,$$

where, of course, $\omega^3 = 0$. *Let us restrict ourselves to elliptic surfaces.* Then it is possible, see [3], to choose the frames in such a way that

(4) $$\omega_1^3 = \omega^1\,, \quad \omega_2^3 = \omega^2\,;$$

(5) $$\omega_1^1 = -\tfrac{1}{2}c\omega^1 + \tfrac{1}{2}b\omega^2\,, \quad \omega_2^2 = \tfrac{1}{2}c\omega^1 - \tfrac{1}{2}b\omega^2\,, \quad \omega_3^3 = 0\,,$$
$$\omega_1^2 = \tfrac{1}{2}b\omega^1 + \tfrac{1}{2}c\omega^2 + \omega\,, \quad \omega_2^1 = \tfrac{1}{2}b\omega^1 + \tfrac{1}{2}c\omega^2 - \omega\,;$$

(6) $$\omega_3^1 = \alpha\omega^1 + \beta\omega^2\,, \quad \omega_3^2 = \beta\omega^1 + \gamma\omega^2$$

with

(7) $$\omega := \tfrac{1}{2}(\omega_1^2 - \omega_2^1)\,.$$

The conditions (3_1) reduce then to

(8) $$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega\,, \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega\,.$$

We get the following invariant forms: the metric quadratic form

(9) $$\mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2\,;$$

the forms (see [2])

(10) $$\mathscr{A} := -\tfrac{1}{2}\{c(\omega^1)^3 - 3b(\omega^1)^2\,\omega^2 - 3c\omega^1(\omega^2)^2 + b(\omega^2)^3\}\,,$$
$$\mathscr{B} := -\omega^1\omega_3^1 - \omega^2\omega_3^2 = -\{\alpha(\omega^1)^2 + 2\beta\omega^1\omega^2 + \gamma(\omega^2)^2\}\,;$$

the form of the lines of affine curvature

(11) $$\mathscr{C} := \beta(\omega^1)^2 + (\alpha - \gamma)\,\omega^1\omega^2 - \beta(\omega^2)^2\,.$$

Further, we get the fundamental invariants: the Pick invariant

(12) $$J = \tfrac{1}{2}(b^2 + c^2)\,,$$

the mean curvature and the affine curvature

(13) $$H = -\tfrac{1}{2}(\alpha + \gamma)\,, \quad K = \alpha\gamma - \beta^2$$

resp., and the Gauss curvature

(14) $$\varkappa = \tfrac{1}{2}(b^2 + c^2 - \alpha - \gamma) = J + H$$

of the metric form (9) defined by

(15) $$\mathrm{d}\omega = -\varkappa\omega^1 \wedge \omega^2\,.$$

Consider a 1-parametric family $M(t)$, $t \in (-\varepsilon, \varepsilon)$ of surfaces such that $M(0) = M$. For each $t \in (-\varepsilon, \varepsilon)$, let an isometry $\iota_t \colon M \to M(t)$ be given. With each surface $M(t)$, let us associate a field of frames $\sigma(t) = \{m(t), v_1(t), v_2(t), v_3(t)\}$ such that $m(t) = \iota_t(m)$, $v_1(t) = \mathrm{d}\iota_t(v_1)$, $v_2(t) = \mathrm{d}\iota_t(v_3)$. Then we may write, for each t,

(16) $$\mathrm{d}m(t) = \omega^1\, v_1(t) + \omega^2\, v_2(t)\,, \quad \mathrm{d}v_i(t) = \omega_i^j(t)\, v_j(t)$$

with

(17) $$\omega_1^1(t) + \omega_2^2(t) + \omega_3^3(t) = 0$$

and $\omega_i^j(0) = \omega_i^j$. Further, let the frames $\sigma(t)$ be chosen in such a way that we have, for each t,

(18) $$\omega_1^3(t) = \omega^1\,, \quad \omega_2^3(t) = \omega^2\,, \quad \omega_3^3(t) = 0\,.$$

Define

(19) $$\varphi_i^j := (\mathrm{d}\omega_i^j(t)/\mathrm{d}t)_{t=0}\,;$$

of course, (18) and (17) imply

(20) $$\varphi_1^3 = \varphi_2^3 = \varphi_3^3 = 0\,, \quad \varphi_1^1 + \varphi_2^2 = 0\,.$$

Taking into account (18), the integrability conditions of (16) are, for t fixed,

(21) $$\begin{aligned}
\mathrm{d}\omega^1 &= \omega^1 \wedge \omega_1^1(t) + \omega^2 \wedge \omega_2^1(t)\,,\\
\mathrm{d}\omega^2 &= \omega^1 \wedge \omega_1^2(t) + \omega^2 \wedge \omega_2^2(t)\,,\\
\mathrm{d}\omega_1^1(t) &= \omega_1^2(t) \wedge \omega_2^1(t) + \omega^1 \wedge \omega_3^1(t)\,,\\
\mathrm{d}\omega_2^2(t) &= \omega_2^1(t) \wedge \omega_1^2(t) + \omega^2 \wedge \omega_3^2(t)\,,\\
0 &= \omega_3^1(t) \wedge \omega^1 + \omega_3^2(t) \wedge \omega^2\,,\\
\mathrm{d}\omega_1^2(t) &= \omega_1^2(t) \wedge (\omega_2^2(t) - \omega_1^1(t)) + \omega^1 \wedge \omega_3^2(t)\,,
\end{aligned}$$

$$
\begin{aligned}
d\omega^1 &= -\omega^1 \wedge \omega_1^1(t) + \omega_1^2(t) \wedge \omega^2 ,\\
d\omega^2 &= -\omega^2 \wedge \omega_2^2(t) + \omega_2^1(t) \wedge \omega^1 ,\\
d\omega_2^1(t) &= \omega_2^1(t) \wedge (\omega_1^1(t) - \omega_2^2(t)) + \omega^2 \wedge \omega_3^1(t) ,\\
d\omega_3^1(t) &= \omega_3^1(t) \wedge \omega_1^1(t) + \omega_3^2(t) \wedge \omega_2^1(t) ,\\
d\omega_3^2(t) &= \omega_3^2(t) \wedge \omega_2^2(t) + \omega_3^1(t) \wedge \omega_1^2(t) .
\end{aligned}
$$

Applying $(d/dt)_{t=0}$ to the equations (21) and taking into account (20), we get

$$
\begin{aligned}
(22)\qquad & 0 = \varphi_1^1 \wedge \omega^1 + \varphi_2^1 \wedge \omega^2 , \quad 0 = \varphi_1^2 \wedge \omega^1 - \varphi_1^1 \wedge \omega^2 ,\\
& d\varphi_1^1 = \varphi_1^2 \wedge \omega_2^1 + \omega_1^2 \wedge \varphi_2^1 + \omega^1 \wedge \varphi_3^1 ,\\
& -d\varphi_1^1 = \varphi_2^1 \wedge \omega_1^2 + \omega_2^1 \wedge \varphi_1^2 + \omega^2 \wedge \varphi_3^2 ,\\
& 0 = \varphi_3^1 \wedge \omega^1 + \varphi_3^2 \wedge \omega^2 ,\\
& d\varphi_1^2 = \varphi_1^2 \wedge (\omega_2^2 - \omega_1^1) - 2\omega_1^2 \wedge \varphi_1^1 + \omega^1 \wedge \varphi_3^2 ,\\
& 0 = \varphi_1^1 \wedge \omega^1 + \varphi_1^2 \wedge \omega^2 , \quad 0 = \varphi_2^1 \wedge \omega^1 - \varphi_1^1 \wedge \omega^2 ,\\
& d\varphi_2^1 = \varphi_2^1 \wedge (\omega_1^1 - \omega_2^2) + 2\omega_2^1 \wedge \varphi_1^1 + \omega^2 \wedge \varphi_3^1 ,\\
& d\varphi_3^1 = \varphi_3^1 \wedge \omega_1^1 + \omega_3^1 \wedge \varphi_1^1 + \varphi_3^2 \wedge \omega_2^1 - \omega_3^2 \wedge \varphi_2^1 ,\\
& d\varphi_3^2 = \varphi_3^2 \wedge \omega_2^2 - \omega_3^2 \wedge \varphi_1^1 + \omega_3^1 \wedge \omega_1^2 + \omega_3^1 \wedge \varphi_1^2 .
\end{aligned}
$$

From $(22_{2,8})$ and $(22_{1,7})$, $(\varphi_1^2 - \varphi_2^1) \wedge \omega^1 = (\varphi_1^2 - \varphi_2^1) \wedge \omega^2 = 0$ follows, i.e.,

$$
(23)\qquad \varphi_2^1 = \varphi_1^2 .
$$

Because of (23), (17) and (18_3), (22) reduce to

$$
\begin{aligned}
(24)\qquad & \varphi_1^1 \wedge \omega^1 + \varphi_1^2 \wedge \omega^2 = 0 , \quad \varphi_1^2 \wedge \omega^1 - \varphi_1^1 \wedge \omega^2 = 0 ,\\
& \varphi_3^1 \wedge \omega^1 + \varphi_3^2 \wedge \omega^2 = 0 , \quad 4\varphi_1^2 \wedge \omega_1^1 + 2(\omega_2^1 + \omega_1^2) \wedge \varphi_1^1 + \varphi_3^2 \wedge \omega^1 - \varphi_3^1 \wedge \omega^2 = 0 ;
\end{aligned}
$$

$$
\begin{aligned}
(25)\qquad d\varphi_1^1 &= \varphi_1^2 \wedge (\omega_2^1 - \omega_1^2) + \omega^1 \wedge \varphi_3^1 ,\\
d\varphi_1^2 &= -2\omega_1^2 \wedge \varphi_1^1 - 2\varphi_1^2 \wedge \omega_1^1 + \omega^1 \wedge \varphi_3^2 ,\\
d\varphi_3^1 &= \omega_3^1 \wedge \varphi_1^1 + \varphi_3^1 \wedge \omega_1^1 + \varphi_3^2 \wedge \omega_2^1 + \omega_3^2 \wedge \varphi_1^2 ,\\
d\varphi_3^2 &= -\omega_3^2 \wedge \varphi_1^1 - \varphi_3^2 \wedge \omega_1^1 + \omega_3^1 \wedge \varphi_1^2 + \varphi_3^1 \wedge \omega_1^2 .
\end{aligned}
$$

Thus it is reasonable to define the *infinitesimal isometries* Φ of our surface M as sets of forms $\{\varphi_i^j\}$ satisfying (20) + (23) + (24) + (25).

From (24_{1-3}), we get the existence of functions a', b', α', β', γ' such that

$$
\begin{aligned}
(26)\qquad & \varphi_1^1 = -\tfrac{1}{2}c'\omega^1 + \tfrac{1}{2}b'\omega^2 , \quad \varphi_1^2 = \tfrac{1}{2}b'\omega^1 + \tfrac{1}{2}c'\omega^2 ,\\
& \varphi_3^1 = \alpha'\omega' + \beta'\omega^2 , \quad \varphi_3^2 = \beta'\omega^1 + \gamma'\omega^2 ;
\end{aligned}
$$

the equation (24_4) reduces then, using (5) + (6), to

$$
(27)\qquad 2(bb' + cc') = \alpha' + \gamma' .
$$

The differential consequences of (26) are, because of (25),

$$
\begin{aligned}
(28)\qquad & (dc' + 3b'\omega) \wedge \omega^1 - (db' - 3c'\omega) \wedge \omega^2 = -2\beta'\omega^1 \wedge \omega^2 ,\\
& (db' - 3c'\omega) \wedge \omega^1 + (dc' + 3b'\omega) \wedge \omega^2 = (\gamma' - \alpha')\,\omega^1 \wedge \omega^2 ,
\end{aligned}
$$

$$(\mathrm{d}\alpha' - 2\beta'\omega) \wedge \omega^1 + (\mathrm{d}\beta' + (\alpha' - \gamma')\,\omega) \wedge \omega^2 =$$
$$= \{\tfrac{1}{2}(\alpha - \gamma)\, b' + \beta c' + \tfrac{1}{2} b(\alpha' - \gamma') + c\beta'\}\, \omega^1 \wedge \omega^2 ,$$
$$(\mathrm{d}\beta' + (\alpha' - \gamma')\,\omega) \wedge \omega^1 + (\mathrm{d}\gamma' + 2\beta'\omega) \wedge \omega^2 =$$
$$= \{-\beta b' + \tfrac{1}{2}(\alpha - \gamma)\, c' + \tfrac{1}{2} c(\alpha' - \gamma') - b\beta'\}\, \omega^1 \wedge \omega^2 .$$

We have $\omega_i^j(t) = \omega_i^j + t\varphi_i^j + O(t^2)$. Comparing (5) + (6) with (26), it turns out that the variations of the forms (10) + (11) are

$$(29)\qquad \delta\mathscr{A} = -\tfrac{1}{2}\{c'(\omega^1)^3 - 3b'(\omega^1)^2\,\omega^2 - 3c'\omega^1(\omega^2)^2 + b'(\omega^2)^3\} ,$$
$$\delta\mathscr{B} = -\{\alpha'(\omega^1)^2 + 2\beta'\omega^1\omega^2 + \gamma'(\omega^2)^2\} ,$$
$$\delta\mathscr{C} = \beta'(\omega^1)^2 + (\alpha' - \gamma')\,\omega^1\omega^2 - \beta'(\omega^2)^2 ;$$

the variations of the invariants (12) + (13) are then

$$(30)\qquad \delta J = bb' + cc' , \quad \delta H = -\tfrac{1}{2}(\alpha' + \gamma') , \quad \delta K = \alpha\gamma' + \gamma\alpha' - 2\beta\beta' .$$

Because of (27), $\delta\varkappa = \delta J + \delta H = 0$ holds, this being the infinitesimal version of the theorema egregium.

Let us restrict ourselves to a coordinate neighbourhood G of our surface M. In G, let us choose coordinates (x, y) such that

$$(31)\qquad \omega^1 = r\,\mathrm{d}x , \quad \omega^2 = r\,\mathrm{d}y ; \qquad r = r(x, y) > 0 .$$

From (8), we get

$$(32)\qquad \omega = -r^{-1}r_y\,\mathrm{d}x + r^{-1}r_x\,\mathrm{d}y ;$$

here, $r_x = \partial r/\partial x$, etc. The equations $(28_{1,2})$ yield the existence of functions $B_1, \ldots, C_2$ such that

$$(33)\qquad \mathrm{d}b' - 3c'\omega = B_1\omega^1 + B_2\omega^2 , \quad \mathrm{d}c' + 3b'\omega = C_1\omega^1 + C_2\omega^2 ;$$

$$(34)\qquad B_1 + C_2 = 2\beta' , \quad C_1 - B_2 = \gamma' - \alpha' .$$

Analogously, equations $(28_{3,4})$ and Cartan's lemma imply the existence of functions $D_1, \ldots, F_2$ such that

$$(35)\qquad \mathrm{d}\alpha' - 2\beta'\omega = D_1\omega^1 + D_2\omega^2 , \quad \mathrm{d}\beta' + (\alpha' - \gamma')\,\omega = E_1\omega^1 + E_2\omega^2 ,$$
$$\mathrm{d}\gamma' + 2\beta'\omega = F_1\omega^1 + F_2\omega^2 ;$$

$$(36)\qquad E_1 - D_2 = \tfrac{1}{2}(\alpha - \gamma)\, b' + \beta c' + \tfrac{1}{2} b(\alpha' - \gamma') + c\beta' ,$$
$$F_1 - E_2 = -\beta b' + \tfrac{1}{2}(\alpha - \gamma)\, c' + \tfrac{1}{2} c(\alpha' - \gamma') - b\beta' .$$

Further, the differential consequences of (5) are

$$(37)\qquad (\mathrm{d}b - 3c\omega) \wedge \omega^1 + (\mathrm{d}c + 3b\omega) \wedge \omega^2 = -(\alpha - \gamma)\,\omega^1 \wedge \omega^2 ,$$
$$-(\mathrm{d}c + 3b\omega) \wedge \omega^1 + (\mathrm{d}b - 3c\omega) \wedge \omega^2 = 2\beta\omega^1 \wedge \omega^2 ;$$

thus there are functions $b_1, \ldots, c_2$ such that

$$(38)\qquad \mathrm{d}b - 3c\omega = b_1\omega^1 + b_2\omega^2 , \quad \mathrm{d}c + 3b\omega = c_1\omega^1 + c_2\omega^2 ;$$

$$(39)\qquad c_1 - b_2 = -(\alpha - \gamma) , \quad b_1 + c_2 = 2\beta .$$

From (27), we get

$$(40) \qquad 2b_1 b' + 2c_1 c' + 2bB_1 + 2cC_1 = D_1 + F_1 ,$$
$$2b_2 b' + 2c_2 c' + 2bB_2 + 2cC_2 = D_2 + F_2 .$$

Now, let us take use of our coordinates (x, y). Inserting (31) and (32) into (33) and (35), we get

$$(41) \qquad rB_1 = b'_x + 3r^{-1}r_y c' , \quad rB_2 = b'_y - 3r^{-1}r_x c' ,$$
$$rC_1 = c'_x - 3r^{-1}r_y b' , \quad rC_2 = c'_y + 3r^{-1}r_x b' ;$$
$$r(D_1 - F_1) = (\alpha' - \gamma')_x + 4r^{-1}r_y\beta' , \quad rE_1 = \beta'_x - r^{-1}r_y(\alpha' - \gamma') ,$$
$$r(D_2 - F_2) = (\alpha' - \gamma')_y - 4r^{-1}r_x\beta' , \quad rE_2 = \beta'_y + r^{-1}r_x(\alpha' - \gamma') .$$

Using (41_{1-4}), the equations (34) turn out to be

$$(42) \qquad b'_x + c'_y + 3r^{-1}r_x b' + 3r^{-1}r_y c' - 2r\beta' = 0 ,$$
$$c'_x - b'_y - 3r^{-1}r_y b' + 3r^{-1}r_x c' + r(\alpha' - \gamma') = 0 .$$

Consider the trivial identities

$$(43) \qquad D_1 + F_1 = D_1 - F_1 + 2(F_1 - E_2) + 2E_2 ,$$
$$D_2 + F_2 = F_2 - D_2 + 2(D_2 - E_1) + 2E_1 ;$$

inserting into them from (40) and (41_{5-8}), we get

$$(44) \qquad (\alpha' - \gamma')_x + 2\beta'_y - 2bb'_x - 2cc'_x + (2r^{-1}r_x + rc)(\alpha' - \gamma') +$$
$$+ 2(2r^{-1}r_y - rb)\beta' - 2(rb_1 - 3r^{-1}r_y c + r\beta) b' -$$
$$- (2rc_1 + 6r^{-1}r_y b - r(\alpha - \gamma)) c' = 0 ,$$
$$2\beta'_x - (\alpha' - \gamma')_y - 2bb'_y - 2cc'_y - (2r^{-1}r_y + rb)(\alpha' - \gamma') +$$
$$+ 2(2r^{-1}r_x - rc)\beta' - (2rb_2 + 6r^{-1}r_x c + r(\alpha - \gamma)) b' -$$
$$- 2(rc_2 - 3r^{-1}r_x b + r\beta) c' = 0 .$$

Thus we have proved

Lemma 1. *Let $G \subset M$ be a coordinate neighbourhood of a surface $M \subset A^3$, let the coordinates (x, y) in G be chosen in such a way that*

$$(45) \qquad ds^2 = r^2(dx^2 + dy^2) ,$$
$$-2\mathscr{A} = r^3(c\, dx^3 - 3b\, dx^2\, dy - 3c\, dx\, dy^2 + b\, dy^3) ,$$
$$-\mathscr{B} = r^2(\alpha\, dx^2 + 2\beta\, dx\, dy + \gamma\, dy^2) .$$

Let Φ be an infinitesimal isometry of G such that

$$(46) \qquad -2\delta\mathscr{A} = r^3(c'\, dx^3 - 3b'\, dx^2\, dy - 3c'\, dx\, dy^2 + b'\, dy^3) ,$$
$$\delta\mathscr{C} = r^2(\beta'\, dx^2 + (\alpha' - \gamma')\, dx\, dy - \beta'\, dy^2) .$$

On G, define the functions

$$(47) \qquad u_1 = b' , \quad u_2 = c' , \quad u_3 = \alpha' - \gamma' , \quad u_4 = 2\beta' ;$$

let

(48) $$u = (u_1, u_2, u_3, u_4)^T .$$

Then

(49) $$u_x + Bu_y + Cu = 0$$

with

(50) $$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2c & 2b & 0 & 1 \\ -2b & -2c & -1 & 0 \end{pmatrix}.$$

Indeed: Consider the equations (42) + (44); in (44), replace b'_x and c'_x by the values calculated from (42).

Lemma 2. *Let the situation be as in Lemma 1. On G, consider the complex variable $z = x + iy$ and the usual operators*

(51) $$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right).$$

Define the functions

(52) $$w_1 := c' + \mathrm{i}b' , \quad w_2 := 2\beta' + \mathrm{i}(\alpha' - \gamma') .$$

Then

(53) $$\frac{\partial w_1}{\partial \bar{z}} + 3r^{-1}\frac{\partial r}{\partial \bar{z}} w_1 - \tfrac{1}{2}\mathrm{i}rw_2 = 0 ,$$

$$\frac{\partial w_2}{\partial \bar{z}} - (b + \mathrm{i}c)\frac{\partial w_1}{\partial z} - \left\{r(\beta - \tfrac{1}{2}\mathrm{i}(\alpha - \gamma)) + \frac{\partial(c - \mathrm{i}b)}{\partial \bar{z}}\right\} w_1 + 2r^{-1}\frac{\partial r}{\partial \bar{z}} w_2 -$$

$$- \left\{3r^{-1}\left(\frac{\partial r}{\partial \bar{z}} b - \mathrm{i}\frac{\partial r}{\partial z} c\right) - \frac{\partial(b - \mathrm{i}c)}{\partial z}\right\} \bar{w}_1 - r(c + \mathrm{i}b)\,\bar{w}_2 = 0 .$$

Proof. Comparing the real and imaginary parts of (53), we get a system equivalent to (42) + (44). The functions $b_1. \ldots, c_2$ are to be calculated from (38). QED.

Theorem 1. *Let G be a simply connected bounded domain on an elliptic quadratic surface $M \subset A^3$. On G, let us choose coordinates (x, y) such that $\mathrm{d}s^2 = r^2(\mathrm{d}x^2 + \mathrm{d}y^2)$, $r = r(u, v) > 0$. Let v_1, v_2 be the unit vector fields (with respect to $\mathrm{d}s^2$) tangent to the curves $y =$ const. or $x =$ const. resp. Let Φ be an infinitesimal isometry of G possessing the variations (46). Suppose:*

(54) $$\delta\mathscr{A}(v_1) = 0 , \quad \delta\mathscr{C}(v_1) = 0 \quad \textit{on} \quad \delta G ,$$

(55) $$\delta\mathscr{A}(v_2) = 0 \quad \textit{at some point} \quad z_0 \in \partial G ,$$

(56) $$\delta\mathscr{C}(v_1 + v_2) = 0 \quad \textit{at some point} \quad z_1 \in \partial G .$$

Then Φ is trivial on G.

Proof. On a quadratic surface,

$$b = c = 0\,, \quad \beta = 0\,, \quad \alpha = \gamma\,; \tag{57}$$

this is well known. Thus the system (53) reduces to

$$\frac{\partial(r^3 w_1)}{\partial \bar{z}} = \tfrac{1}{2} i r^4 w_2\,, \quad \frac{\partial(r^2 w_2)}{\partial \bar{z}} = 0\,. \tag{58}$$

This means that $r^2 w_2$ is a holomorphic function on G; (54_2) reads $\operatorname{Re}(r^2 \omega_2) = 0$ on ∂G, (56) is then $\operatorname{Im}(r^2 w_2) = 0$ at the point $z_1 \in \partial G$. But this means $w_2 \equiv 0$ in G. Now, $r^3 w_1$ is holomorphic in G, and we apply the same procedure to ensure $w_1 \equiv 0$ in G. Thus $b' = c' = 0$, $\beta' = 0$, $\alpha' = \gamma'$ in G. Finally, from (27) we get $\alpha' = \gamma' = 0$ in G. QED.

Theorem 2. *Let G be a simply connected bounded domain on an elliptic analytic surface $M \subset A^3$. Let Φ be an infinitesimal isometry of G. Let $\gamma \subset G$ be an arc, and let $\delta\mathcal{A} = 0$ and $\delta\mathcal{C} = 0$ on γ. Then Φ is trivial in G.*

Proof. The system (49) is clearly elliptic, and we may use Carleman's theorem claiming that the zeroes of a non-trivial solution are isolated. For this, see [4], Theorem 5.4.1. Thus $b' = c' = \beta' = \alpha' - \gamma' = 0$ in G, and we are finished. QED.

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