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ON H-CLOSED GRAPHS

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0. INTRODUCTION

We start with four questions. All these questions are in a special way similar. It is this similarity which is of interest.

a) For which Hamiltonian graphs G, each subgraph of G, isomorphic to a connected subgraph of a Hamiltonian cycle of G, can be extended to a Hamiltonian cycle of G?

The answer was given by Chartrand and Kronk [1] in 1968. Namely, G is iso-isomorphic to one of K_n, C_n and $K_{n/2,n/2}$.

b) For which Eulerin graphs G, each trail of G can be extended to an Eulerian circuit of G?

The answer was given by Ore [8] in 1951.

c) For which perfect matchable graphs G, each subgraph of G, isomorphic to a subgraph of a perfect matching of G without isolated vertices, can be extended to a perfect matching of G?

The answer was given by Sumner [12] in 1979. G is isomorphic to one of K_{2n} or $K_{n,n}$.

d) For which graphs G containing a path of length $n - 1$, each such path can be extended to an n-cycle in G?

The answer was given by Dirac and Thomassen [5] in 1973. G is isomorphic to K_p, $p \geqq n$, C_n if n is odd and K_p, $p \geqq n$, C_n and $K_{r,s}$, $r + s = p$; $r, s \geqq \frac{1}{2}n$ if n is even.

One could continue asking and answering analogous questions. See, for example, [2], [3], [6], [7], [9]. However, for our purposes this sample of questions is quite sufficient.

Now we are coming to the crucial point. Let us generalize these questions in a natural manner. First we make the following important agreement:

From now on, all graphs considered will be tacitly assumed to be without isolated vertices.

Question. *For which graphs G, each subgraph of G, isomorphic to a subgraph of H, can be extended to H in G?*

This question was for the first time proposed by Chartrand, Oellermann and Ruiz [4].

Definition. A non-empty graph G is said to be *H-closed* if and only if
F is a subgraph of G that is isomorphic to a subgraph of H implies F can be extended to a subgraph of G which is isomorphic to H.

Obviously, every graph G is K_2-closed and also every graph G is G-closed. Further, K_n is H-closed for every $H \subset K_n$. A little bit less elementary results concerning a characterization of H-closed graphs for certain H of small size as well as when H is a star, can be found in [4] ("randomly H" is used there instead of the term H-closed). There is also a characterization of C_n-closed graphs. The authors essentially used the following result of Dirac and Thomassen [5].

C_p-closed graphs for $p \geqq 5$ are K_n and C_p if p is odd and K_n, C_p and $K_{p/2,p/2}$ if p is even.

In the study of H-closed graphs rather surprising facts occur. For example, intuitively it seems true that if the only H-closed graphs are complete graphs then the only F-closed graphs, for $H \subset F$, are complete graphs as well. However, this intuitive feeling does not work. The only C_5-closed graph on five vertices different from C_5 is K_5. However, for G equal to C_5 with one chord, 5-vertex G-closed graphs may be essentially different from K_5, see Fig. 1.

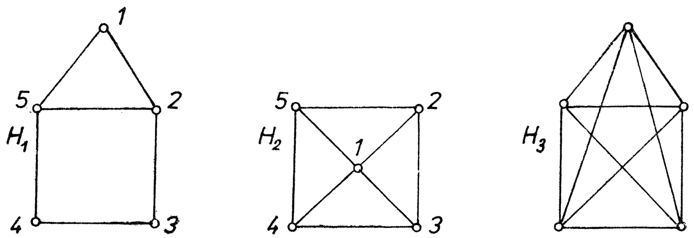

Fig. 1. 5-vertex G-closed graphs.

For six vertices, the G-closed and C_5-closed graphs are also essentially different. For the G-closed graphs on 6 vertices see Fig. 2.

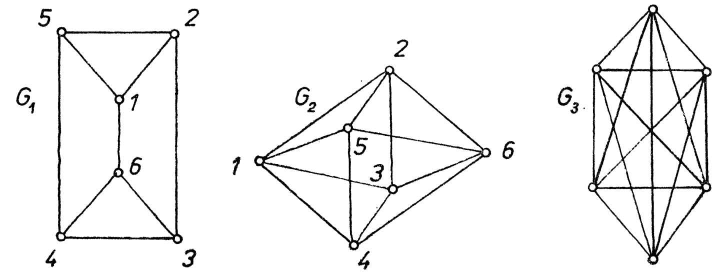

Fig. 2. 6-vertex G-closed graphs.

On the other hand, there is a single C_5-closed graph on six vertices, namely K_6. However, up from 7 vertices the only G-closed graph and C_5-closed graph is the complete graph.

Up to now, as far as we know, there is no satisfactory theory of H-closed graphs, although the present state calls for it. We try to establish modest beginnings of such a theory. Then we apply this theory to obtain several results concerning the characterization of H-closed graphs for certain infinite families of H's. At the end we indicate some open problems and further possible development of the theory.

1. MAIN RESULTS

As we remarked above, this article revolves primarily around the theory of H-closed graphs. As a consequence we characterize H-closed graphs for H regular of degree $r \geqq 2$. Another infinite class of graphs for which we obtain the characterization results is a family of cycles with one special chord, the so called triangle chord, as well as a family of cycles with two special triangle chords. We also find a large family of connected free graphs, all with degrees at least two, for which there exist closed graphs different from the complete graphs K_n for every sufficiently large natural n.

Before we state the results let us fix the notation used throughout the paper.

The family of all H-closed graphs will be denoted by $\mathfrak{S}(H)$ and the family of all n-vertex H-closed graphs by $\mathfrak{S}_n(H)$. The H-cloure $\mathfrak{S}_n(H, G)$ of an n-vertex graph G will be the set of all minimal graphs from $\mathfrak{S}_n(H)$ containing G.

1.1. Closedness Criterion. Lemma 1. *If $G \in \mathfrak{S}(H)$ then $\mathfrak{S}(G) \subset \mathfrak{S}(H)$.*

Proof. Let $M \in \mathfrak{S}(G)$. We shall prove $M \in \mathfrak{S}(H)$. That is, we have to show: if F is a subgraph of M that is isomorphic to a subgraph of H, then F can be extended to a subgraph of M which is isomorphic to H.

Given a subgraph F of M that is isomorphic to a subgraph of H, then F is a subgraph of M that is isomorphic to a subgraph of G, since $G \in \mathfrak{S}(H)$. Thus $M \in \mathfrak{S}(G)$ implies that F can be extended to a subgraph G' of M which is isomorphic to G Since $F \subset G'$, from $G \in \mathfrak{S}(H)$ we have that F can be extended to a subgraph H' of G' which is isomorphic to H and this completes the proof.

Corollary. *If $G \in \mathfrak{S}_n(H)$ then $\mathfrak{S}_n(G) \subset \mathfrak{S}_n(H)$.*

The proof is evident and is left to the reader.

The main problem in the theory of H-closed graphs is the following:

We are given two graphs G and H. How to find out whether the graph G is H-closed?

One can proceed by definition. However, in most cases this procedure requires a great amount of time and is horrible even for small graphs. This can be demonstrated for example if one tries to prove the graph G in Fig. 3 to be H-closed.

We will now direct our effort to developing some useful criterion for H-closedness.

Let G and H be given graphs. The notation $H \subset G$ will be an abbreviation for the correct but needlessly pedantic formulation: there exists a subgraph H' of G that is

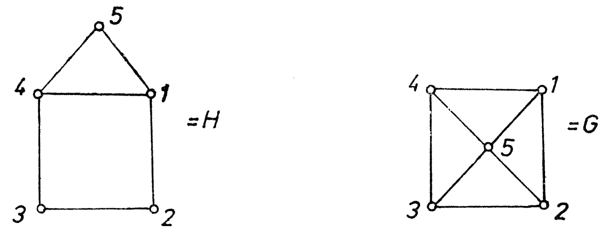

Fig. 3

isomorphic to H. Assign to every edge $(ab) \in G$ a boolean variable x_{ab} and construct the following boolean expression V:

$$V = \prod_{H \subset G} \sum_{e \in E(G) - E(H)} x_e \, .$$

Let $S = \{x_{e_1}, x_{e_2}, \ldots, x_{e_k}\}$ be a family of variables for which the expression V is true. Denote by F_S the graph consisting of the edges $\{e_1, e_2, \ldots, e_k\}$ which correspond to the variables in S.

Now we can state

Proposition 1. *Let G and H be graphs, $H \subset G$. If $F_S \not\subset H$ holds for every family $S = \{x_{e_1}, x_{e_2}, \ldots, x_{e_k}\}$ of boolean variables for which the boolean expression*

$$V = \prod_{H \subset G} \sum_{e \in E(G) - E(H)} x_e$$

is true, then G is H-closed.

Proof. Assume G is not H-closed. By definition there exists $F \subset H \subset G$ such that F is not extendible to H in G. Let us interpret the boolean variables x_e as follows: set

$$x_e = 1 \quad \text{if} \quad e \in F$$
$$x_e = 0 \quad \text{otherwise} \, .$$

If we take some $H \subset G$, the graph F must contain at least one edge from $E(G) - E(H)$, since F is not extendible to any H in G. Thus the expression

$$V_H = \sum_{e \in E(G) - E(H)} x_e$$

is true. This conclusion is true for any subgraph H of G and we conclude that the boolean expression

$$V = \prod_{H \subset G} V_H$$

is true.

Since F is not extendible to H in G and V is true, there exists a system $S =$

$= \{x_{e_1}, x_{e_2}, \ldots, x_{e_k}\}$ such that $F_S \subset F$ and V is true for S. However, $F_S \subset H$, since $F \subset H$ and we have a contradiction with the assumption. The proof is complete.

Let G be an H-closed graph. Assume that V is true for $S = \{x_{e_1}, x_{e_2}, \ldots, x_{e_k}\}$ and $F_S \subset H$ for some H in G. But then the boolean expression V contains an expression $\sum_{e \in E(G)-E(H)} x_e$ which has no member from S. Thus V is not true, a contradiction. This proves the converse implication in Proposition 1 and we have the desired criterion:

Closedness Criterion: *G is H-closed if and only if $F_S \not\subset H$ holds for every (minimal) system $S = \{x_{e_1}, x_{e_2}, \ldots, x_{e_k}\}$ for which the boolean expression*

$$V = \prod_{H \subset G} \sum_{e \in E(G)-E(H)} x_e$$

is true.

Now we illustrate the usefulness of Closedness Criterion on some examples. Later, in Section 1.3, we use this criterion to demonstrate the H-closedness for an infinite class of graphs.

Example 1. Let G and H be the two graphs in Fig. 4.

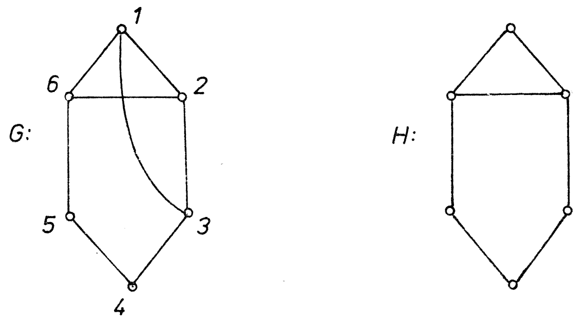

Fig. 4

We prove that G is H-closed. Let us apply Closedness Criterion. The graph G contains exactly four graphs H:

$$H_1 = \{(12), (23), (34), (45), (56), (16), (26)\},$$
$$H_2 = \{(12), (26), (56), (45), (34), (13), (16)\},$$
$$H_3 = \{(12), (26), (56), (45), (34), (13), (23)\},$$
$$H_4 = \{(13), (16), (56), (45), (34), (23), (12)\}.$$

Thus the boolean expression V has the form

$$V = x_{13} \wedge x_{23} \wedge x_{16} \wedge x_{26}.$$

There exists only one minimal system S for which V is true namely $S = \{(13), (23),$

$\{(16), (26)\}$. The resulting graph F_S is the cycle C_4 on four vertices. Since $C_4 \nsubseteq H$, we obtain by Closedness Criterion that the graph G is H-closed.

Example 2. Let G and H be graphs in Fig. 3. We prove G is H-closed. The graph G contains exactly twelve graphs H. The corresponding sets $E(G) - E(H)$ are the following:

$$\{(15), (25)\}, \ \{(25), (35)\}, \ \{(35), (45)\}, \ \{(45), (15)\}\,,$$
$$\{(12), (35)\}, \ \{(12), (45)\}, \ \{(23), (45)\}, \ \{(23), (15)\}\,,$$
$$\{(34), (15)\}, \ \{(34), (25)\}, \ \{(14), (35)\}, \ \{(14), (25)\}\,.$$

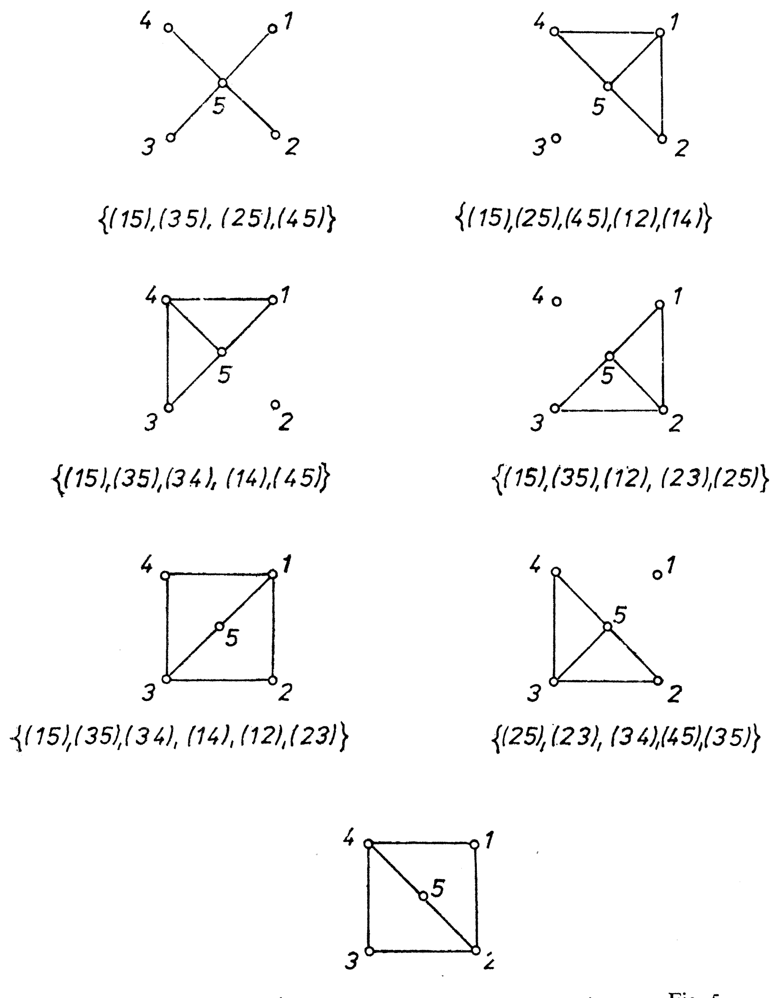

Fig. 5

The boolean expression V has the form

$$V = (x_{15} \vee x_{25}) \wedge (x_{25} \vee x_{35}) \wedge (x_{35} \vee x_{45}) \wedge (x_{45} \vee x_{15}) \wedge$$
$$\wedge (x_{12} \vee x_{35}) \wedge (x_{12} \vee x_{45}) \wedge (x_{23} \vee x_{45}) \wedge (x_{23} \vee x_{15}) \wedge$$
$$\wedge (x_{34} \vee x_{15}) \wedge (x_{34} \vee x_{25}) \wedge (x_{14} \vee x_{35}) \wedge (x_{14} \vee x_{25}) =$$
$$= [x_{15} \vee (x_{25} \wedge x_{23} \wedge x_{34} \wedge x_{45})] \wedge [x_{35} \vee (x_{25} \wedge x_{45} \wedge x_{12} \wedge x_{14})] \wedge$$
$$\wedge [x_{25} \vee (x_{34} \wedge x_{14})] \wedge [x_{45} \vee (x_{12} \wedge x_{23})] .$$

If $S_1 \subset S_2$ then $F_{S_1} \not\subset H$ implies $F_{S_2} \not\subset H$. Consequently, it is sufficient to investigate only minimal S for which V is true. There are exactly seven such S. The corresponding F_S are shown in Fig. 5.

None of these F_S are contained in H. Thus, by Closedness Criterion, G is H-closed.

1.2. Regular graphs. Lemma 2. *Let H be a connected graph on at least four vertices which is different from a star. Then every H-closed graph is connected.*

Proof. Let G be an H-closed graph which has at least two components. Take two edges from different components of G. These two edges form the graph $F = K_2 \cup K_2$ which is a subgraph of H. Since G is H-closed, F must be extendible to H in G. But H is connected and we have a contradiction. This completes the proof.

Assume we are given a connected n-vertex regular graph H_n^r of degree $r \geqq 2$. Let G_p be an H_n^r-closed graph on p vertices, $p > n$. Since K_3-closed graphs and C_4-closed graphs are characterized in [4], from now on we will suppose $H_n^r \neq K_3, C_4$. Denote by $\Gamma(a)$ the neighbourhood of a vertex a.

Lemma 3. *Let v, x and w be three vertices of G_p with the following properties:*
(i) $w \in G_p - H_n^r$,
(ii) $v, x \in H_n^r$,
(iii) *the edges* (wv), (vx) *are in* G_p.
Then $\Gamma(v) \subset \Gamma(w)$.

Proof. Form a graph F as follows: from H_n^r delete the vertex v and add the edge (wx). Obviously, F is a subgraph of H_n^r and thus it can be extended to H_n^r in G_p. However, the only possibility to extend F to H_n^r is to add precisely those edges (wa) for which the edge (va) was deleted from H_n^r. This completes the proof.

Lemma 4. *Let w be a vertex of G_p which is joined to some vertex of H_n^r $(w \notin H_n^r)$. Then*

$$V(H_n^r) \subset \Gamma(w) .$$

Proof. Put $V_1 = \{v \in H_n^r, (wv) \in G_p\}$, $V_2 = V(H_n^r) - V_1$.

Let $v_1, v_2 \in V_1$ and $(v_1 v_2) \in E(H_n^r)$. If $V_2 = \emptyset$ there is nothing to prove. Thus we assume $z \in V_2$. Since H_n^r is connected, we can suppose without loss of generality that there exists an edge which joins z and v_1 or v_2, say v_2. However, the vertices w, v_1 and v_2 fulfil the assumptions of Lemma 3. Thus $\Gamma(v_2) \subset \Gamma(w)$. This implies $(zw) \in G_p$, which is a contradiction and the set V_1 is independent.

Let $v_1, v_2 \in V_2$ and $(v_1v_2) \in E(H_n^r)$. As above, we can assume that there exists an edge which joins v_1 or v_2, say v_2, with a vertex $z \in V_1$. However, the vertices w, z and v_2 fulfil the assumptions of Lemma 3. Thus $\Gamma(v_2) \subset \Gamma(w)$. This implies $(v_1w) \in G_p$, which is a contradiction and the set V_2 is also independent.

Since V_1 and V_2 are independent, the graph H_n^r is bipartite. However, the graph H_n^r is regular, hence $|V_1| = |V_2|$. But in this case the well known König's theorem implies the existence of an 1-factor of H_n^r. Denote it by F. Let (v_iu_i) be edges of F, $v_i \in V_1$ and $u_i \in V_2$. Since $H_n^r \neq K_3$, C_4 we can suppose without loss of generality that the edge $(v_2u_3) \in H_n^r$ exists. Form a graph F' as follows (see Fig. 6): add to F the edges (wv_1) and (v_2u_3) and delete the vertex v_3 from F.

Obviously, F' is a subgraph of H_n^r. Thus, F' can be extended to H_n^r in G_p. The vertices v_1 and w must belong to different parts of H_n^r. The same is true for the vertices v_2 and w. Thus v_1 and v_2 are contained in the same part of H_n^r. Hence the parts of H_n^r have different cardinality, which is a contradiction. Hence the proof is complete.

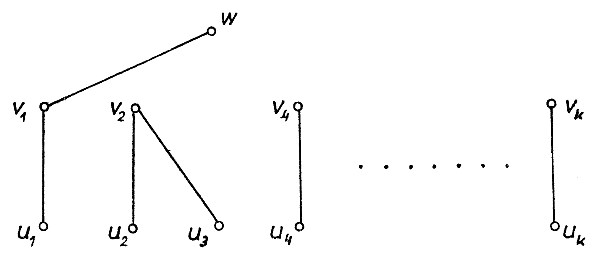

Fig. 6

Let two graphs G and H be given, $H \subset G$. The graph G will be called the *roof with respect to H* if for $H \subset G$,

$$v \in (V(G) - V(H)) \wedge \Gamma(v) \cap V(H) \neq 0 \quad \text{implies} \quad V(H) \subset \Gamma(v) .$$

Lemma 5. *Let H_n be a given n-vertex graph and let G be H_n-closed and the roof with respect to H_n. Moreover, let G contain a complete n-vertex graph K_n. Then G is a complete graph.*

Proof. Let K be a maximal complete subgraph of G. Suppose $K \neq G$. Then there exists $v \in G - K$ and $x, y \in K$ such that $(vx) \notin G$ and $(vy) \in G$. The inclusion $K_n \subset G$ implies that there exists $H_n \subset K$ containing x and y. Since G is the roof with respect to H_n, $V(H_n) \subset \Gamma(v)$. Thus $(vx) \in G$, a contradiction. This completes the proof.

Theorem 1. *Let H_n^r be a connected n-vertex regular graph of degree $r \geqq 2$, different from K_3 and C_4. Then*

$$\mathfrak{S}_p(H_n^r) = \{K_p\} \quad \textit{for every} \quad p > n .$$

Proof. Let $G_p \in \mathfrak{S}_p(H_n^r)$. By Lemma 2, G_p is connected. Take any $H_n^r \subset G_p$ and any $w \in G_p - H_n^r$ such that w is adjacent to H_n^r. By Lemma 4, $V(H_n^r) \subset \Gamma(w)$.

This means G_p is the roof with respect to H_n^r. We shall prove that the graph H induced on $V(H_n^r) \cup \{w\}$ is K_{n+1}. Suppose this is not the case. Then there exist u, v such that $(uv) \notin G_p$. Form a graph F as follows: delete the edges (vx) from H_n^r and add the edges (wx). Clearly, F is isomorphic to H_n^r. The graph G_p is H_n^r-closed and v is joined to F, hence Lemma 4 yields $V(F) \subset \Gamma(v)$, which implies $(vu) \in G_p$, a contradiction. Hence $K_{n+1} \subset G_p$. By Lemma 5, G_p is a complete graph. The proof of Theorem 1 is complete.

Remark. Theorem 1 generalizes a previous result of Chartrand, Oellermann and Ruiz. Their paper [4] contains a characterization of cycle-closed graphs.

1.3. Cycles with one or two triangle chords. We introduce a suitable notation which will be used throughout this section. Denote by $[a_1, a_2, \ldots, a_n]$ the cycle C_n on n vertices $a_1, a_2, \ldots, a_n$, $n \geqq 3$. The edges of C_n are $(a_i a_{i+1})$ taken modulo n. Put $TC_n = C_n \cup (a_2 a_n)$. The sequence

$$a_i a_{i+1} \ldots a_{n-1} [a_n a_1 a_2] \, a_3 \ldots a_{i-1} \, , \quad 3 \leqq i \leqq n \, ,$$

denotes the graph $TC_n - (a_{i-1} a_i)$, and the sequence

$$a_1 [a_n a_2 a_3 \ldots a_{n-1}]$$

denotes the graph $TC_n - (a_1 a_2)$.

Before investigating TC_n-closed graphs for general n, we deal with the cases TC_4 and TC_5 separately.

Lemma 6. $\mathfrak{S}_n(TC_4) = \{K_n - e, K_n\}$ *for* $n \geqq 4$.

Proof. For $n = 5$ the statement of Lemma 6 is trivial. Take $n = 5$. Let $M \in \mathfrak{S}_5(TC_4)$. Then M contains one of the two graphs

i) $G_1 = [1234] \cup \{(24), (15)\} \cup \{5\}$,

ii) $G_2 = [1234] \cup \{(24), (25)\} \cup \{5\}$

since by Lemma 2, M is connected.

(i) Let $G_1 \subset M$. Then $5[124] \subset M$. Thus $(25) \in M$ or $(45) \in M$, say $(25) \in M$. Then $3[215] \subset M$ and we have

$(35) \in M$ or $(13) \in M$.

(i1) Put $(35) \in M$. Then $[1534] \subset M$ implies

$(13) \in M$ or $(45) \in M$.

(i11) Put $(45) \in M$. Then $M \supseteq K_5 - e \in \mathfrak{S}_5(TC_4)$.

(i12) Put $(13) \in M$. Then also $M \subset K_5 - e \in \mathfrak{S}_5(TC_4)$.

(i2) Put $(13) \in M$. Then $5[134] \subset M$ implies

$(45) \in M$ or $(35) \in M$.

In both cases $M \supseteq K_5 - e \in \mathfrak{S}_5(TC_4)$.

(ii) Let $G_2 \subset M$. Then $5[241]$. Thus $(15) \in M$ or $(45) \in M$.

(ii1) Let $(15) \in M$. Then $G_1 \subset M$. As in the case (i) this implies $M \supseteq K_5 - e$.

(ii2) Let $(45) \in M$. The graph $\{(12), (23), (25)\}$ is a subgraph of TC_4. Thus it can be extended to TC_4 in M.

Hence

$$(13) \in M \quad \text{or} \quad (15) \in M \quad \text{or} \quad (35) \in M .$$

In all these cases there exists $G_1 \subset M$. Thus $M \subset K_5 - e$ by (i).

Conclusion: $\mathfrak{S}_5(TC_4) = \{K_5 - e, K_5\}$.

Now take $n \geqq 6$. Let $M \in \mathfrak{S}(TC_4)$, $M \neq K_n$ and let $Q \subset M$ be the maximal complete graph without one edge in M. If $Q = M$, there is nothing to prove. Thus, assume $Q \neq M$. Since M is connected by Lemma 2, there exists a vertex $v \neq Q$ which is joined with a vertex of Q, say w. However, since Q is the maximal complete graph without one edge in M, there is a vertex $z \in Q$ with $(vz) \notin M$. Moreover, let $x, y \in Q$ and $(xy) \notin Q$. Obviously, the vertices v, w, z, x, y are contained in a 5-vertex subgraph G' of M which is isomorphic to G_1 or G_2. In both cases, as we have seen above, G' can be extended to $K_5 - e$ in M. Thus $(vz) \in M$, a contradiction. Consequently, we have $Q = M$ and $\mathfrak{S}_n(TC_4) = \{K_n - e, K_n\}$. Hence, Lemma 6 is proved.

Proposition 2.

(i) $\mathfrak{S}_5(TC_5) = \{H_1, H_2, H_3\}$, *see Fig.* 1.

(ii) $\mathfrak{S}_6(TC_5) = \{G_1, G_2, G_3\}$, *see Fig.* 2.

(iii) $\mathfrak{S}_n(TC_5) = \{K_n\}$, $n \geqq 7$.

Proof. (i) Let $M \in \mathfrak{S}_5(TC_5, H_1 \cup (13))$ be given. Since $4[5132] \subset M$, we have $(41) \in M$ or $(42) \in M$. In both cases M contains H_2. Thus $M = H_2$, since H_2 is TC_5-closed, as we have proved in Example 2.

Suppose $H_2 = TC_5 \cup \{(13), (14)\}$. Let $M \in \mathfrak{S}_5(TC_5, H_2 \cup (24))$. Since $5[412]\,3 \subset$ $\subset M$, we have $(35) \in M$. Thus $M = K_5$. Now let $M \in \mathfrak{S}_5(TC_5, H_1 \cup (24))$. Since $1[524]\,3 \subset M$, we have $(13) \in M$. From the above considerations we may conclude $M = K_5$. The first part of Proposition 2 is proved.

(ii) Let $M \in \mathfrak{S}_6(TC_5, H_1 \cup \{6\} \cup (16))$. Since $6[125]\,4 \subset M$. we have $(46) \in M$. $6[152]\,3 \subset M$ implies $(63) \in M$. Thus $G_1 = M$. We prove G_1 is TC_5-closed. In fact, $G_1 - x = TC_5$ for any vertex $x \in G_1$.

Let $M \in \mathfrak{S}_6(TC_5, G_1 \cup (13))$. Since $2[136]\,4 \subset M$, we have $(24) \in M$. Since $6[342]\,5 \subset M$, we have $(56) \in M$. For any $x \in M$, $M - x$ is equal to H_2. Since H_2 is TC_5-closed by (i), $H = G_2$.

Let $M \in \mathfrak{S}_6(TC_5, H_1 \cup \{6\} \cup (26))$. Since $6[2345] \subset M$, we have $(36) \in M$ or $(56) \in M$.

I. Suppose $(36) \in M$. $1[263]\,4$ implies $(14) \in M$. Further, $62[154] \subset M$ implies $(46) \in M$ or $(56) \in M$.

(1) Put $(46) \in M$. $63[215] \subset M$ implies $(16) \in M$ or $(56) \in M$.

(1a) Let $(16) \in M$. $5[162]\,3 \subset M$ implies $(53) \in M$. Thus $G_2 \subset M$. This implies $G_2 = M$, since G_2 is TC_5-closed. In fact, $G_2 - x$ is TC_5-closed for every $x \in G_2$ as follows from the case (i).

(1b) Let $(56) \in M$. $1[564]\,3 \subset M$ implies $(13) \in M$. Thus $G_2 \subset M$. This implies $G_2 = M$, since G_2 is TC_5-closed, as we have seen before.

(2) Put $(56) \in M$. $1[256]\,3 \in M$ implies $(13) \in M$. Further, $4[123]\,6 \subset M$ implies $(46) \in M$. Thus $G_2 \subset M$ and we have $G_2 = M$, since G_2 is TC_5-closed.

II. Now suppose $(56) \in M$. $4[562]\,1 \subset M$ implies $(14) \in M$. $1[562]\,3 \subset M$ implies $(13) \in M$. $4[152]\,6 \subset M$ implies $(46) \in M$ and $3[125]\,6 \subset M$ implies $(36) \in M$. Thus $G_2 \subset M$ and we have $G_2 = M$, since G_2 is TC_5-closed.

Let $M \in \mathfrak{S}_6(TC_5, TC_5 \cup \{6\} \cup (36))$. $63[215]$ implies $(56) \in M$ or $(16) \in M$. If $(56) \in M$ we have

$$M \in \mathfrak{S}_6(TC_5, H_1 \cup \{6\} \cup (56)),$$

and if $(16) \in M$ we have

$$M \in \mathfrak{S}_6(TC_5, TC_5 \cup \{6\} \cup (16)).$$

Thus we have $M = G_1$ or $M = G_2$, as we have seen above.

Now, let $M \in \mathfrak{S}_6(TC_5, G_2 \cup (24))$, where G_2 is labeled as in Fig. 2. Since $6[254]\,1 \subset \subset M$, we have $(16) \in M$. Since $5[126]\,3 \subset M$, we have $(35) \in M$. Thus $M = K_6$. Since $M \in \mathfrak{S}_6(TC_5)$ is connected by Lemma 2, we have completed the proof of the second part of Proposition 2.

(iii) Consider G_1 from Fig. 2. Let $M \in \mathfrak{S}_7(TC_5, G_1 \cup \{7\} \cup (17))$.
$4[521]\,7 \subset M$ and $7[152]\,3 \subset M$ imply $(47) \in M$ and $(37) \in M$.
$5[463]\,7 \subset M$ and $7[463]\,2 \subset M$ imply $(57) \in M$ and $(27) \in M$.
$6[152]\,7 \subset M$ implies $(67) \in M$. Thus the vertex 7 is adjacent to each vertex of G_1. Since $M - x$ contains G_1 for each vertex $x \in M$, we conclude that x is adjacent to each vertex of G_1 for any $x \in G_1$. This implies $M = K_7$.

Consider G_2 from Fig. 2. Let $M \in \mathfrak{S}_7(TC_5, G_2 \cup \{7\} \cup (17))$. Then $7[123]\,4 \subset M$, $7[132]\,5 \subset M$, $7[154]\,3 \subset M$, $7[145]\,2 \subset M$, $7[145]\,6 \subset M$ imply $\{(72), (73), (74), (75), (76)\} \subset M$.

Thus the vertex 7 is adjacent to each vertex of G_2. Since $M - x$ contains G_2 for each vertex $x \in M$ we conclude that x is adjacent to each vertexof G_2 for any $x \in G_2$. This implies $M = K_7$.

Let $G \in \mathfrak{S}_n(TC_5)$, $n \geqq 7$ and take any copy $T = TC_5$ in G. By Lemma 2, G is connected. Thus there is a vertex $v \in G - T$ that is joined to T by an edge e. However, we have $\mathfrak{S}_6(TC_5, T \cup v \cup e) = G_1$, G_2 or G_3 by the case (ii). There are x_i and e_i such that $G_i \cup x_i \cup e_i$ $(i = 1, 2)$ are connected.

As we have seen above, $\mathfrak{S}_7(TC_5, G_i \cup x_i \cup e_i) = K_7$, $i = 1, 2$. Thus G is the roof with respect to TC_5. From Lemma 5 we may conclude that $G = K_n$ and the proof of Proposition 2 is complete.

Theorem 2. *For every natural numbers $n > p \geqq 6$ we have*

$$\mathfrak{S}_n(TC_p) = K_n.$$

Proof. (1) Let $M \in \mathfrak{S}_{p+1}(TC_p, TC_p \cup \{p + 1\} \cup (1\ p + 1))$.

(1a) We prove $(i\ p + 1) \in M$, $1 \leqq i \leqq p$.
Since $p + 1[12p]\ p - 1\ p - 2 \dots 54 \subset M$, we have $(p + 1\ 4) \in M$.
Since $p + 1[45 \dots p - 1\ p23] \subset M$, we have $(p + 1\ 3) \in M$ or $(p + 1\ 5) \in M$.

Put $(p+1\ 5) \in M$. Since $3[21p]\ p-1 \ldots 65\ p+1 \subset M$, we have $(p+1\ 3) \in \in M$.

Thus we may assume $(p+1\ 3) \in M$.

Suppose $(p+1\ i) \in M$ for some $3 \leqq i \leqq p-3$. Then $p+1\ i\ i-1 \ldots \ldots 3[21p]\ p-1\ p-2 \ldots i+2 \subset M$ implies $(p+1\ i+2) \in M$.

Thus we have proved $(p+1\ i) \in M$ for any $3 \leqq i \leqq p-1$.

$12[3\ p+1\ 4]\ 56 \ldots p-1 \subset M$ implies $(p-1\ 1) \in M$.

$2[1p\ p-1]\ p-2 \ldots 54\ p+1 \subset M$ implies $(p+1\ 2) \in M$.

$1[p+1\ 56]\ 78 \ldots p23 \subset M$ implies $(13) \in M$.

$p[123]\ 45 \ldots p-2\ p+1 \subset M$ implies $(p\ p+1) \in M$.

We conclude $(i\ p+1) \in M$ for any $1 \leqq i \leqq p$.

(1b) The graph $\{[2\ p+1\ 3]\ 45 \ldots p \cup (2p) \cup \{1\} \cup (1\ p+1)\} \subset M$ is isomorphic to $TC_p \cup \{p+1\} \cup \{1\ p+1\}$. Since the roles of the vertices $p+1$ and 1 are interchanged, by the case (1a) we can conclude $(1\ i) \in M$ for any $2 \leqq \leqq i \leqq p+1$.

(1c) The graph $\{[p\ p+1\ 2]\ 34 \ldots i-1\ 1\ i+1 \ldots p-1 \cup (p-1\ p) \cup \{i\} \cup \cup (i\ p+1)\} \subset M$, $3 \leqq i \leqq p-1$, is isomorphic to $TC_p \cup \{p+1\} \cup \cup (1\ p+1)$. The roles of $p+1$ and i are interchanged. Thus by the case (1a) we can conclude $(ij) \in M$ for any $1 \leqq j \leqq p+1$, $i \neq j$.

(1d) The graph $[3\ p+1\ 4]\ 56 \ldots p1 \cup (13) \cup \{2\} \cup (2\ p+1)$ and the graph $[3\ p+1\ 4]\ 56 \ldots p-1\ 12 \cup (23) \cup \{p\} \cup (p\ p+1)$ are isomorphic to $TC_p \cup \cup \{p+1\} \cup (1\ p+1)$. As above, we conclude $(2i) \in M$ for any $1 \leqq i \leqq p+1$, $i \neq 2$, $(pi) \in M$ for any $1 \leqq i \leqq p+1$, $i \neq p$.

Thus we have proved $M = K_{p+1}$.

(2) Let $M \in \mathfrak{S}_{p+1}(TC_p, TC_p \cup \{p+1\} \cup (2\ p+1))$. Since $p+1[21p]\ p-1 \ldots \ldots 4 \subset M$, we have $(4\ p+1) \in M$. $34\ p+1[21p]\ p-1 \ldots 5 \subset M$ implies $(35) \in M$.

From $p+1[345]\ 6 \ldots p\ 1 \subset M$ we have $(1\ p+1) \in M$. Thus by the case (1) we can conclude $M = K_{p+1}$.

(3) Let $M \in \mathfrak{S}_{p+1}(TC_p, TC_p \cup \{p+1\} \cup (3\ p+1))$. $p+1[345 \ldots p2] \subset M$ implies $(2\ p+1) \in M$ or $(4\ p+1) \in M$. If $(2\ p+1) \in M$, then by the case (2), $M = K_{p+1}$. Suppose $(4\ p+1) \in M$. Since $[3\ p+1\ 4]\ 56 \ldots p1 \subset M$, we have $(1\ p+1) \in M$ or $(13) \in M$.

If $(1\ p+1) \in M$, then by the case (1), $M = K_{p+1}$.

If $(13) \in M$, then by the case (2), $M = K_{p+1}$.

(4) Let $M \in \mathfrak{S}_{p+1}(TC_p, TC_p \cup \{p+1\} \cup (i\ p+1))$ imply $M = K_{p+1}$ for any $i \geqq 3$ smaller than $j \leqq [\frac{1}{2}p]$. We prove that the same is true for j. Assume $M \in \mathfrak{S}_{p+1}(TC_p, TC_p \cup \{p+1) \cup (j\ p+1))$. Since $p+1[j\ j+1 \ldots p\ 23 \ldots \ldots j-1] \in M$, we have $(j-1\ p+1) \in M$ or $(j+1\ p+1) \in M$. If $(j-1\ p+1) \in M$, the proof is complete. Suppose $(j+1\ p+1) \in M$. Since $p+1$

$j+1\ j+2 \ldots p-1[p12]\ 34 \ldots j-1 \in M$, we have $(j-1\ p+1) \in M$ and again the proof is complete.

Now take $M \in \mathfrak{S}_{p+1}(TC_p)$. By Lemma 2, M must be connected. However, from (1), (2), (3) and (4) we can conclude $M = K_{p+1}$.

Suppose we are given a TC_p in some $M \in \mathfrak{S}_n(TC_p)$. By Lemma 2, M is connected. Thus, take any vertex $v \notin TC_p$ which is joined with TC_p by an edge e. From the above we have that the TC_p-closure of $TC_p \cup v \cup e$ is K_{p+1}. The preceding considerations also imply $(vi) \in M$ for any $i \in TC_p$. Thus M is the roof with respect to TC_p. By Lemma 5, $M = K_n$ and the proof of Theorem 2 is complete.

Denote by T_2C_p the graph $TC_p \cup (a_1a_3)$. We will characterize T_2C_p-closed graphs with more vertices than p.

Theorem 3. *For every natural numbers $n > p \geqq 6$ we have*

$$\mathfrak{S}_n(T_2C_p) = K_n .$$

First we prove

Proposition 3. *T_2C_p is TC_p-closed for $p \geqq 6$.*

Proof. We apply Closedness Criterion. Consider T_2C_p as in Fig. 7.

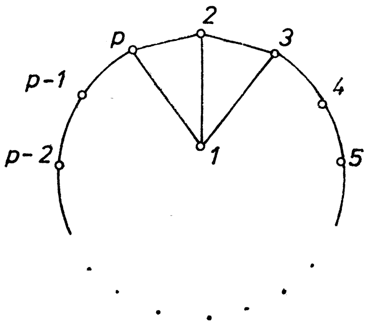

Fig. 7. T_2C_p is TC_p-closed graph.

There are exactly four TC_p's in T_2C_p. Namely,

$$[12 \ldots p] \cup (p2), \quad [134 \ldots p2] \cup (23),$$
$$[12 \ldots p] \cup (13), \quad [134 \ldots p2] \cup (1p).$$

The boolean expression V has the form

$$V = X_{1p} \wedge X_{13} \wedge X_{23} \wedge X_{2p}.$$

Thus the unique minimal F_S is C_4. Since $C_4 \not\subset TC_p$, we conclude by Closedness Criterion that T_2C_p is TC_p-closed.

Proposition 3 is proved.

Proof of Theorem 3. From Proposition 3 we have

$$T_2C_p \in \mathfrak{S}(TC_p)\,.$$

By Corollary of Lemma 1, $\mathfrak{S}_n(T_2C_p) \in \mathfrak{S}_n(TC_p)$. However, from Theorem 2 we have $\mathfrak{S}_n(TC_p) = K_n$. Thus $\mathfrak{S}_n(T_2C_p) = K_n$. Theorem 3 is proved.

1.4. $K_n - e$ **and free graphs.** A graph H is said to be *free* if for any edge $e \in H$ there exists an edge $f \notin H$ such that

$$H - e + f \cong H\,.$$

The first who had introduced the notion of free graphs was Sheehan [10], [11]. However, his motivation had been quite different from ours. He had investigated "fixing subgraphs" of a given graph. The graphs which contain exactly one fixing subgraph are just the free graphs [11].

The class of free graphs is sufficiently abundant. For example, the following proposition is true:

Proposition 4. *Let H be an edge-transitive graph. Then $H - e$ is free for any edge $e \in H$.*

Proof. The proof is evident and is left to the reader.

Theorem 4. *Let H_p be a given p-vertex free graph. Then*

$$\{K_n - e, K_n\} \subset \mathfrak{S}_n(H_p)\,, \quad n \geqq p\,.$$

Proof. Evidently, K_n is H_p-closed. We prove $K_n - e$ is H_p-closed. Let F with $F \subset H_p$ and $F \subset K_n - e$ be given. If F does not contain the end vertex of e, then evidently F can be extended to H_p in $K_n - e$. Assume that F contains both end vertices of e. F can be extended to some H_p, say $\bar{H}_p$, in K_n. If $e \notin \bar{H}_p$ then the proof is complete.

Let $e \in \bar{H}_p$. Since H_p is free, there exists f such that $\bar{H}_p - e + f \cong \bar{H}_p$. But $F \subset \bar{H}_p - e + f$. Hence the proof follows.

The following theorem gives a characterization of $(K_p - e)$-closed graphs.

Theorem 5. *Let $n \geqq p \geqq 4$ be given natural numbers. Then*

$$\mathfrak{S}_n(K_p - e) = \{K_n - e, K_n\}\,.$$

Proof. Since $K_p - e$ is free, Theorem 4 yields

$$\{K_n - e, K_n\} \subset \mathfrak{S}_n(K_p - e)\,.$$

Let $M \in \mathfrak{S}_n(K_p - e)$ be given. Since M is $(K_p - e)$-closed, each induced graph of M on p vertices is $K_p - e$ or K_p. The only such graphs are $K_n - e$ and K_n. Thus the proof is complete.

2. CONCLUDING REMARKS AND OPEN PROBLEMS

Let G_p be a p-vertex graph with at least one end vertex. Then $\mathfrak{S}_n(G_p) \neq \{K_n\}$ for every $n \geqq p$. For example, K_{n-1} with one "hanging" edge is G_p-closed. On the other hand, let F be a graph with no end vertex which is not free. We have not succeeded in looking for F for which $\mathfrak{S}_n(F) \neq \{K_n\}$ for sufficiently large n.

One can obtain similar results as in Proposition 2 and Theorem 2 for cycles with one chord of length greater than two. Even if the number of chords is much smaller then the length of the cycle, it is not difficult to find out that for sufficiently large n the only graph closed under such graph is the complete graph.

Problem 1. *Let F be a graph with no end vertex which is not free. Is it true that there exists n_0 such that*

$$\mathfrak{S}_n(F) = K_n \quad \textit{for every} \quad n \geqq n_0?$$

If the answer is no, the following problem would be of interest.

Problem 2. *Is there an algorithm which decides whether or not for a given graph G there exists n_0 such that*

$$\mathfrak{S}_n(G) = K_n\,, \quad \textit{for} \quad n \geqq n_0?$$

Sections 1.2 and 1.3 contain characterizations of H-closed graphs with more vertices than $|H|$ for H regular. However, there are no results about H-closed graphs on $|H|$ vertices. This task seems to be more difficult, although some partial results can be obtained.

Problem 3. *Characterize H^r_p-closed graphs on p vertices.*

Closedness Criterion works very well if G has not much more edges than H. This calls for further development of the theory of H-closed graphs in order to obtain a more suitable criterion in the general case.

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