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ASYMPTOTIC BEHAVIOUR OF RICCATI'S DIFFERENTIAL EQUATION ASSOCIATED WITH SELF-ADJOINT SCALAR EQUATIONS OF EVEN ORDER

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1. INTRODUCTION

In this paper we study the behaviour of Hermitian matrix-solutions $Q(x) = Q(x; \lambda)$ of the Hermitian Riccati matrix equation

$$Q' + A^T Q + QA + QBQ - C + \lambda C_0^* = 0$$

when the parameter $\lambda \to -\infty$ in case that the (n, n)-matrices A, B, C and C_0^* are of a special form (as described in section 2, formula (2)). More precisely: a matrix $Q(x)$, which solves the Riccati equation above, is of the form $Q(x) = V(x)\, U^{-1}(x)$ where $U(x)$, $V(x)$ solve the corresponding Hamiltonian system

$$U' = AU + BV\,, \quad V' = (C - \lambda C_0^*)\, U - A^T V\,.$$

The matrices A, B, C, C_0^* are given by formula (2) such that this Hamiltonian system corresponds to a self-adjoint scalar differential equation of even order $2n$, i.e.

$$(0) \qquad L(y) = \sum_{\nu=0}^{n} (-1)^\nu \left(r_\nu y^{(\nu)}\right)^{(\nu)} = \lambda r y$$

with realvalued functions $r_\nu \in C(\mathbb{R})$, $r \in C(\mathbb{R})$, $r(x) > 0$ and $r_n(x) > 0$ on $\mathbb{R}$. For fixed $x_0 \in \mathbb{R}$ we consider solutions $Q = VU^{-1}$ of the Riccati equation, for which U, V satisfy (with respect to λ fixed) initial conditions at x_0, such that U, V form a so-called conjoined basis of the Hamiltonian system (see [6]). Our main result (Theorem 1) describes the asymptotic behaviour of $Q(x; \lambda) = V(x; \lambda)\, U^{-1}(x; \lambda)$ as $\lambda \to -\infty$ for all $x \neq x_0$ (note that $Q(x_0; \lambda)$ is not defined if the fixed initial value $U(x_0)$ is a singular matrix).

This matrix $Q(x; \lambda)$ occurs in the treatment of variational problems (where (0) is the corresponding Euler equation) via Picone's identity (see [10, 6]); and an essential aid of that treatment is the asymptotic behaviour of $Q(x; \lambda)$ as $x \to x_0$ or $\lambda \to \lambda_0$ (this is discussed in [5, 6]), and also as $\lambda \to -\infty$ with x fixed. It is shown in [6, Theorem 11] that $Q(x; \lambda) \to \infty$ as $\lambda \to -\infty$, $x > x_0$, and this crude result is improved in this paper by deriving the precise asymptotic behaviour. Actually the asymptotic behaviour of solutions of (0) is treated extensively in the literature (see e.g. [1, 4, 7, 12] for the case $\lambda \to -\infty$ or [3] for the case $x \to \infty$, λ fixed). But these results do not lead to the results below (Theorems 1 and 3) on $Q(x; \lambda)$. Moreover, the methods in [4, 7, 12] need stronger smoothness conditions on the coefficients $r_\nu(x)$, $r(x)$; essentially r and $r_n \in C_n(\mathbb{R})$ is needed (then the equation (0) may be transformed such that $r = r_n \equiv 1$).

The setup of this paper is as follows. In § 2 we introduce the necessary notation and assumptions, and the main result (Theorem 1) is stated. In § 3 precise estimates for $Q(x; \lambda)$ (Theorem 2) are derived in case that (0) is an equation with constant coefficients r_ν $(\nu = 0, \ldots, n)$, r. These estimates combined with inequalities for the Riccati equation [6, 9] lead to our main results (Theorems 1 and 3) in § 4. In the last § Theorem 3 is applied to derive estimates of solutions of (0) (Prop. 3 and 4). On the one hand these estimates do not imply the asymptotic results from [4 or 7], but on the other hand one does not obtain uniform estimates for solutions of (0) with fixed initial values (as in Prop. 3) from those known results.

2. NOTATION, ASSUMPTIONS, AND MAIN RESULT

We consider Hamiltonian systems of ordinary differential equations, which correspond to self-adjoint scalar equations of even order, i.e.

(1) $$u' = Au + Bv\,, \quad v' = (C - \lambda C_0^*)\,u - A^T v\,,$$

where the (n, n)-matrices A, B, C, C_0^* are of the special form

(2) $$A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \quad B = \frac{1}{r_n(x)}\, B_0\,, \quad B_0 = \begin{pmatrix} 0 & \ldots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \ldots & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} r_0(x; \lambda) & & 0 \\ & \ddots & \\ 0 & & r_{n-1}(x; \lambda) \end{pmatrix}, \quad C_0^* = r(x)\, C_0\,, \quad C_0 = \begin{pmatrix} 1 & \ldots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \ldots & 0 \end{pmatrix},$$

and where λ is a real parameter. We assume thoughout this paper that the realvalued functions $r(x)$, $r_n(x)$, $r_\nu(x; \lambda)$ satisfy

(3) $$r(x),\, r_n(x) \in C(\mathbb{R})\,, \quad r_\nu(x; \lambda) \in C(\mathbb{R}^2)\,,$$
$$\nu = 0, \ldots, n-1 \quad \text{and} \quad r(x),\, r_n(x) > 0 \quad \text{on} \quad \mathbb{R}\,.$$

Observe, that a function $y\colon \mathbb{R} \to \mathbb{R}$ solves the scalar equation

(4) $$L(y) = \sum_{\nu=0}^{n} (-1)^\nu \left(r_\nu y^{(\nu)}\right)^{(\nu)} = \lambda r y$$

on $\mathbb{R}$, if and only if the vectors $u = (u_k(x))$, $v = (v_k(x))$ given by

(5) $$u_k = y^{(k)}\,, \quad v_k = \sum_{\nu=k+1}^{n} (-1)^{\nu-k-1} \left(r_\nu y^{(\nu)}\right)^{(\nu-k-1)}\,, \quad k = 0, \ldots, n-1$$

are well-defined on $\mathbb{R}$ and solve the differential system (1) [6, 8]. Moreover, we assume throughout that the (n, n)-matrices $U = U(x; \lambda)$, $V = V(x; \lambda)$ are the solution of the following initial value problem

$$(6)\qquad U' = AU + BV, \quad V' = (C - \lambda C_0^*)\,U - A^T V,$$
$$U(x_0) = U_0, \quad V(x_0) = V_0,$$

where $x_0 \in \mathbb{R}$ is fixed, and where the (complex) (n, n)-matrices U_0, V_0 satisfy

$$(7)\qquad \operatorname{rank}(\bar{U}_0^T, \bar{V}_0^T) = n, \quad \bar{U}_0^T V_0 = \bar{V}_0^T U_0.$$

Then, by [6], the matrices U, V are a so-called 'conjoined basis' of the Hamiltonian system (1); we have $\bar{U}^T V \equiv \bar{V}^T U$ on $\mathbb{R}^2$; the focal points of U (i.e. those $x \in \mathbb{R}$, for which $U(x; \lambda)$ is singular when $\lambda \in \mathbb{R}$ is fixed) are isolated; and then the Hermitian matrix $Q(x; \lambda) := V(x; \lambda)\, U^{-1}(x; \lambda)$ solves the Hermitian Riccati matrix equation [6]

$$(8)\qquad Q' + A^T Q + QA + QBQ - C + \lambda C_0^* = 0,$$

whenever it exists (i.e. for all $x \in \mathbb{R}$, for which $U(x; \lambda)$ is regular).

For the formulation of our main result we need some further notation. Let

$$(9)\qquad \varrho_0 := \sqrt[2n]{-\lambda}, \quad \delta(x) := \sqrt[2n]{(r(x)/r_n(x))}, \quad \varrho(x; \lambda) := \varrho_0\, \delta(x),$$
$$\text{if} \quad \lambda < 0, \quad x \in \mathbb{R}.$$

Moreover, let

$$(10)\quad \varepsilon_k := \exp\left\{ i\pi \left(\frac{k}{n} - \frac{n-1}{2n} \right) \right\}, \quad k = 0, \dots, 2n-1 \quad (\text{observe } \varepsilon_k^{2n} = (-1)^{n-1}),$$

$$(11)\quad \Phi^0 := \begin{pmatrix} 1 & \dots & 1 \\ \varepsilon_0 & & \varepsilon_{2n-1} \\ \dots & \dots & \dots \\ \varepsilon_0^{2n-1} & \dots & \varepsilon_{2n-1}^{2n-1} \end{pmatrix}_{(2n,2n)} = \begin{pmatrix} \Phi_{11}^0 & \Phi_{12}^0 \\ \Phi_{21}^0 & \Phi_{22}^0 \end{pmatrix} \text{ (with } (n, n)\text{-matrices } \Phi_{kl}^0),$$

and finally we introduce the (n, n)-matrices

$$(12)\qquad G_i := E^0 \Phi_{2i}^0 \Phi_{1i}^{0\,-1} \quad \text{for} \quad i = 1, 2 \quad \text{with} \quad E^0 = \begin{pmatrix} 0 & \dots & (-1)^{n-1} \\ \dots & \dots & \dots \\ 1 & \dots & 0 \end{pmatrix},$$

$$(13)\qquad D_\alpha := \operatorname{diag}(1, \alpha, \dots, \alpha^{n-1}), \quad \tilde{D}_\alpha := \operatorname{diag}(\alpha^{n-1}, \dots, \alpha, 1) \quad \text{for} \quad \alpha \in \mathbb{C},$$

where diag denotes diagonal matrices. Observe that Φ^0 is a (regular) Vandermonde matrix and that the matrix G_1 is real, symmetric, and positive definite by [6; p. 140]. This implies that the matrix G_2 is also real and symmetric, but negative definite, since $G_2 = -D_{-1} G_1 D_{-1}$, which follows from a simple calculation.

Now, our basic result is given by

Theorem 1. *If* $|r_\nu(x; \lambda)| \leqq R\varrho_0^{2(n-\nu-1)}$ *for* $\nu = 0, \dots, n-1$, $x \in \mathbb{R}$, $\varrho_0 \geqq 1$, *and some* $R > 0$, *then* $\lim_{\lambda \to -\infty} (1/\varrho_0)\, \tilde{D}_{1/\varrho_0} Q(x; \lambda)\, \tilde{D}_{1/\varrho_0} = r_n(x)\, \delta(x)\, \tilde{D}_{\delta(x)} G \tilde{D}_{\delta(x)}$ *for all* $x \neq x_0$, *where* $G = G_1$ *in case* $x > x_0$ *and* $G = G_2$ *in case* $x < 0$.

(Of course, $Q(x; \lambda)$ exists, i.e. $U(x; \lambda)$ is regular, for $x \neq x_0$ if λ is sufficiently small, i.e. $\lambda \leqq \lambda_0 = \lambda_0(x)$. Note, moreover, that $Q(x_0; \lambda)$ is not defined if the fixed initial value U_0 is a singular matrix.)

Finally we mention, that $\|\cdot\|$ always denotes the Euclidean norm of a vector resp. the induced matrix norm (spectral norm) of a matrix. For quadratic matrices Q_1, Q_2 we write $Q_1 < Q_2$ (resp. $Q_1 \leqq Q_2$) if Q_1 and Q_2 are Hermitian matrices and if $Q_2 - Q_1$ is positive definite (resp. nonnegative definite); and Q^T (resp. $\bar{Q}$) denotes the transpose (resp. complex conjugate) of a matrix Q; I denotes the identity matrix.

3. CONSTANT COEFFICIENTS

In this section we assume that the differential equation (4) has constant coefficients, i.e.

$$r_\nu(x; \lambda) \equiv r_\nu(\lambda) \in \mathbb{R}\,, \quad \nu = 0, \ldots, n-1\,, \quad r_n(x) \equiv r_n\,,$$
$$r(x) \equiv r \quad \text{on} \quad \mathbb{R} \quad \text{with} \quad r > 0\,, \quad r_n > 0\,.$$

Thus, by (9), $\varrho(x; \lambda) \equiv \varrho(\lambda) = \varrho = \sqrt[2n]{(-\lambda r/r_n)}$. We shall derive estimates of the matrix

$$\tilde{Q}(x; \lambda) = \frac{1}{r_n \varrho}\, \tilde{D}_{1/\varrho} Q(x; \lambda)\, \tilde{D}_{1/\varrho} \quad \text{with} \quad Q(x; \lambda) = V(x; \lambda)\, U^{-1}(x; \lambda)\,.$$

(Note the corresponding definition of $Q(x; \lambda)$ and (13) of section 2.)

Theorem 2. *Suppose that R, α are positive constants such that $|r_\nu(\lambda)| \leqq$ $\leqq R r_n \varrho^{2(n-\nu-1)}$, $\nu = 0, \ldots, n-1$ for $\varrho \geqq 1$, $(1/r_n)\, \bar{U}_0^T V_0 \geqq -\alpha \bar{U}_0^T U_0$. Then, for any $0 < c < 1$, there exist positive constants $K = K(n, R, \alpha, c)$, $K_1 = K_1(n, R)$ (depending on n, R, α, c resp. n, R only) such that the following holds: $Q(x; \lambda)$ exists for all $x > x_0$, $\varrho \geqq K$, and*

$$\|\tilde{Q}(x; \lambda) - G_1\| \leqq K_1 \varrho^{-2} \quad \textit{for all} \quad \varrho \geqq K\,,$$
$$x \geqq x_0 + x_1(\varrho) \quad \textit{with} \quad x_1(\varrho) = \left(c \sin \frac{\pi}{2n}\right)^{-1} \frac{\log \varrho}{\varrho} > 0\,.$$

If the inequality above on U_0, V_0 holds for $-V_0$ instead of V_0, then the assertions hold for $x < x_0$ resp. $x \leqq x_0 - x_1(\varrho)$ with G_2 instead of G_1.

Proof. We prove the result for $x > x_0$, since the case $x < x_0$ may be obtained from $x > x_0$ substituting x by $-x$, and then the matrices U, V, and G_1 must be replaced by $D_{-1}U$, $-D_{-1}V$ (according to (5)), resp. $G_2 = -D_{-1}G_1D_{-1}$.

First, it follows from [6; Prop. 4 with $\tilde{r}_\nu = -R r_n \varrho^{2(n-\nu-1)}$, $\nu = 0, \ldots, n-1$, $\tilde{r} = r$, $\tilde{r}_n = r_n$, $\tilde{\lambda} = \lambda < 0$, and $\tilde{U}(x_0) = I$, $\tilde{V}(x_0) = -\alpha r_n I$] (using our assumptions on $r_\nu(\lambda)$, $U_0 V_0$) that $Q(x; \lambda)$ exists on $(x_0, x_0 + \delta]$ for all $\varrho \geqq 1$ (i.e. $\lambda \leqq -r_n/r < 0$) with a positive constant $\delta = \delta(n, R, \alpha)$, so that $Q(x; \lambda)$ exists on $(x_0, x_0 + x_1(\varrho)]$

for $\varrho \geqq K$. Here, and in the following K resp. K_1 denote different, positive constants, which depend on n, R, α, c resp. n, R only.

Now, the characteristic polynomial of the equation (4) (with constant coefficients) is given by $P(t) = (-1)^n r_n \varrho^{2n} P_1(t/\varrho)$, where

$$P_1(t) = t^{2n} - \varrho^{-2} \sum_{\nu=0}^{n-1} (-\varrho^2)^{\nu+1-n} \frac{r_\nu}{r_n} t^{2\nu} + (-1)^n .$$

Therefore $P(t)$ has the zeros $\tilde{\varrho}_k = \varrho\delta_k$, where the δ_k are the zeros of $P_1(t)$.

It follows from [11; 66 pp.] (using the assumption on the $r_\nu(\lambda)$) that (observe (10))

$$(14) \qquad |\delta_k - \varepsilon_k| \leqq K_1 \varrho^{-2} \quad \text{if} \quad \varrho \geqq K_1 \quad \text{for} \quad k = 0, \ldots, 2n-1 .$$

This implies, in particular, that the δ_k (and then also the ϱ_k) are distinct for $\varrho \geqq K_1$, so that the functions $\exp(\tilde{\varrho}_k x)$, $k = 0, \ldots. 2n-1$ form a fundamental system of (4). Then the matrix

$$W(x) = W(x; \lambda) = \begin{pmatrix} I & 0 \\ F & E \end{pmatrix} \begin{pmatrix} D_\varrho & 0 \\ 0 & \varrho^n D_\varrho \end{pmatrix} \Phi \begin{pmatrix} D_1(x) & 0 \\ 0 & D_2(x) \end{pmatrix}$$

is a fundamental matrix of the corresponding Hamiltonian system (1). This is obtained from the transformation formulas (5), when the following notation is used:

$$F = \begin{pmatrix} 0 & r_1 & & * \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & r_{n-1} \\ 0 & \ldots & & 0 \end{pmatrix}, \quad E = r_n(E^0 + E^*) \text{ with } E^* = \begin{pmatrix} * & & 0 & 0 \\ & & \cdot & \\ 0 & \ldots & & 0 \\ 0 & \ldots & & 0 \end{pmatrix}$$

(E^0, D_ϱ as in (12), (13)),

$$\Phi = \begin{pmatrix} 1 & \ldots & 1 \\ \delta_0 & \ldots & \delta_{2n-1} \\ \ldots & \ldots & \ldots \\ \delta_0^{2n-1} & \ldots & \delta_{2n-1}^{2n-1} \end{pmatrix},$$

$$D_1(x + x_0) = \operatorname{diag}\left(e^{\tilde{\varrho}_0 x}, \ldots, e^{\tilde{\varrho}_{n-1} x}\right), \quad D_2(x + x_0) = \operatorname{diag}\left(e^{\tilde{\varrho}_n x}, \ldots, e^{\tilde{\varrho}_{2n-1} x}\right).$$

Note, that $|f_{ij}| \leqq r_n K_1$, $|e_{ij}^*| \leqq K_1$ (for $1 \leqq i < j \leqq n$, $1 \leqq i \leqq n-1-j \leqq n-2$ resp.) It follows that the solution U, V of our initial value problem (6) is given by

$$(15) \qquad \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} =$$

$$= \begin{pmatrix} I & 0 \\ F & E \end{pmatrix} \begin{pmatrix} D_\varrho & 0 \\ 0 & \varrho^n D_\varrho \end{pmatrix} \Phi \begin{pmatrix} D_1(x) & 0 \\ 0 & D_2(x) \end{pmatrix} \Phi^{-1} \begin{pmatrix} D_{1/\varrho} & 0 \\ 0 & \varrho^{-n} D_{1/\varrho} \end{pmatrix} \begin{pmatrix} U_0 \\ E^{-1}V_0 - E^{-1}FU_0 \end{pmatrix},$$

i.e.

$$U(x) = D_\varrho \Phi_{11} D_1(x) A_1 + D_\varrho \Phi_{12} D_2(x) A_2 ,$$
$$V(x) = F\, U(x) + \varrho^n E D_\varrho (\Phi_{21} D_1(x) A_1 + \Phi_{22} D_2(x) A_2) ,$$

where

and $A_i = \varrho^{-n} \tilde{\Phi}_{i2} D_{1/\varrho} E^{-1}(V_0 + S_i U_0)$, $S_i = \varrho^n E D_\varrho \tilde{\Phi}_{i2}^{-1} \tilde{\Phi}_{11} D_{1/\varrho} - F$, $i = 1, 2$,

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \quad \Phi^{-1} = \begin{pmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} \end{pmatrix}.$$

Since $\Phi^0 \overline{\Phi^{0T}} = 2n\, I$ (i.e. $\Phi^{0^{-1}} = (1/2n)\, \overline{\Phi^{0T}}$), which follows from (10), (11) by direct calculation, we obtain from (14)

$$(16) \qquad \|\Phi_{kl} - \Phi^0_{kl}\| \leqq K_1 \varrho^{-2}, \quad \left\|\tilde{\Phi}_{kl} - \frac{1}{2n}\, \overline{\Phi^{0T}_{lk}}\right\| \leqq K_1 \varrho^{-2} \quad \text{for} \quad \varrho \geqq K_1 .$$

Our definitions imply that $E = r_n(E^0 + E^*)$, $D_{1/\varrho} \tilde{D}_{1/\varrho} = \varrho^{-n+1} I$, and $\tilde{D}_{1/\varrho} E^0 D_\varrho = E^0$. These formulas, and the formulas above for $U(x)$, $V(x)$ imply the following representation of $\tilde{Q}(x; \lambda)$:

$$\tilde{Q}(x; \lambda) = \frac{1}{r_n \varrho}\, \tilde{D}_{1/\varrho} F \tilde{D}_{1/\varrho} + (E^0 + \tilde{D}_{1/\varrho} E^* D_\varrho)(I + \varepsilon_2(x))\, \Phi_{21} \Phi_{11}^{-1} (I + \varepsilon_1(x))^{-1},$$

with $\varepsilon_i(x) = \Phi_{i2}\, D_2(x)\, A_2 A_1^{-1}\, D_1^{-1}(x)\, \Phi_{i1}^{-1}$, $i = 1, 2$, if the matrices A_1 and $I + \varepsilon_1(x)$ are regular (which implies that $U(x)$ is regular too, since the Φ_{ij} are regular for $\varrho \geqq K_1$ by (16) and (10), (11)). We have that $\operatorname{Re} \varepsilon_k = -\operatorname{Re} \varepsilon_{n+k} \geqq \sin(\pi/2n) > 0$ for $k = 0, \ldots, n-1$ by (10), and we obtain from (14), (16), and the definition of F and E^* the following estimates: $\|D_1^{-1}(x)\| \leqq 1/\varrho$, $\|D_2(x)\| \leqq 1/\varrho$, $\|\tilde{D}_{1/\varrho} E^* D_\varrho\| \leqq K_1 \varrho^{-2}$, $\|(1/r_n)\, \tilde{D}_{1/\varrho} F \tilde{D}_{1/\varrho}\| \leqq K_1 \varrho^{-1}$ and $\|E^0 \Phi_{21} \Phi_{11}^{-1} - G_1\| \leqq K_1 \varrho^{-2}$ for all $x \geqq x_0 + x_1(\varrho)$, $\varrho \geqq K$. These estimates and the formula for $\tilde{Q}(x; \lambda)$ above imply the assertions of Theorem 2 for $x > x_0$ (observe that we have already shown that $Q(x; \lambda)$ exists on $(x_0, x_0 + x_1(\varrho)]$), if we prove the following:

$$(17) \qquad A_1 \text{ is regular, and } \quad \|A_2 A_1^{-1}\| \leqq K_1 \quad \text{for} \quad \varrho \geqq K,$$

since this implies $\|\varepsilon_i(x)\| \leqq K_1 \varrho^{-2}$ for $x \geqq x_0 + x_1(\varrho)$, $\varrho \geqq K$, from which the regularity of $I + \varepsilon_1(x)$ follows, in particular. Thus, it remains to prove (17). First, we show that the matrices $S_i = S_i(\varrho)$, $i = 1, 2$ are Hermitian for $\varrho \geqq K_1$. Therefore, we consider the matrix solutions U_i, V_i of (1) which satisfy the initial conditions $U_1(x_0) = V_2(x_0) = 0$, $V_1(x_0) = -U_2(x_0) = I$. Then U_i, V_i are 'normalized conjoined bases' of (1), so that $U_2(x)\, U_1^T(x)$ is symmetric (and real) for $x \in \mathbb{R}$ by [6; Def. 2, (7'), (8)]. Our representation (15) of the solution U, V applied to U_i, V_i, $i = 1, 2$ yields:

$$(E D_\varrho \tilde{\Phi}_{12}^{-1}\, D_1^{-1}(x)\, \Phi_{11}^{-1} D_{1/\varrho})\, U_2(x)\, U_1^T(x)\, \overline{(E D_\varrho \tilde{\Phi}_{12}^{-1}\, D_1^{-1}(x)\, \Phi_{11}^{-1} D_{1/\varrho})}^T =$$

$$= (-S_1 - E D_\varrho\, D_1^{-1}(x)\, \Phi_{11}^{-1} \Phi_{12}\, D_2(x)\, \tilde{\Phi}_{22} D_{1/\varrho} E^{-1} S_2) \times$$

$$\times\, \overline{(I + E D_\varrho \tilde{\Phi}_{12}^{-1}\, D_1^{-1}(x)\, \Phi_{11}^{-1} \Phi_{12}\, D_2(x)\, \tilde{\Phi}_{22} D_{1,\varrho} E^{-1})}^T \to -S_1(\varrho) \quad \text{as} \quad x \to \infty,$$

since $-\operatorname{Re} \varepsilon_k = \operatorname{Re} \varepsilon_{n+k} < 0$ for $k = 0, \ldots, n-1$ and $\varrho \geqq K_1$. Since the lefthand side of the limit is for all x a Hermitian matrix, we obtain that S_1 is Hermitian; and a similar argument (let $x \to -\infty$) shows that S_2 is Hermitian, too. Next, we have

$$\tilde{S}_i = \frac{1}{r_n \varrho}\, \tilde{D}_{1/\varrho} S_i \tilde{D}_{1/\varrho} = (E^0 + \tilde{D}_{1/\varrho} E^* D_\varrho)\, \tilde{\Phi}_{i2}^{-1} \tilde{\Phi}_{11} - \frac{1}{r_n \varrho}\, \tilde{D}_{1/\varrho} F \tilde{D}_{1/\varrho},$$

which implies by (16) and (12) that

(18) $$\|\tilde{S}_i - G_i^{-1}\| \leqq K_1\varrho^{-2} \quad \text{for} \quad \varrho \geqq K_1\,, \quad i = 1,2\,.$$

Hence, $(1/r_n)\, S_1 > \alpha I$ for $\varrho \geqq K$, so that the matrix

$$V_0 + S_1 U_0 = r_n\left(\left\{\frac{1}{r_n} V_0 - \alpha U_0\right\} + \left\{\frac{1}{r_n} S_1 - \alpha I\right\} U_0\right)$$

is regular by [6; Prop A 1], since

$$\overline{\left(\frac{1}{r_n} V_0 - \alpha U_0\right)}^T U_0 \geqq 0$$

by our assumption. This implies that A_1 is regular for $\varrho \geqq K$ (observe that the cited Prop. A 1 remains true for complex, Hermitian matrices instead of real, symmetric matrices). Now, we obtain for $\varrho \geqq K$ that

$$A_2 A_1^{-1} = \tilde{\Phi}_{22}(\tilde{D}_{1/\varrho} E^* D_\varrho + E^0)^{-1}\,.$$
$$.\,(\tilde{V}_0 + \varrho \tilde{S}_2 \tilde{U}_0)(\tilde{V}_0 + \varrho \tilde{S}_1 \tilde{U}_0)^{-1}(\tilde{D}_{1/\varrho} E^* D_\varrho + E^0)\,\tilde{\Phi}_{12}^{-1}$$

with

$$\tilde{V}_0 = \frac{1}{r_n} \tilde{D}_{1/\varrho} V_0 \tilde{D}_{1/\varrho}\,, \quad \tilde{U}_0 = \tilde{D}_\varrho U_0 \tilde{D}_{1/\varrho}\,.$$

Next, we have $(\tilde{V}_0 + \varrho \tilde{S}_2 \tilde{U}_0)(\tilde{V}_0 + \varrho \tilde{S}_1 \tilde{U}_0)^{-1} = I + \varrho(\tilde{S}_2 - \tilde{S}_1)\, Q$, where the matrix $Q = \tilde{U}_0(\tilde{V}_0 + \varrho \tilde{U}_0)^{-1}$ is Hermitian, since $\bar{\tilde{U}}^T \tilde{V}_0 = \bar{\tilde{V}}^T \tilde{U}_0$ by (7). Moreover,

$$Q \geqq 0 \text{ (i.e. } \overline{(\tilde{V}_0 + \varrho \tilde{S}_1 \tilde{U}_0)}^T \tilde{U}_0 \geqq 0)\,, \quad Q \leqq \frac{1}{\varrho} \tilde{S}_1^{-1}$$

$$\left(\text{i.e. } \overline{(\tilde{V}_0 + \varrho \tilde{S}_1 \tilde{U}_0)}^T \tilde{U}_0 \geqq \frac{1}{\varrho} \overline{(\tilde{V}_0 + \varrho \tilde{S}_1 \tilde{U}_0)}^T \tilde{S}_1^{-1}(\tilde{V}_0 + \tilde{S}_1 \tilde{U}_0)\right)$$

for $\varrho \geqq K$ (use (18) and the assumption $(1/r_n)\, \bar{U}_0^T V_0 \geqq -\alpha \bar{U}_0^T U_0$). These inequalities show that $\|A_2 A_1^{-1}\| \leqq K_1$ for $\varrho \geqq K$, which completes the proof. □

The proof above shows, that one may also obtain an inequality similar to Theorem 2 for values of x nearer x_0 (this follows simply from estimates of $\|D_2(x)\|$, $\|D_1^{-1}(x)\|$ using (14)), namely:

Corollary 1. *Under the assumptions of Theorem 2 there exist positive constants* $K = K(n, R, \alpha, c)$, $K_1 = K_1(n, R)$ *such that*

$$\|\tilde{Q}(x; \lambda) - G_1\| \leqq K_1 e^{-\gamma} \quad \textit{for all} \quad \gamma \geqq 1\,, \quad \varrho \geqq \max\{K, e^{\gamma/2}\}\,,$$

and $x \geqq x_0 + x_1^*(\varrho)$ *with* $x_1^*(\varrho) = (c \sin(\pi/2n))^{-1}\, \gamma/\varrho$, *for any* $0 < c < 1$; *and if the inequality of the assumption of Theorem 2 holds for* $-V_0$ *instead of* V_0, *then the assertion holds for* $x \leqq x_0 - x_1^*(\varrho)$ *with* G_2 *instead of* G_1.

Next, we estimate $\tilde{Q}(x; \lambda)$ for $|x - x_0| \leqq x_1^*(\varrho)$ as $\varrho \to \infty$ (i.e. $\lambda \to -\infty$), if U_0 is regular, i.e. if $Q(x_0; \lambda) = V_0 U_0^{-1}$ exists.

Proposition 1. *Suppose that U_0 is regular, and that R, α are positive constants with*

$$|r_\nu(\lambda)| \leqq R r_n \varrho^{2(n-\nu-1)}, \quad \nu = 0, \dots, n-1 \quad \text{for} \quad \varrho \geqq 1, \quad \text{and} \quad \left\| \frac{1}{r_n} V_0 U_0^{-1} \right\| \leqq \alpha.$$

Then, for any $c > 0$, there exist positive constants $K = K(n, R, \alpha, c)$, $K_1 = K_1(n, c)$, such that $Q(x; \lambda)$ exists and

$$\|\tilde{Q}(x; \lambda)\| \leqq K_1 \quad \text{for all} \quad \varrho \geqq K, \quad |x - x_0| \leqq c/\varrho.$$

Proof. Since

$$\left\| \frac{1}{r_n} V_0 U_0^{-1} \right\| \leqq \alpha$$

is equivalent with

$$-\alpha \bar{U}_0^T U_0 \leqq \frac{1}{r_n} \bar{U}_0^T V_0 \leqq \alpha \bar{U}_0^T U_0,$$

the existence of $Q(x; \lambda)$ on $\mathbb{R}$ for $\varrho \geqq K$ follows from Theorem 2. Now, we consider the matrix $Q^*(x) = \tilde{Q}(x_0 + x/\varrho; \lambda)$ for $\varrho \geqq K$, $x \in \mathbb{R}$. Because

$$\tilde{D}_\varrho A \tilde{D}_{1/\varrho} = \varrho A, \quad r_n \tilde{D}_\varrho B \tilde{D}_\varrho = B_0, \quad \frac{1}{r_n \varrho} \tilde{D}_{1/\varrho} \lambda C_0^* \tilde{D}_{1/\varrho} = -\varrho C_0,$$

and $\|C^*(\varrho)\| \leqq R\varrho^{-2}$ with

$$C^*(\varrho) = \frac{1}{r_n \varrho^2} \tilde{D}_{1/\varrho} C \tilde{D}_{1/\varrho}$$

by (2), (13), and (9), it follows from (8) nad (6) that $Q^*(x)$ is the solution of the following initial value problem:

$$Q^{*\prime} + A^T Q^* + Q^* A + Q^* B_0 Q^* - C_0 - C^*(\varrho) = 0, \quad Q^*(0) = Q_0^*(\varrho)$$

with

$$Q_0^*(\varrho) = \frac{1}{r_n \varrho} \tilde{D}_{1/\varrho} V_0 U_0^{-1} \tilde{D}_{1/\varrho}.$$

Since $\|C^*(\varrho)\| \leqq R\varrho^{-2} \to 0$, and $\|Q_0^*(\varrho)\| \leqq \alpha\varrho^{-1} \to 0$ as $\varrho \to \infty$, the results on the continuous dependence of solutions of initial value on parameters and initial values (compare e.g. [2; 22 pp.]) imply that $\lim\limits_{\varrho\to\infty} \sup\limits_{|x|\leqq c} \|Q^*(x; \varrho) - Q_0(x)\| = 0$ (in particular $\|Q^*(x; \varrho) - Q_0(x)\| \leqq 1$ for $|x| \leqq c$, $\varrho \geqq K(n, R, \alpha, c)$ and any $c > 0$), where $Q_0(x)$ is the solution of the initial value problem

$$Q_0' + A^T Q_0 + Q_0 A + Q_0 B_0 Q_0 - C_0 = 0, \quad Q_0(0) = 0.$$

Note that $Q_0(x)$ must exist on $[-c, c]$ for any $c > 0$, thus on $\mathbb{R}$, which is also included in the results mentioned above [2; 22 pp.]. It follows that

$$\|Q^*(x; \varrho)\| \leqq 1 + \sup_{|x|\leqq c} \|Q_0(x)\| = K_1(n, c) \quad \text{for} \quad \varrho \geqq K, \quad |x| \leqq c,$$

hence $\|\tilde{Q}(x; \lambda)\| = \|Q^*(\varrho(x - x_0), \lambda)\| \leqq K_1(n, c)$ for $\varrho \geqq k$, $|x - x_0| \leqq c/\varrho$, which is our assertion. □

Remark. It follows from results on Hermitian systems with constant coefficients (compare e.g. [9; p. 161]) that any Hermitian solution of $Q' + A^TQ + QA + QB_0Q - C_0 = 0$ on $(-\infty, \infty)$ satisfies $G_2 \leqq Q(x) \leqq G_1$ for $x \in \mathbb{R}$. Moreover, the particular solution $Q_0(x)$ with $Q_0(0) = 0$ satisfies $\lim_{x\to\infty} Q_0(x) = G_1$, $\lim_{x\to-\infty} Q_0(x) = G_2$, and the following algebraic equations hold:

$$A^TG_i + G_iA + G_iB_0G_i - C_0 = 0\,, \quad i = 1, 2\,.$$

Of course, these results may be verified directly in our special situation by a rather tedious caclulation.

A direct consequence of Cor. 1 (with $\gamma = 1$, $c = \frac{1}{2}$) and this Prop. 1 (with $c = (\frac{1}{2}\sin(\pi/2n))^{-1}$ is the

Corollary 2. *Under the assumptions of Proposition* 1 *there exist positive constants* $K = K(n, R, \alpha)$, $K_1 = K_1(n, R)$ *such that*

$$\|\tilde{Q}(x; \lambda)\| \leqq K_1 \quad \textit{for all} \quad \varrho \geqq K\,, \quad x \in \mathbb{R}\,.$$

4. VARIABLE COEFFICIENTS

In this section we consider the behaviour of $V(x; \lambda)\, U^{-1}(x; \lambda)$ as $\lambda \to -\infty$ on a compact interval $[x_0 - a, x_0 + a]$. Therefore, we fix $a > 0$ and introduce the following further notation:

(19) $$0 < r_* \leqq r(x) \leqq r^*\,, \quad 0 < r_{n*} \leqq r_n(x) \leqq r_n^* \quad \text{for} \quad |x - x_0| \leqq a$$
$$\text{with suitable positive constants} \quad r_*, \ldots, r_n^*\,.$$

Observe that these constants exist by (3). Moreover, we need the common modulus of continuity of $r(x)$ and $r_n(x)$, namely:

(20) $$\omega(h) = \max\{(|r(x) - r(y)| + |r_n(x) - r_n(y)|) : |x - y| \leqq h\,,$$
$$|x - x_0| \leqq a\,, \quad |y - x_0| \leqq a\} \quad \text{for} \quad h \geqq 0\,.$$

We shall derive estimates of the matrix (compare section 3)

(21) $$\tilde{Q}(x; \lambda) = \frac{1}{r_n(x)\,\varrho(x; \lambda)}\, \tilde{D}_{1/\varrho(x;\lambda)}\, Q(x; \lambda)\, \tilde{D}_{1/\varrho(x;\lambda)}$$

with $Q(x; \lambda) = V(x; \lambda)\, U^{-1}(x; \lambda)$, $\varrho(x; \lambda) = \sqrt[2n]{(-\lambda\, r(x)/r_n(x))}$ (according to (9)). Our main result in this section is

Theorem 3. *Suppose that* R, α *are positive constants such that* $|r_\nu(x; \lambda)| \leqq R\varrho_0^{2(n-\nu-1)}$, $\nu = 0, \ldots, n-1$, $|x - x_0| \leqq a$, $\varrho_0 = \sqrt[2n]{-\lambda} \geqq 1$, *and* $\bar{U}_0^T V_0 \geqq -\alpha \bar{U}_0^T U_0$. *Then there exist positive constants* K, K_1 (*depending on* $n, r_*, r^*, r_{n*}, r_n^*, R, \alpha$ *resp.* $n, r_*, r^*, r_{n*}, r_n^*, R$ *only*) *such that the following holds*: $Q(x; \lambda)$ *exists for* $x_0 < x \leqq x_0 + a$, $\lambda \leqq -K$, *and*

$$\|\tilde{Q}(x; \lambda) - G_1\| \leqq \omega^*(\varrho_0) := K_1(\varrho_0^{-2} + \omega(\log \varrho_0/\varrho_0)) \quad \textit{for all} \quad \lambda \leqq -K\,,$$

$$x_0 + x_1(\lambda) \leqq x \leqq x_0 + a \quad \textit{with} \quad x_1(\lambda) = c \log \varrho_0/\varrho_0 \,,$$
$$c = (\tfrac{1}{2} \sin(\pi/2n) \min_{|x-x_0| \leqq a} \sqrt[2n]{(r(x)/r_n(x))})^{-1} > 0 \,.$$

If the inequality above on U_0, V_0 holds with $-V_0$ instead of V_0, then the assertions hold for $x < x_0$ resp. $x \leqq x_0 - x_1(\lambda)$ $(x \geqq x_0 - a)$ with G_2 instead of G_1.

(Note that the constants K, K_1 do not depend on a 'directly', but, of course, in general the quantities $r_*, r^*, r_{n*}, r_n^*, R$, and also $\omega(h)$ depened on a, i.e. the interval $[x_0 - a, x_0 + a]$.)

Proof. As in the proof of Theorem 2 we may restrict ourselves to the case $x > x_0$, First, we introduce the solution $U_*(x)$, $V_*(x)$ of the initial value problem:

$$U'_* = AU_* + B_* V_* \,, \quad V'_* = (C_* - \lambda C_{0*}) U_* - A^T V_* \,,$$
$$U^*(x_0) = U_0, \; V_*(x_0) = V_0 \,,$$

where

$$B_* = \frac{1}{r_{n*}} B_0 \geqq B(x) \,, \quad C_* = -R\tilde{D}^2_{\varrho 0} \leqq C(x) \,, \quad \text{and} \quad C_{0*} = r_* C_0 \leqq C_0^*(x)$$

for $|x - x_0| \leqq a$ by (19) and (2). This initial value problem has constant coefficients, and therefore we obtain the existence of $Q_*(x; \lambda) = V_*(x; \lambda) U_*^{-1}(x; \lambda)$ on $(x_0, x_0 + a]$ for $\lambda \leqq -K$ from Theorem 2. Now, the inequalities $B_* \geqq B(x)$, $C_* \leqq C(x)$, $C_{0*} \leqq C_0^*(x)$ and the fact that $U(x_0) = U_*(x_0)$, $V(x_0) = V_*(x_0)$ imply that $Q(x; \lambda)$ exists with $Q(x; \lambda) \geqq Q_*(x; \lambda)$ for $x \in (x_0, x_0 + a]$, $\lambda \leqq -K$. This follows from [6, Prop. 4 and Theorem 7, which implies $\lim_{x \to x_0+} (\bar{U}^T(x) V_*(x) U_*^{-1}(x) \,.$ $. \, U(x) - \bar{U}^T(x) V(x)) = 0$]. Then Cor. 1 (applied to Q_*) and $G_1 > 0$ imply that

$$(22) \qquad Q(x; \lambda) \geqq Q_*(x; \lambda) > 0 \quad \text{for} \quad x_0 + \tfrac{1}{4} x_1(\lambda) \leqq x \leqq x_0 + a \,, \quad \lambda \leqq -K \,.$$

Next, we fix any $x \in [x_0 + x_1(\lambda), x_0 + a]$ and introduce the following quantities (depending on x): $x^* = x - \frac{3}{4} x_1(\lambda) \geqq x_0 + \frac{1}{4} x_1(\lambda)$, and

$$r^- = \min_{[x^*,x]} r(t) \,, \quad r^+ = \max_{[x^*,x]} r(t) \,, \quad r_n^- = \min_{[x^*,x]} r_n(t) \,, \quad r_n^+ = \max_{[x^*,x]} r_n(t) \,,$$

$\varrho^- = \sqrt[2n]{(-\lambda r^-/r_n^-)}$, $\varrho^+ = \sqrt[2n]{(-\lambda r^+/r_n^+)}$. We consider the solutions $U^\pm(t)$, $V^\pm(t)$ of the initial value problem (with constant coefficients):

$$U^{\pm\prime} = AU^\pm + B^\pm V^\pm \,, \quad V^{\pm\prime} = (C^\pm - \lambda C_0^\pm) U^\pm - A^T V^\pm \,,$$
$$U^\pm(x^*) = U(x^*; \lambda) \,, \quad V^\pm(x^*) = V(x^*; \lambda) \,,$$

where $B^\pm, C^\pm, C_0^\pm$ are defined according to (2) with $r^\pm, r_n^\pm, r_\nu^\pm = \pm R\varrho_0^{2(n-\nu-1)}$ $(\nu = 0, \ldots, n-1)$ instead of r, r_n, r_ν resp.. Then $B_* \geqq B^- \geqq B(t) \geqq B^+$, $C_* \leqq$ $\leqq C^- \leqq C(t) \leqq C^+$, and $C_{0*} \leqq C_0^- \leqq C_0(t) \leqq C_0^+$ on $[x^*, x]$, $Q_*(x^*; \lambda) \leqq$ $\leqq Q^\pm(x^*; \lambda) = Q^\pm(x^*; \lambda)$. Hence, we may apply [6; Prop. 4] again, and we get: $Q^\pm(t; \lambda) = V^\pm(t; \lambda) U^{\pm -1}(t; \lambda)$ exists on $[x^*, x]$ with

$$(23) \quad Q_*(t; \lambda) \leqq Q^-(t; \lambda) \leqq Q(t; \lambda) \leqq Q^+(t; \lambda) \quad \text{on} \quad [x^*, x] \quad \text{for} \quad \lambda \leqq -K \,.$$

Since $Q(x^*; \lambda) = Q^{\pm}(x^*; \lambda) > 0$ by (22), Theorem 2 with $\alpha = 0$ (and $c = \frac{3}{4}$) can be applied to $Q^{\pm}(t; \lambda)$. (Note here, that the initial values $U(x^*; \lambda)$, $V(x^*; \lambda)$ depend on λ, of course, but the constants in Theorem 2 do not depend on U_0, V_0 but only on α, which may be 0 here since $\overline{U(x^*; \lambda)}^T V(x^*; \lambda) \geqq 0$.) Let $\tilde{Q}^{\pm}(t; \lambda) = (1/r_n^{\pm} \varrho^{\pm}) \tilde{D}_{1/\varrho^{\pm}} Q^{\pm}(t; \lambda) \tilde{D}_{1/\varrho^{\pm}}$, then Theorem 2 yields:

$$\text{(24)} \qquad \|\tilde{Q}^{\pm}(t; \lambda) - G_1\| \leqq K_1 \varrho_0^{-2} \quad \text{for} \quad x^* + x_1^* \leqq t \leqq x\,, \quad \lambda \leqq -K$$

with $x_1^* = (\frac{3}{4} \sin(\pi/2n))^{-1} \log \varrho^{\pm}/\varrho^{\pm}$. Observe that (24) holds for $t = x$, in particular, since $x^* + x_1^* \leqq x$ for $\lambda \leqq -K$, which follows from:

$$x^* + x_1^* = x - \frac{3}{2}\left(\sin\frac{\pi}{2n}\right)^{-1} \frac{\log \varrho_0}{\varrho_0} \frac{1}{c^*} + \frac{4}{3}\left(\sin\frac{\pi}{2n}\right)^{-1} \frac{\log(c^{\pm}\varrho_0)}{c^{\pm}\varrho_0}$$

with

$$c^* = \min_{|t-x_0| \leqq a} \sqrt[2n]{(r(t)/r_n(t))} \leqq c^{\pm} = \sqrt[2n]{(r^{\pm}/r_n^{\pm})} \leqq \sqrt[2n]{(r^*/r_{n*})}\,.$$

Now, (23) and (24) for $t = x$ yield for $\lambda \leqq -K$:

$$\tilde{Q}(x; \lambda) \leqq \frac{1}{r_n \varrho} \tilde{D}_{1/\varrho} Q^+(x; \lambda) \tilde{D}_{1/\varrho} = \frac{r_n^+ \varrho^+}{r_n \varrho} \tilde{D}_{\varrho^+/\varrho} \tilde{Q}^+(x; \lambda) \tilde{D}_{\varrho^+/\varrho} \leqq$$

$$\leqq \frac{r_n^+ \varrho^+}{r_n \varrho} \tilde{D}_{\varrho^+/\varrho}(G_1 + K_1 \varrho_0^{-2} I) \tilde{D}_{\varrho^+/\varrho}\,,$$

and this is (with another K_1) $\leqq G_1 + \omega^*(\varrho_0) I$, since $\omega(h) \leqq 2(r^* + r_n^*)$ by (19), (20) and

$$\frac{\varrho^+}{\varrho} = \frac{\varrho^+(x; \lambda)}{\varrho(x; \lambda)} = \sqrt[2n]{\left(\frac{r^+}{r_n^+}(x) \frac{r_n}{r}(x)\right)} \leqq 1 + K_1 \log \varrho_0/\varrho_0\,, \quad \frac{r_n^+}{r_n} \leqq 1 + K_1 \log \varrho_0/\varrho_0$$

(use that $x - x^* \leqq K_1 \log \varrho_0/\varrho_0$, and that the modulus of continuity $\omega(h)$ has the obvious properties: $\omega(h) \leqq \omega(h')$ if $h \leqq h'$, $\omega(nh) \leqq n\, \omega(h)$ for $n \in \mathbb{N}$, thus, $\omega(h) \leqq \omega(ch) \leqq (c+1)\, \omega(h)$ for all $c \geqq 1$, $h \geqq 0$).

Similarly, we obtain that $\tilde{Q}(x; \lambda) \geqq G_1 - \omega^*(\varrho_0) I$, which completes the proof. □

Remark. If we have additionally that $r_n(x), r(x) \in C_1(\mathbb{R})$, then we obtain (under the assumptions of Theorem 3) the estimate

$$\|\tilde{Q}(x; \lambda) - G_1\| \leqq K_1 \frac{\log \varrho_0}{\varrho_0} \quad \text{for all} \quad \lambda \leqq -K\,, \quad x_0 + x_1(\lambda) \leqq x \leqq x_0 + a$$

with $x_1(\lambda)$ as in Theorem 3, but where the constant K_1 may depend on $c_1 := \max\{|r_n'(x)| + |r'(x)| : |x - x_0| \leqq a\} < \infty$ also.

Proof of Theorem 1. Fix any $x \neq x_0$, choose $a \geqq |x - x_0|$, and choose a constant $\alpha > 0$ (which is obviously always possible) such that the assumptions of Theorem 3 hold. Since $x_1(\lambda) \to 0$ as $\lambda \to -\infty$, we have that $|x - x_0| \geqq x_1(\lambda)$ for λ sufficiently small. Moreover, $\lim_{\lambda \to -\infty} \omega(\log \varrho_0/\varrho_0) = 0$ by the continuity of $r_n(x)$

and $r(x)$, and therefore the estimate of Theorem 3 implies the assertion of Theorem 1. □

If U_0 is regular, the proof above and Cor. 2 yield the following

Corollary 3. *Suppose that U_0 is regular, and that R, α are positive constants such that*

$$|r_\nu(x;\lambda)| \leqq R\varrho_0^{2(n-\nu-1)}, \quad \nu = 0, \ldots, n-1, \quad |x - x_0| \leqq a, \quad \varrho_0 \geqq 1,$$
$$\|V_0 U_0^{-1}\| \leqq \alpha.$$

Then there exist positive constants $K(n, r_*, r^*, r_{n*}, r_n^*, R, \alpha)$ and $K_1(n, r_*, r^*, r_{n*}, r_n^*, R)$ such that $\tilde{Q}(x;\lambda)$ exists on $[x_0 - a, x_0 + a]$ with $\|\tilde{Q}(x;\lambda)\| \leqq K_1$ for $x \in [x_0 - a, x_0 + a]$, if $\lambda \leqq -K$.

5. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE CORRESPONDING SCALAR EQUATION

In this section we use the same notation as in the previous section, and we study the asymptotic behaviour of the particular conjoined bases U_i, V_i, $i = 1, 2$ of (1) with the initial conditions

$$(25) \qquad U_1(x_0) = V_2(x_0) = 0, \quad V_1(x_0) = -U_2(x_0) = I.$$

This leads to asymptotic properties of any solution (u, v) of (1) resp. (4) with fixed initial conditions at x_0. First, we derive from Theorem 3 the

Corollary 4. *With the same assumptions and notation as in Theorem 3 the following inequalities hold:*

$$\left\| \frac{1}{\varrho(x_0;\lambda)\, r_n(x_0)} \tilde{D}_{1/\varrho(x_0;\lambda)} U_1^{-1}(x;\lambda)\, U_2(x;\lambda)\, \tilde{D}_{1/\varrho(x_0;\lambda)} - G \right\| \leqq K_1 \omega^*(\varrho_0);$$
$$\left\| \frac{1}{\varrho(x_0;\lambda)\, r_n(x_0)} \tilde{D}_{1/\varrho(x_0;\lambda)} V_1^{-1}(x;\lambda)\, V_2(x;\lambda)\, \tilde{D}_{1/\varrho(x_0;\lambda)} - G \right\| \leqq K_1 \omega^*(\varrho_0)$$

for all $\lambda \leqq -K$ and $x_1(\lambda) \leqq |x - x_0| \leqq a$, where $G = G_2$ for $x > x_0$ and $G = G_1$ for $x < x_0$. Moreover,

$$\|U_2^{-1}(x;\lambda)\, U_1(x;\lambda)\| \leqq K_1 \varrho_0^{-1} \quad \textit{for all} \quad 0 \leqq |x - x_0| \leqq a, \quad \lambda \leqq -K$$

(Observe that the asymptotic behaviour does not depend on x, e.g. $\varrho(x;\lambda)$, but only on x_0 in contrast to Theorem 3 and (21).)

Proof. For any fixed $x \in [x_0 - a, x_0 + a]$ consider the particular conjoined bases $\tilde{U}_i(t) = \tilde{U}_i(t;\lambda, x)$, $\tilde{V}_i(t) = \tilde{V}_i(t;\lambda, x)$, $i = 1, 2$ of (1) with the initial conditions $\tilde{U}_1(x) = \tilde{V}_2(x) = 0$, $\tilde{U}_2(x) = \tilde{V}_1(x) = I$. Then, it follows from [6; (7′) and the proof of Cor. 13] and (25) that

$$\tilde{Q}_1(x_0) = U_1^{-1}(x)\, U_2(x) \quad \text{and} \quad \tilde{Q}_2(x_0) = V_1^{-1}(x)\, V_2(x) \quad \text{hold for}$$
$$\tilde{Q}_i(t) = \tilde{V}_i(t)\, \tilde{U}_i^{-1}(t).$$

Hence, Theorem 3 applied to $\tilde{Q}_i$ (with x instead of x_0) yields the first two inequalities. The additional inequality follows from these inequalities and from the fact that

$$U_2^{-1}(x_0)\, U_1(x_0) = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}x}\, U_2^{-1}(x)\, U_1(x) \leqq 0$$

for all x (by [6; Prop. 3]). □

Remark. For any solution $y = y(x; \lambda)$ of (4) with fixed initial conditions $u(x_0) = u_0$, $v(x_0) = v_0$ we have by (25) that

$$u(x) = U_2(x)\,(-u_0 + U_2^{-1}(x)\, U_1(x)\, v_0), \quad v(x) = V_2(x)\,(-u_0 + V_2^{-1}(x)\, V_1(x)\, v_0)\,.$$

Hence, in view of Theorem 3 and this Cor. 4 it suffices to discuss the asymptotic bebaviour of $U_2(x; \lambda)$ if one wants to analyze $y(x; \lambda)$. We shall restrict this discussion to the case $x \geqq x_0$.

First, we need some identities, which follow from (2), (10), (11), and (12) by a rather tedious calculation

$$(26)\qquad \Phi_{11}^0 \operatorname{diag}(\varepsilon_0, \ldots, \varepsilon_{n-1})\, \Phi_{11}^{0^{-1}} = A + B_0 G_1 = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & & \ldots & 1 \\ \gamma_0 & & \ldots & \gamma_{n-1} \end{pmatrix}$$

and this Frobenius matrix has the characteristic polynomial $P_0(z) = z^n - \sum_{v=0}^{n-1} \gamma_v z^v = (z - \varepsilon_0) \ldots (z - \varepsilon_{n-1})$, so that e.g.

$$(27)\qquad \gamma_{n-1} = \sum_{v=0}^{n-1} \varepsilon_v = \frac{1}{\sin(\pi/2n)}\,, \quad \gamma_k = (-1)^{n-k-1} \prod_{v=0}^{n-1} \frac{\cos(v\pi/2n)}{\sin((v+1)\,\pi/2n)}\,,$$

and

$$(28)\qquad 0 < \gamma_* := \sin(\pi/2n) = \min_{k=0,\ldots,n-1} \operatorname{Re}(\varepsilon_k) \leqq \gamma_{n-1}/n \leqq$$

$$\leqq \max_{k=0,\ldots,n-1} \operatorname{Re}(\varepsilon_k) = : \gamma^* = \sin\left(\pi\left(\frac{1}{2n} + \frac{[(n-1)/2]}{n}\right)\right).$$

Moreover, these identities, (25) (use also the notation (9), (21)) imply that $U_2 = U_2(x; \lambda)$ satisfies the initial value problem

$$(29)\qquad U_2' = (A + BQ_2)\, U_2\,, \quad U_2(x_0) = -I\,,$$

where

$$(30)\qquad A + BQ_2 = \varrho \tilde{D}_{1/\varrho}(A + B_0 G_1 + \Delta_1)\, \tilde{D}_\varrho\,, \quad \Delta_1 = B_0(\tilde{Q}_2(x; \lambda) - G_1)\,.$$

Proposition 3. *With the same assumptions and notation as in Theorem* 3 *and Cor.* 3 (*but with* $U_0 = -I$, $V_0 = 0$, $\alpha = 0$) *we have*

$$\det U_2(x; \lambda) = (-1)^n \exp\{\varrho_0 \gamma_{n-1} \textstyle\int_{x_0}^x \delta(t)\, \mathrm{d}t\}\, \alpha(x; \lambda) \quad \textit{for}$$

$$x_0 \leqq x \leqq x_0 + a\,, \quad \lambda \leqq -K\,,$$

where

$$1/\beta(x; \lambda) \leqq \alpha(x; \lambda) \leqq \beta(x; \lambda) := \exp\{\varrho_0 \textstyle\int_{x_0}^x \varepsilon(t; \lambda)\, \mathrm{d}t\}\,,$$

$$\varepsilon(t;\lambda) \equiv K_1 \quad for \quad x_0 \leqq t \leqq x_1(\lambda)\,,$$
$$\varepsilon(t;\lambda) \equiv K_1\,\omega^*(\varrho_0) \quad for \quad x_1(\lambda) \leqq t \leqq x_0 + a\,.$$

Proof. By Theorem 3, Cor. 3, and (30) we have $\|\Delta_1(t;\lambda)\| \leqq \varepsilon(t;\lambda)$. Hence, we obtain from (26), (29), and (30) that

$$\det U_2(x;\lambda) = (-1)^n \exp\{\int_{x_0}^x \operatorname{trace}(A + BQ_2)(t;\lambda)\,\mathrm{d}t\} =$$
$$= (-1)^n \exp\{\varrho_0\gamma_{n-1}\int_{x_0}^x \delta(t)\,\mathrm{d}t\} \exp\{\int_{x_0}^x \varepsilon^*(t;\lambda)\,\mathrm{d}t\}$$

where

$$\varepsilon^*(t;\lambda) = \operatorname{trace}(\varrho\tilde{D}_{1/\varrho}\Delta_1\tilde{D}_\varrho)\,, \quad \text{thus} \quad |\varepsilon^*(t;\lambda)| \leqq \varrho_0\,\delta(t)\,\|\Delta_1(t;\lambda)\|\,,$$

and the factor $\delta(t)$ can be included in the constant K_1. □

Remark. If we have additionally r_n, $r \in C_1(\mathbb{R})$, then

$$\varrho_0^{-K_1(1+x-x_0)} \leqq \beta(x;\lambda) \leqq \varrho_0^{K_1(1+x-x_0)} \quad \text{for} \quad x_0 \leqq x \leqq x_0 + a\,, \quad \lambda \leqq -K\,,$$

where K_1 may also depend on $\max\{|r'(x)| + |r_n'(x)| : |x - x_0| \leqq a\}$. Observe also, that Prop. 3 yields lower bounds for $\|U_2(x;\lambda)\|$ and $\|U_2^{-1}(x;\lambda)\|$, since for any matrix $H = (h_1, \ldots, h_n)$, $h_i \in \mathbb{C}^n$ we have by Hadamard's inequality:

$$\|H\| \geqq \max_{i=1,\ldots,n} \|h_i\| \geqq |\det H|^{1/n}\,.$$

Finally, we derive upper bounds for $\|U_2(x;\lambda)\|$, $\|U_2^{-1}(x;\lambda)\|$, namely:

Proposition 4. *With the assumptions of Prop.* 3 *and with* r_n, $r \in C_1(\mathbb{R})$ *there exist constants* K, K_1 *(depending on* $n, r_*, r^*, r_{n*}, r_n^*, R$ *as in Prop.* 3 *and* K_1 *depending also on* $\max\{|r_n'(x)| + |r'(x)| : |x - x_0| \leqq a\}$*), such that*

$$\|U_2(x;\lambda)\| \leqq \varrho_0^{K_1(1+x-x_0)} \exp\{\varrho_0\gamma^*\int_{x_0}^x \delta(t)\,\mathrm{d}t\}\,,$$

and

$$\|U_2^{-1}(x;\lambda)\| \leqq \varrho_0^{K_1(1+x-x_0)} \exp\{-\varrho_0\gamma_*\int_{x_0}^x \delta(t)\,\mathrm{d}t\} \quad for$$
$$x_0 \leqq x \leqq x_0 + a\,, \quad \lambda \leqq -K\,.$$

Proof. Put $\tilde{U}_2 := \Phi_{11}^{0^{-1}}\tilde{D}_\varrho U_2$, then by (26), (30) $\tilde{U}_2' = \varrho(\operatorname{diag}(\varepsilon_0, \ldots, \varepsilon_{n-1}) + \Delta_2)\tilde{U}_2$ with $\Delta_2 = \Phi_{11}^{0^{-1}}\Delta_1\Phi_{11}^0 + \tilde{\Delta}_2$ where

$$\tilde{\Delta}_2 = \frac{1}{\varrho(x;\lambda)}\frac{\delta'(x)}{\delta(x)}\,\Phi_{11}^{0^{-1}}\operatorname{diag}((n-1), \ldots, 1, 0)\,\Phi_{11}^0\,.$$

Hence, by Theorem 3 and its remark we obtain that $\|\Delta_2(x;\lambda)\| \leqq \tilde{\varepsilon}(x;\lambda)$ with

$$\tilde{\varepsilon}(x;\lambda) \equiv K_1 \quad \text{for} \quad x_0 \leqq x \leqq x_1(\lambda) \quad \text{and} \quad \equiv K_1 \log\varrho_0/\varrho_0 \quad \text{for}$$
$$x_1(\lambda) \leqq x \leqq x_0 + a\,.$$

Now the matrix $P := \bar{\tilde{U}}_2^T\tilde{U}_2$ is positive definite and satisfies

$$P' = \varrho\bar{\tilde{U}}_2^T(2\operatorname{diag}(\operatorname{Re}\varepsilon_0, \ldots, \varepsilon_{n-1}) + \Delta_2 + \bar{\Delta}_2^T)\,\tilde{U}_2\,.$$

Hence by (28), $P' \leqq \varrho(2\gamma^* + \tilde{\varepsilon})\,P$ and $P' \geqq \varrho(2\gamma_* - \tilde{\varepsilon})\,P$, which yields our assertions. □

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