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COMPACTNESS TYPE PROPERTIES IN TOPOLOGICAL GROUPS

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1. Introduction. Pseudocompactness is a purely topological property. Being considered in the class of topological groups, it acquires specific features that are of the most interest for an investigation. It is known that the product of two pseudocompact (even countably compact) spaces need not be pseudocompact [19]. However the product of an arbitrary many pseudocompact topological groups is pseudocompact, too [7, Theorem 1.4]. This Comfort and Ross's theorem is a base point of our research.

1.1. The notion of pseudocompactness admits the following generalization. A subset X of a space Y is said to be *bounded* in Y if any continuous real-valued function defined on Y is bounded on X [4, 5]. In Section 2 we spread the Comfort and Ross's theorem on bounded subsets of topological groups: if a set X_α is bounded in a topological group G_α, $\alpha \in A$, then $\Pi\{X_\alpha: \alpha \in A\}$ is a bounded subset of $\Pi\{G_\alpha: \alpha \in A\}$ (Theorem 2.2). Some applications of this result to (free) topological groups are given here.

Let f be any continuous real-valued function on a pseudocompact group G. Then f is extendable to a continuous function $\hat{f}$ on the completion $\hat{G}$ of G, hence f is uniformly continuous [7, Theorems 1.2 and 1.5]. If, however, a topological group G is not pseudocompact, there may exist continuous functions which do not admit to a continuous extension to $\hat{G}$. Nevertheless, a "bounded" analogy of the uniform continuity theorem holds (see Theorem 2.27 and Corollary 2.29).

In Section 3 we consider the inverse spectra consisting of pseudocompact groups and open homomorphisms. Recently W. W. Comfort and L. C. Robertson [10] (and the author independently) proved that if the groups G/N and N are pseudocompact then G has the same property. A generalization of the above result is given in Section 2 (Theorem 2.18). It is shown also that the natural quotient map $p: G \to G/N$ is z-closed in this case (Lemma 3.1.) With the help of these facts we prove that the limit group of a countable spectrum $\mathscr{L} = \{G_n, p_{n,m}\}_{m,n\in N}$ is pseudocompact if the groups G_n are pseudocompact and $p_{n+1,n}$ are open epimorphisms with pseudocompact kernels (Theorem 3.4). The conditions of this theorem are essential, that follows from Examples 4.2 and 4.3. It is necessary to mention that a closed subgroup K

of a pseudocompact group G does not need even to be locally pseudocompact and moreover, any totally bounded group is embeddable to some pseudocompact group as a closed subgroup. This fact is an easy generalization of [8, Th. 2.4].

The case of the uncountable well-ordered inverse spectrum with pseudocompact groups is much more simple (see Theorem 3.6).

It is known [20] that for each continuous real-valued function f on a compact (or even pseudocompact [7]) group G there exist a continuous homomorphism π of G onto a separable metrizable group H and a continuous function h on H such that $f = h \circ \pi$. It seems to be surprising that this assertion holds for any totally bounded group G (Theorem 3.8). The equivalent formulation of Theorem 3.8 is the following: Any continuous real-valued function defined on a subgroup of a product of compact groups, depends on countably many coordinates (Assertion 3.7).

Recently M. V. Matveev [16] and E. Reznichenko have shown that there exists a Tychonoff pseudocompact space X such that every countable subset of X is closed and C^*-embedded in X. We claim in Example 4.5 that there exists a pseudocompact group G with $w(G) = |G| = 2^{\aleph_0}$ such that every countable subgroup of G is closed and h-embedded in G (see Definition 4.4). Besides the group G is zerodimensional and contains no non-trivial convergent sequences. It is interesting to note that every infinite pseudocompact group has a countable non-closed subset (Remark 4.8).

1.2. Our terminology and notations are standard. The set of all positive integers (with zero) denoted by $\boldsymbol{N}$, $\boldsymbol{N}^+ = \boldsymbol{N} \setminus \{0\}$.

The symbol $\boldsymbol{c}$ stands for the power of continuum, $\boldsymbol{c} = 2^{\aleph_0}$. All spaces under consideration are assumed to be Tychonoff and topological groups are Hausdorff.

2. Bounded subsets in topological groups.

2.1. Definition. A subset X of a space Y is said to be *bounded in* Y provided that any continuous real-valued function defined on Y is bounded on X.

The main result of this section is the following theorem.

2.2. Theorem. *Let X_α be a bounded subset of a topological group G_α, where $\alpha \in A$. Then $\Pi\{X_\alpha: \alpha \in A\}$ is bounded in $\Pi\{G_\alpha: \alpha \in A\}$.*

Comfort and Ross's theorem on a product of pseudocompact groups follows easily from Theorem 2.2 (put $X_\alpha = G_\alpha$ for each $\alpha \in A$). Here we shall prove a more general result, Theorem 2.11 (note that Lemma 2.10 answers partially the question: what properties of topological groups are responsible for the bounded subsets product theorem). Some definitions and preliminary results are necessary.

2.3. Definition. A subset B of a space X is *strongly bounded* in X if each infinite family of open subsets of X meeting B, contains an infinite subfamily $\{U_n: n \in \boldsymbol{N}\}$ which satisfies $(*)$, where

(∗) For each filter Φ consisting of infinite subsets of N,

$$\bigcap_{P\in\Phi} \mathrm{cl}_X(\bigcup_{n\in P} U_n) \neq \emptyset .$$

Obviously, each strongly bounded subset of X is bounded in X, for any locally finite family of open sets in X has at most finitely many members which intersect a bounded subset B of X. The following result is quite analogous to the Lemma of [17].

2.4. Lemma. *Let Φ be a filter on N, $U_n \cap B_0 \neq \emptyset$, $x \in X_0 \supseteq B_0$ and U_n open in X_0 for each $n \in N$, where $x \in \mathrm{cl}_{X_0}(\bigcup_{n\in P} U_n)$ for each $P \in \Phi$. If B_1 is strongly bounded in a space X_1, V_n is open in X_1 and $V_n \cap B_1 \neq \emptyset$ for each $n \in N$ then there exists a point $y \in X_1$ such that $(x, y) \in \mathrm{cl}_{X_0\times X_1}(\bigcap_{n\in Q} U_n \times V_n)$ for each $Q \in \Phi$.* ■

Lemma 2.4 implies easily the following corollary.

2.5. Corolary. *If B_0 is bounded in X_0 and B_1 is strongly bounded in X_1, then $B_0 \times B_1$ is bounded in $X_0 \times X_1$.* ■

The following theorem can be proved in the same manner as Theorem 3.1 [17], therefore we omit its proof.

2.6. Theorem. *Let B_α be a strongly bounded subset of a space X_α, for each $\alpha \in A$. Then $\Pi\{B_\alpha: \alpha \in A\}$ is strongly bounded in $\Pi\{X_\alpha: \alpha \in A\}$.* ■

Now let $f: X \to Y$ and $g: X \to Z$ be continuous mappings. The inequality $f \leqq g$ means that there exists a continuous mapping $h: Y \to Z$ such that $g = h \circ f$.

2.7. Definition. Let $\mathscr{X}$ be a family of continuous mappings of X to some spaces. We shall say that $\mathscr{X}$ is $\aleph_0$-directed lattice for X provided that $\mathscr{X}$ generates the original topology of X and each countable subfamily of $\mathscr{X}$ has an lower bound in $\mathscr{X}$.

Due to the Definition 4 from [22], a continuous mapping $f: X \to Y$ is said to be *d-open* if for open set O of X there exists an open set V of X such that $f(O)$ is a dense subset of V. The crucial step toward Theorem 2.11 is the following lemma.

2.8. Lemma. *Let $\mathscr{X}$ be an $\aleph_0$-directed lattice for X consisting of d-open mappings onto Dieudonné-complete spaces. Then each bounded subset of X is strongly bounded in X.*

Proof. Let B be a bounded subset of X and $\{U_n: n \in N\}$ a family of open subsets of X each of which meets B. Pick a point $x_n \in B \cap U_n$, $n \in N$. As $\mathscr{X}$ generates the original topology of X, for each $n \in N$ there exist a mapping $\varphi_n \in \mathscr{X}$, $\varphi_n: X \to X_n$, and an open subset $V_n \subseteq X_n$ such that $x_n \in \varphi_n^{-1}(V_n) \subseteq U_n$. We choose $\varphi \in \mathscr{X}$ with $\varphi \leqq \varphi_n$ for every $n \in N$.

There exist open subsets W_n of the space $Y = \varphi(X)$ such that $x_n \in \varphi^{-1}(W_n) \subseteq U_n$ for each $n \in N$. The set $\varphi(B)$ is bounded in the Dieudonné-complete space Y, hence

$K = \mathrm{cl}_Y\, \varphi(B)$ is compact. Consequently for each filter Φ on N the set $\bigcap_{P \in \Phi} \mathrm{cl}_Y(\bigcup_{n \in P} W_n)$ is not empty. The mapping φ is d-open, therefore one can apply Lemma 5 of [22] to conclude that $\varphi^{-1}(\mathrm{cl}_Y W) = \mathrm{cl}_X\, \varphi^{-1}(W)$ for each open subset $W \subseteq Y$. In particular, $\bigcap_{P \in \Phi} \mathrm{cl}(\bigcup_{n \in P} U_n) \supseteq \bigcap_{P \in \Phi} \mathrm{cl}(\bigcup_{n \in P} \varphi^{-1}(W_n)) = \varphi^{-1}(\bigcap_{P \in \Phi} \mathrm{cl}(\bigcup_{n \in P} W_n)) \neq \emptyset$. ∎

2.9. Definition. A subset Y of a space X is said to be a *dop-subset of* X (Dense in OPen) provided that there exists an open set $V \subseteq X$ with $Y \subseteq V \subseteq \mathrm{cl}_X Y$.

2.10. Lemma. *Each dop-subset of a topological group G has an $\aleph_0$-directed lattice consisting of d-open mappings onto Dieudonné-complete spaces.*

Proof. Here we use Definition 2.19 of an admissible subgroup. Let X be a dop-subset of G, $\mathscr{D}$ a family of all admissible subgroups of G and π_N a quotient mappings of G onto a coset space G/N, for each $N \in \mathscr{D}$. Now we define $\mathscr{L}^* = \{\pi_N : N \in \mathscr{D}\}$ and $\mathscr{L} = \{\pi_N|_X : N \in \mathscr{D}\}$. The family $\mathscr{L}^*$ is $\aleph_0$-directed and consists of open mappings. Moreover, for each $N \in \mathscr{D}$ there exists a continuous one-to-one mapping of a coset space G/N onto a metrizable space (see [1], or our Lemma 2.21), hence G/N is a Dieudonné-complete space [12, Exercise 8.5.13 (g)] and any subspace of G/N has the same property. It remains to note that a restriction of an open mapping to a dop-subset is d-open [22; Lemma 7], hence $\mathscr{L}$ is an $\aleph_0$-directed lattice consisting of d-open mappings onto Dieudonné-complete spaces. ∎

Theorem 2.6 and Lemma 2.8, 2.10 imply together:

2.11. Theorem. *Let X_α be a dop-subset of some topological group and B_α bounded in X_α, where $\alpha \in A$. Then $\Pi\{B_\alpha : \alpha \in A\}$ is bounded in $\Pi\{X_\alpha : \alpha \in A\}$.* ∎

Lemma 8 from [25] implies that a dense (dop-) subset of any $\varkappa$-metrizable*) compact space has an $\aleph_0$-directed lattice consisting of d-open mappings onto separable metrizable spaces. Thus one can apply Theorem 2.6 and Lemma 2.8 to prove the following.

2.12. Theorem. *Let X_α be a dense (dop-) subset of some $\varkappa$-metrizable compact space and B_α bounded in X_α, $\alpha \in A$. Then $\Pi\{B_\alpha : \alpha \in A\}$ is bounded in $\Pi\{X_\alpha : \alpha \in A\}$.* ∎

2.13. Theorem. *Let X be a pseudocompact $\varkappa$-metrizable space. Then $X \times Y$ is pseudocompact for each pseudocompact space Y.*

Proof. Theorem 2 of [6] implies that the Cech-Stone compactification βX of X is $\varkappa$-metrizable, hence the space X has an $\aleph_0$-directed lattice consisting of d-open mappings onto separable metrizable spaces [25, Lemma 8]. It remains to apply Lemma 2.8 and Lemma 2.4 (with $X_1 = B_1 = Y$ and $X_0 = B_0 = X$). ∎

If G is a pseudocompact group, then the completion $\hat{G}$ of G is a compact group and $\hat{G}$ is $\varkappa$-metrizable [25, p. 201]. Thus G is $\varkappa$-metrizable as a dense subspace of $\hat{G}$. With the help of Theorem 2.13 we have proved the following.

*) The notion of $\varkappa$-metrizable space is defined in [25], [27].

2.14. Corollary. *If G is a pseudocompact topological group then $G \times Y$ is pseudocompact for each pseudocompact space Y.* ■

Theorem 2.2 has a few corollaries.

2.15. Corollary. *For any bounded subsets X, Y of a topological group G the group product $X \,.\, Y$ is bounded in G.*

Proof. Consider the bounded subset $X \times Y$ of $G \times G$ and the continuous mapping $\Theta: G \times G \to G$, $\Theta(x, y) = x \,.\, y$. ■

A subset X of a space Y is said to be *σ-bounded* in Y, if X is an union of countably many bounded in Y subsets.

2.16. Corollary. *Let a topological group G be generated by its σ-bounded subset. Then G is σ-bounded, too.* ■

For free topological groups, Corollary 2.16 admits the following improvement.

2.17. Theorem. *The following conditions are equivalent for a space X:*

(i) *the free topological group $F(X)$ is σ-bounded;*

(ii) *the free abelian topological group $A(X)$ is σ-bounded;*

(iii) *the space X is σ-bounded.*

Proof. If X is σ-bounded then $F(X)$ and $A(X)$ are σ-bounded by Corollary 2.17. Inversely, assume that the group $F(X)$ is σ-bounded, i.e. $F(X) = \bigcup\{M_n: n \in N\}$, where each M_n is bounded in $F(X)$. For any $M \subseteq F(X)$ let $\alpha(M)$ be the set of all the elements $x \in X$ which occur in the reduced words of M. Proposition 2 of [3] implies that $\alpha(M_n)$ is bounded in X for each $n \in N$, hence $X = \bigcup\{\alpha(M_n): n \in N\}$ is σ-bounded. Analogous arguments are applicable to the group $A(X)$. ■

It was mentioned above that the class of pseudocompact groups is closed under the operation of extension [10, Theorem 6.3]. A considerably more general version of this result is valid.

2.18. Theorem. *Let K be a closed subgroup of a topological group G, K bounded in G and X bounded subset of the quotient space G/K. Then the set $p^{-1}(X)$ is bounded in G, where $p: G \to G/K$ is a quotient mapping.*

To prove this theorem we need some preliminary results.

2.19. Definitiin. A subgroup H of a topological group G is said to be *admissible*, if there exists a sequence $\{U_n: n \in N\}$ of open neighbourhoods of identity such that $U_n^{-1} = U_n$, $U_{n+1}^3 \subseteq U_n$ for each $n \in N$ and $H = \bigcap\{U_n: n \in N\}$.

It is easily seen that any admissible subgroup of G is closed in G. An intersection, of countably many admissible subgroups is an admissible subgroup, too. Finally, each open neighborhood of identity contains some admissible subgroup.

2.20 Lemma. *If X is not bounded in a topological group G then there exists an admissible subgroup N of G such that $\pi(X)$ is not bounded in G/N, where $\pi: G \to G/N$ is a quotient mapping.*

Proof. Let G/N be a left coset space. As X is not bounded in G, there exists a locally finite family $\{V_n: n \in N\}$ of open subsets of G each element of which intersects X. For each $n \in N$ we pick a point $x_n \in V_n \cap X$ and an open neighborhood of identity W_n such that $x_n W_n^2 \subseteq V_n$. Let H_n be an admissible subgroup of G with $H_n \subseteq W_n$ and $H = \bigcap\{H_n: n \in N\}$. Then H is admissible in G. For the quotient mapping $\pi_n: G \to G/H_n$ we have

$$\pi_n^{-1}\pi_n(x_n W_n) = x_n W_n H_n \subseteq x_n W_n^2 \subseteq V_n .$$

Consequently $\pi^{-1}\pi(x_n W_n) \subseteq V_n$ for each $n \in N$, where $\pi: G \to G/H$. The mapping π is open, hence the previous inclusion implies that the family $\{\pi(x_n W_n): n \in N\}$ of open subsets is locally finite in G/H. Evidently, each element of this family meets $\pi(X)$, hence $\pi(X)$ is not bounded in G/H. ∎

2.21. Lemma. (See also [1]). *For any admissible subgroup H of a topological group G there exists a continuous one-to-one mapping of the quotient space G/H onto a metrizable space.*

Proof. There exists a sequence $\{U_n: n \in N\}$ of open neighborhoods of identity such that $U_n^{-1} = U_n$, $U_{n+1}^3 \subseteq U_n$ for each $n \in N$ and $H = \bigcap\{U_n: n \in N\}$. We define

$$\mathscr{V}_n = \{(x, y) \in G \times G: x^{-1}y \in U_n\}, \quad n \in N .$$

From Theorem 8.1.10 of [12] it follows that there exists a continuous pseudometric d on G such that

$$\mathscr{V}_{n+1} \subseteq \{(x, y) \in G \times G: d(x, y) \leqq 2^{-n-1}\} \subseteq \mathscr{V}_n \quad \text{for each} \quad n \in N .$$

Obviously $d(x, y) = 0$ iff $x^{-1}y \in H$. Thus there exists a metric d^* on G/H with $d^*(\pi(x), \pi(y)) = d(x, y)$ for any $x, y \in G$, where $\pi: G \to G/H$. A continuity of the metric d^* on the quotient space G/H follows from the fact that π is an open mapping. ∎

2.22. Lemma. *Suppose that there exists a continuous one-to-one mapping of X onto a metrizable space and B is a bounded subset of X. Then $\mathrm{cl}_X B$ is compact.*

Proof. Exercise 8.5.13 (g) of [12] implies that X is a Dieudonné-complete space and so we have done it. ∎

The following lemma is a consequence of Lemma 1.3 from [13].

2.23. Lemma. *Let N be a closed subgroup of a topological group G, $\hat{G}$ the completion of G and $\hat{N} = \mathrm{cl}_{\hat{G}} N$. Then G/N is embeddable naturally into $\hat{G}/\hat{N}$.* ∎

2.24. Lemma. *Let K be a bounded subset of a topological group G with the identity e and $\{V_n: n \in N\}$ a sequence of open subsets of G such that $e \in V_n = V_n^{-1}$ and $V_{n+1}^2 \subseteq V_n$ for each $n \in N$. Then the set KP is dense in $F = \bigcap\{KV_n: n \in N\}$, where $P = \bigcap\{V_n: n \in N\}$.*

Proof. Assume the contrary and pick a point $x \in F \setminus \mathrm{cl}_G(KP)$. Then $(xV_n) \cap K \neq \emptyset$ for each $n \in N$ and there exists an open symmetric subset $U \ni e$ of G such that

$U^2x \cap KP = \emptyset$ (hence $UxP \cap UK = \emptyset$). Now define $\gamma^* = \{xV_n\colon n \in N\}$ and $\gamma = \{xV_n \setminus \mathrm{cl}_G(UxP)\colon n \in N\}$. Then γ is an infinite locally finite family consisting of open non-void subsets of G. Indeed, each element of γ is contained in a single element of γ^* and all limit points of the family γ^* are in xP. It remains to note that $W \cap K \neq \emptyset$ for any $W \in \gamma$. It contradicts the boundness of K in G. ∎

2.25. Lemma. *Let K be a bounded subset of a topological group G with the identity e. Then for each open set $U \ni e$ there exists an open set $V \ni e$ such that $VK \subseteq KU$.*

Proof. Clearly the set K is totally bounded in G, i.e. for any open subset $U \ni e$ there exists a finite subset $M \subseteq G$ with $UM \supseteq K$. One can prove this assertion in a fashion nearly identical to that of Theorem 1.1 in [7]. Now it is enough to note that each totally bounded subset of G has the required property. ∎

Proof of Theorem 2.18. We assume that $p^{-1}(X)$ is not bounded in G. Then Lemma 2.20 implies that there exist an admissible subgroup $P = \bigcap\{V_n\colon n \in N\}$ of G and a locally finite family $\{U_n\colon n \in N\}$ consisting of open subsets of G/P such that $\pi^{-1}(U_n) \cap Y \neq \emptyset$ for each $n \in N$, where $Y = p^{-1}(X)$ and $\pi\colon G \to G/P$. Using Lemma 2.25 one can assume that open sets V_n are chosen so that

$$KV_{n+1} \subseteq V_nK \quad \text{and} \quad V_{n+1}K \subseteq KV_n$$

for each $n \in N$. Clearly, $F = \bigcap\{KV_n\colon n \in N\}$ is a closed subgroup of G and $K \subseteq F$. Therefore there exist continuous mappings $q\colon G/P \to G/F$ and $w\colon G/K \to G/F$ such that $w \circ p = q \circ \pi =^{\mathrm{def}} \lambda$. The quotient mappings p, π and $\lambda\colon G \to G/F$ are open, hence the mappings q, w are open, too. From Lemma 2.24 it follows that KP is dense in F. Consequently $q^{-1}(\bar{e}) = \mathrm{cl}_{G/P}\, \pi(K)$, where $\bar{e} = \lambda(e)$ and e is the identity of G. Indeed, $\pi^{-1}\, q^{-1}(\bar{e}) = \lambda^{-1}(\bar{e}) = F$, and using the fact that π is an open mapping, we conclude that

$$\pi^{-1}(\mathrm{cl}_{G/P}\, \pi(K) = \mathrm{cl}_G \pi^{-1}\, \pi(K) = \mathrm{cl}_G(KP) = F\,.$$

Thus the equality $q^{-1}(\bar{e}) = \mathrm{cl}_{G/P}\, \pi(K)$ is proved.

Now Lemmas 2.21 and 2.22 imply that the set $B = \mathrm{cl}_{G/P}\, \pi(K)$ is compact. We claim that q is a perfect mapping. Obviously, all fibers of q are homeomorphic to the compact set $B = q^{-1}(\bar{e})$. To show that q is a closed mapping, we choose an arbitrary open subset O of G/P with $B \subseteq O$. Using Lemma 2.23 one can identify G/P with the corresponding subspace of $\hat{G}/\hat{P}$. Let $\tilde{O}$ be an open subset of $\hat{G}/\hat{P}$ with $\tilde{O} \cap G/P = O$ and $\hat{\pi}\colon \hat{G} \to \hat{G}/\hat{P}$. Then $\hat{K}$ is a compact subgroup of $\hat{G}$ and $\hat{\pi}(\hat{K}) = \mathrm{cl}_{\hat{G}/\hat{P}}\, \pi(K) = B \subseteq \tilde{O}$, hence $\hat{\pi}^{-1}(\tilde{O})$ is an open neighborhood of the compact set $\hat{K}$ in $\hat{G}$. Consequently there exists an open subset $\tilde{V} \ni e$ of $\hat{G}$ such that $\tilde{V}\hat{K} \subseteq \hat{\pi}^{-1}(\tilde{O})$. For the open subset $V = \tilde{V} \cap G$ we have $\pi(VK) \subseteq \hat{\pi}(\tilde{V}\hat{K}) \cap G/P \subseteq \tilde{O} \cap G/P = O$. It is easily seen that the point $\bar{e}$ belongs to the open subset $U = \lambda(V)$ of G/F and $q^{-1}(U) \subseteq O$. Indeed, $\pi^{-1}q^{-1}(U) = \lambda^{-1}(U) = \lambda^{-1}\, \lambda(V) = VF$ and $VF = VKP$, for KP is dense in F (Lemma 2.24). Further, $\pi(VK) \subseteq O$, hence $\pi^{-1}\pi(VK) \subseteq \pi^{-1}(O)$ and $VKP \subseteq \pi^{-1}(O)$. Thus we have $\pi^{-1}\, q^{-1}(U) \subseteq \pi^{-1}(O)$ whence it follows $q^{-1}(U) \subseteq O$.

So, the mapping q is closed at the point $\bar{e} \in G/F$. Analogous arguments show that q is closed at each point of G/F. Thus q is perfect.

The family $\{U_n: n \in N\}$ of open subsets of G/P is locally finite whence it follows that the family $\{q(U_n): n \in N\}$ is locally finite in G/F, and in turn $\{w^{-1}\, q(U_n): n \in N\}$ is a locally finite family consisting of open subsets of G/K. Further, we claim that $(*)$ $w^{-1}\, q(U) = p\, \pi^{-1}(U)$ for each open subset $U \subseteq G/P$. Indeed, the equality $(*)$ is equivalent to $p^{-1}w^{-1}\, q(U) = p^{-1}p\, \pi^{-1}(U)$, or to $WF = WK$, where $W = \pi^{-1}(U)$. However KP is dense in F (Lemma 2.24), hence $(KP)^{-1} = PK$ is dense in $F^{-1} = F$. Consequently $WF = WPK = WK$, for W is open in G and $W = \pi^{-1}\, \pi(W) = WP$, and $(*)$ is proved.

From the choice of the sets U_n it follows that $\pi^{-1}(U_n) \cap p^{-1}(X) \neq \emptyset$ for each $n \in N$. Therefore the equality $(*)$ implies that the open set $p\pi^{-1}(U_n) = w^{-1}\, q(U_n)$ meets X for each $n \in N$. It contradicts to the boundness of X in G/K. ■

2.26. Corollary. *The class of pseudocompact groups is closed under the operation of extension. Moreover, if K is a closed pseudocompact subgroup of a group G and the coset space G/K is pseudocompact, then G has the same property.* ■

It is well-known that any real-valued continuous function on a topological group is uniformly continuous on a compact subset of this group. In the Comfort and Ross's paper [7] it is shown that any real-valued continuous function on a pseudocompact group is uniformly continuous. The following theorem generalizes these results via bounded subsets of topological groups.

2.27. Theorem. *Let d be any continuous pseudometric on a topological group G and X a bounded subset of G. Then d is uniformly continuous on X.*

To prove Theorem 2.27 we need one lemma.

2.28. Lemma. *Let $H = \bigcap\{V_n: n \in N\}$ be an admissible subgroup of a topological group G and Y a bounded subset of the coset space G/H. Then for any neighborhood $\mathscr{V}$ of the diagonal Δ in $G/H \times G/H$ there exists $n \in N$ such that $Y^2 \cap \pi^2(U_{V_n}) \subseteq \mathscr{V}$, where $\pi: G \to G/H$, π^2 is a square of π and $U_V = \{(x, y) \in G \times G: x^{-1}y \in V\}$ for each $V \subseteq G$ with $e \in V$.*

Proof. Lemmas 2.21 and 2.22 imply jointly that the set $B = \mathrm{cl}_{G/H} Y$ is compact. We claim that the family $\gamma = \{B^2 \cap \pi^2(U_{V_n}): n \in N\}$ is a base of the diagonal Δ_B in B^2 (the sets V_n are chosen in accordance with the definition of an admissible subgroup). Indeed, B is a compact space, hence it is sufficient to show that $\Delta_B = \bigcap\gamma$ and the closure of the set $B^2 \cap \pi^2(U_{V_{n+1}})$ is contained in $B^2 \cap \pi^2(U_{V_n})$ for each $n \in N$. However the first fact is obvious and the second one requires standard arguments (see [21]). ■

Proof of Theorem 2.27. Assume that the pseudometric d is not (left) uniformly continuous. Then we can find $\varepsilon > 0$ such that for any open neighborhood U of the identity in G there exist elements $x, y \in X$ with $x^{-1}y \in U$ and $d(x, y) \geqq \varepsilon$. By induction one can easily define a sequence $\{V_n: n \in N\}$ of open neighborhoods of the identity in G and a sequence $\{(x_n, y_n): n \in N\} \subseteq X^2$ satisfying the conditions

(i) $V_{n+1}^3 \subseteq V_n$ and $V_n^{-1} = V_n$;

(ii) $d(x_n, y_n) \geqq \varepsilon$;

(iii) if $x_n^{-1}z \in V_n$ then $d(z, x_n) < \varepsilon/4$,
if $y_n^{-1}z \in V_n$ then $d(z, y_n) < \varepsilon/4$;

(iv) $x_{n+1}^{-1}y_{n+1} \in V_n$,

where $n \in N$. Then $H = \bigcap\{V_n: n \in N\}$ is an admissible subgroup of G. Let π be a quotient mapping of G onto left coset space G/H, $U = \{(x, y) \in G \times G: d(x, y) < \varepsilon/2\}$ and $\mathscr{V} = \pi^2(U)$ an open neighborhood of the diagonal Δ in $G/H \times G/H$. Then

(*) $(Y^2 \cap \pi^2(U_{V_n})) \setminus \mathscr{V} \neq \emptyset$ for each $n \in N$, where $Y = \pi(X)$.

Indeed, the condition (iv) implies $(\pi(x_{n+1}), \pi(y_{n+1})) \in Y^2 \cap \pi^2(U_{V_n})$ and we claim that $(\pi(x_{n+1}), \pi(y_{n+1})) \notin \mathscr{V}$. For if $(\pi(x_{n+1}), \pi(y_{n+1})) \in \mathscr{V}$ then there exists a pair $(x, y) \in U$ such that $\pi(x) = \pi(x_{n+1})$ and $\pi(y) = \pi(y_{n+1})$, i.e. $x_{n+1}^{-1}x \in H \subseteq V_{n+1}$ and $y_{n+1}^{-1}y \in H \subseteq V_{n+1}$.

Thus we have

$$d(x, y) < \varepsilon/2\,, \quad d(x_{n+1}, x) < \varepsilon/4 \quad \text{and} \quad d(y_{n+1}, y) < \varepsilon/4\,,$$

whence it follows that $d(x_{n+1}, y_{n+1}) < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon$. The last inequality contradicts the condition (ii) and the property (*) is proved. However, (*) contradicts Lemma 2.28 that completes the proof. ∎

2.29 Corollary. *Let X be a bounded subset of a topological group G. Then any continuous real-valued function f defined on G is uniformly continuous on X.*

Proof. Apply Theorem 2.27 to a continuous pseudometric d on G defined by the rule $d(x, y) = |f(x) - f(y)|$ for each $x, y \in G$. ∎

2.30. Corollary. *Let $\mathscr{U}$, $^*\mathscr{U}^*$, $^*\mathscr{U}$ and $\mathscr{U}^*$ be universal, two-sided, left and right uniformities, resp., of a topological group G. Then $\mathscr{U}|_X = {}^*\mathscr{U}^*|_X = {}^*\mathscr{U}|_X = \mathscr{U}^*|_X$ for any bounded subset X of G.* ∎

3. Inverse spectra and pseudocompact groups. Being given a countable inverse spectrum $\mathscr{X} = \{G_n, p_{n,m}\}_{m,n \in N}$ with pseudocompact groups G_n and continuous epimorphisms $p_{n,m}$, we consider the question whether the limit group $\varprojlim \mathscr{X}$ is pseudocompact. We begin with the following lemma, which gives an additional information on homomorphisms of pseudocompact groups.

3.1. Lemma. *Let K be a closed pseudocompact subgroup of a group G having the Souslin property. Then the natural quotient mapping $p: G \to G/K$ is z-closed, i.e. p transforms zero-sets to zero-sets.*

Proof. Let Φ be a zero-set in G, i.e. $\Phi = f^{-1}(0)$ for some continuous real-valued function f on G. One can assume that $f \geqq 0$ (replace f by $|f|$). Theorem 6 of [26] implies that there exists an open homomorphism $\pi: G \to G_1$ and a continuous function g on G_1 such that G_1 is of countable pseudocharacter and $f = g \circ \pi$. Let $N =$

$= \ker \pi$ and $H = KN$ $(=NK)$. Now we claim that H is a closed subgroup of G. Indeed, the group G_1 is of countable pseudocharacter, whence it follows that N is an admissible normal subgroup of G (see Definition 2.19). Consequently Lemma 2.21 implies that there exists a continuous one-to-one mapping of the quotient space $G/N \cong G_1$ onto a metrizable space. So, each subspace of G_1 is Dieudonné-complete (see Exercise 8.5.13 (g) from [12]), whence it follows that the pseudocompact subspace $\pi(K) \subseteq G_1$ is compact and closed in G_1. Clearly, $KN = \pi^{-1}\,\pi(K)$, hence KN is a closed subgroup of G.

Let λ be a quotient mapping of G onto $L = G/H$. Then there exist open mappings $q\colon G_1 \to L$ and $w\colon G/K \to L$ such that $q \circ \pi = w \circ p = \lambda$. It is easily seen that the following condition is satisfied:

(*) $p\,\pi^{-1}(x) = w^{-1}\,q(x)$ for each point $x \in G_1$.

Let e be the identity of G and $\bar{e} = \lambda(e)$. Obviously, $\pi(K) = q^{-1}(\bar{e})$ is a compact subgroup of G_1, hence $q^{-1}\,q(P) = P\,\pi(K)$ is closed in G_1 for any closed subset $P \subseteq G_1$. As q is a quotient mapping, so the last equality implies that q is closed. Thus q is a perfect and open mapping.

Now we define $F = g^{-1}(0)$. Then the equality $f = g \circ \pi$ implies that $\Phi = \pi^{-1}(F)$. Using the properties of the mapping q one can easily varify that $q(F)$ is a zero-set in L (define a continuous function h on L by $h(y) = \inf\{g(x)\colon x \in q^{-1}(y)\}$ for each $y \in L$). An application of the property $(*)$ gives us the equality $p(\Phi) = p\,\pi^{-1}(F) = w^{-1}\,q(F)$ whence it follows that $p(\Phi)$ is a zero-set in G/K. ∎

3.2. Remark. It seems to be interesting to find out whether the Souslin property of the group G is necessary in Lemma 3.1.

3.3. Corollary. *Let K be a closed pseudocompact subgroup of a group G with $c(G) \leqq \aleph_0$, and $p\colon G \to G/K$ a quotient mapping. Then $p(\bigcap_{n=0}^{\infty} \mathrm{cl}_G V_n) = \bigcap_{n=0}^{\infty} \mathrm{cl}_{G/K}\, p(V_n) = \bigcap_{n=0}^{\infty} p(\mathrm{cl}_G V_n)$ for any decreasing sequence $\{V_n\colon n \in N\}$ of open subsets in G.*

Proof. As the group G has the Souslin property, so for any open subset $V \subseteq G$ there exists a closed normal subgroup $N \subseteq G$ of type G_δ in G such that $\mathrm{cl}_G V = \pi_N^{-1}\pi_N(\mathrm{cl}_G V)$, where π_N is a quotient mapping of G onto G/N (see f.e. § 2 of [26]). Consequently there exists a closed normal subgroup $\Pi \subseteq G$ of type G_δ in G such that $\mathrm{cl}_G V_n = \pi_\Pi^{-1}\pi_\Pi(\mathrm{cl}_G V_n)$ for each $n \in N$. Then we define a closed subgroup $H = K\Pi \subseteq G$, the mappings $q\colon G/\Pi \to G/H$, $w\colon G/K \to G/H$ and complete the proof as in Lemma 3.1. ∎

The following theorem is the main result of this section.

3.4. *Let $\mathscr{X} = \{G_n, p_{n,m}\}_{m,n \in N}$ be an inverse spectrum consisting of topological groups G_n and open epimorphisms $p_{n,m}$, where each* $\ker p_{n+1,n}$ *be pseudocompact.*

If the group G_0 is pseudocompact then the limit group $G = \varprojlim \mathscr{X}$ is pseudocompact, too.

Proof. Assume that G_0 is a pseudocompact group. By induction on n (with an aid of Corollary 2.26) one can easily show that G_n and $\ker p_{n,m}$ are pseudocompact groups for each $m, n \in N$. The completion of any pseudocompact group is a compact group, hence pseudocompact groups have the Souslin property. Let $p_n\colon G \to G_n$ be a limit homomorphism, $n \in N$, $\{V_m\colon m \in N\}$ a decreasing sequence of open non-empty subsets in G and $\Phi_n = \bigcap\{\operatorname{cl} p_n(V_m)\colon m \in N\}$. The homomorphisms $p_{n,m}$ are z-closed (Lemma 3.1) and open, hence Corollary 3.3 implies that $p_{n,m}(\Phi_n) = \Phi_m$ for each $m, n \in N$ with $m < n$. Consequently there exists an element $x \in G$ such that $p_n(x) \in \Phi_n$ for each $n \in N$. Now we fix an integer $m \in N$. From the definition of the sets Φ_n it follows that $p_n(x) \in \operatorname{cl} p_n(V_m)$ for each $n \in N$, i.e. $x \in \operatorname{cl} V_m$. Thus we have shown that $x \in \bigcap\{\operatorname{cl}_G V_m\colon m \in N\}$, i.e. an intersection of any decreasing sequence of non-empty regular closed subsets in G is not empty. So the limit group $G = \varprojlim \mathscr{X}$ is pseudocompact. ■

Examples 4.2 and 4.3 below show that the conditions of Theorem 3.4 on epimorphisms $p_{n,m}$ can not be weakened.

It seems that an inverse spectrum ordered by N is a very partial case of an inverse well-ordered spectrum. However the case of uncountable well-ordered spectrum with pseudocompact spaces is much more clear. We omit the proof of the following obvious result.

3.5. Assertion. *Let $\mathscr{X} = \{X_\alpha, \pi_{\beta,\alpha}\}_{\alpha,\beta<\tau}$ be a well-ordered inverse spectrum consisting of pseudocompact spaces X_α and continuous open mappings $\pi_{\beta,\alpha}$. If $cf(\tau) > \aleph_0$ and all limit projections $\pi_\alpha\colon \lim \mathscr{X} \to X_\alpha$ are mappings "onto", then the limit space $X = \varprojlim \mathscr{X}$ is pseudocompact, too.* ■

In the previous assertion one can replace the condition on mappings $\pi_{\beta,\alpha}$ by the weaker condition "the mappings $\pi_{\beta,\alpha}$ are d-open" (see the comment after Definition 2.6). Clearly, if $p\colon G \to H$ is a continuous epimorphism of totally bounded topological groups, then p is a d-open mapping. Indeed, let $\hat{p}\colon \hat{G} \to \hat{H}$ be an extension of p to a continuous homomorphism of completions $\hat{G}$ and $\hat{H}$. Any continuous epimorphism of compact groups is open, whence it follows that the restriction $p = \hat{p}|_G$ of open mapping $\hat{p}$ to the dense subspace $G \subseteq \hat{G}$ is d-open [22, Lemma 7]. Thus we have proved the following theorem.

3.6. Theorem. *Let $\mathscr{X} = \{G_\alpha, p_{\beta,\alpha}\}_{\alpha,\beta<\tau}$ be an inverse well-ordered spectrum consisting of pseudocompact groups G_α and continuous homomorphisms $p_{\beta,\alpha}$. If $cf(\tau) > \aleph_0$ and all limit projections $p_\alpha\colon \varprojlim \mathscr{X} \to G_\alpha$ are epimorphisms, then the limit group $G = \varprojlim \mathscr{X}$ is also pseudocompact.* ■

The following two results seem to be surprising.

3.7. Assertion. *Let f be a continuous real-valued function defined on a subgroup G*

of a product $\Pi = \prod_{\alpha \in A} K_\alpha$ with compact groups K_α. Then there exists a countable subset $B \subseteq A$ and continuous function g defined on $p_B(G)$ such that $f = g \circ p_B|_G$, where $p_B\colon \Pi \to \prod_{\alpha \in B} K_\alpha$.

Proof. The group $G^* = \mathrm{cl}_\Pi G$ is compact, hence the homomorphism $p_B|_{G^*}$ is open for each $B \subseteq A$. Thus the family $\{p_B|_G\colon B \subseteq A, |B| \leqq \aleph_0\}$ is $\aleph_0$-complete and consists of d-open homomorphisms. Any compact group embeds into some product of compact metrizable groups (see f.e. Theorem 1 of [2]), hence we can assume that all factors K_α are metrizable. Now Theorem 1 of [24] implies that f depends on at most countably many coordinates. ■

3.8. Theorem. *Let f be a continuous real-valued function on a totally bounded group G. Then there exists a continuous homomorphism π of G onto separable metrizable group H and a continuous function h on H such that $f = h \circ \pi$.*

Proof. The completion $\hat{G}$ of the group G is a compact group, hence $\hat{G}$ embeds into some product of compact metrizable groups. An application of Assertion 3.7 completes the proof. ■

A spectral form of Assertion 3.7 is obvious now.

3.9. Remark. The following result generalizes Assertion 3.7:

Let a topological group G_α be generated by its Lindelöf Σ-space for each $\alpha \in A$ and S a dense subspace of the product $\Pi = \prod_{\alpha \in A} G_\alpha$. Then any continuous function $f\colon S \to \boldsymbol{R}$ depends on at most countably many coordinates. In addition, there exists a continuous homomorphism $\pi\colon \Pi \to H$ onto separable metrizable group H and a continuous function $g\colon \pi(S) \to \boldsymbol{R}$ such that $f = g \circ \pi|_S$.

We omit the proof of this theorem which based on some Uspenskiĭ's ideas [28] and Gleason's factorization theorem.

4. Some examples. Examples 4.2 and 4.3 below show that all conditions of Theorem 3.4 are necessary. Moreover, we will see that the limit group of countable inverse sequence of pseudocompact groups may even have a countable pseudocharacter. Here the sketches of the corresponding constructions are given. We need some notations. Let B be a set with $|B| = \boldsymbol{c}$, $H = \Pi\{T_\alpha\colon \alpha \in B\}$, where $T_\alpha = T$ the circle group for each $\alpha \in B$, e the identity of T and $\bar{e}$ the identity of H. For any point $x \in H$, let $k(x)$ denote the set $\{\alpha \in B\colon \pi_\alpha(x) \neq e\}$ and $\pi_\alpha\colon H \to T_\alpha$ the natural projection. The symbol $\Sigma = \Sigma(B, \bar{e})$ is used to denote the set $\{x \in H\colon |k(x)| \leqq \aleph_0\}$, the Σ-product of spaces T_α, $\alpha \in B$. For each $X \subseteq H$, $\mathrm{gp}(X)$ is the subgroup of H generated by X. Finally, the projection of H onto $H_M = \Pi\{T_\alpha\colon \alpha \in M\}$ is denoted by π_M, where $M \subseteq B$.

4.1. Lemma. *There exists a sequence $\{Y_n\colon n \in N\}$ of disjoint subsets in Σ satisfying the conditions*

(1) *$\pi_M(Y_n) = H_M$ for each $n \in N$ and each countable $M \subseteq B$;*

(2) *the set* $Y = \bigcup_{n=0}^{\infty} Y_n$ *generates the group* $G = \mathrm{gp}(Y)$ *isomorphic to the free group on* Y.

Proof. Let $\Sigma = \{x_\alpha: \alpha < \boldsymbol{c}\}$. For each $\alpha < \boldsymbol{c}$ one can choose a sequence $A_\alpha = \{b_{\alpha,n}: n \in N\} \subseteq B$ with $b_{\alpha,n} \neq b_{\alpha,m}$ whenever $n \neq m$, such that $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$ and $A_\alpha \cap \bigcup\{k(x_\beta): \beta \leqq \alpha\} = \emptyset$ (it is possible because $|B| = \boldsymbol{c}$). Pick an element $t^* \in T$ of infinite order. For each $\alpha < \boldsymbol{c}$ and $n \in N$ let $y_{\alpha,n}$ be a point of H such that $\pi_\beta(y_{\alpha,n}) = \pi_\beta(x_\alpha)$ if $\beta \in B \setminus \{b_{\alpha,n}\}$, and $\pi_\beta(y_{\alpha,n}) = t^*$ if $\beta = b_{\alpha,n}$. Now define $Y_n = \{y_{\alpha,n}: \alpha < \boldsymbol{c}\}$ for every $n \in N$. Clearly, $Y_n \subseteq \Sigma$. It is easy to check that the sequence $\{Y_n: n \in N\}$ satisfies the conditions (1) and (2). ■

4.2. Example. Let G be a free abelian group of cardinality $\boldsymbol{c}$. There exists a strictly increasing sequence $\{\mathscr{T}_n: n \in N\}$ of topological group topologies on G such that

(i) the group $(G, \mathscr{T}_n)$ is pseudocompact for each $n \in N$;

(ii) the group $G_\omega = (G, \mathscr{T})$ endowed with the topological group topology $\mathscr{T} = \sup_n \mathscr{T}_n$ is of countable pseudocharacter, hence it is not pseudocompact;

(iii) the group G_ω has the Frechét-Urysohn property;

(iv) the completion $\hat{G}_\omega$ of G_ω is topologically isomorphic to the compact group $T^{\boldsymbol{c}}$.

The sketch of the construction. Let B_0 be a set of cardinality $\boldsymbol{c}$, $H_0 = \Pi\{T_\alpha: \alpha \in B_0\}$, $\bar{e}$ the identity of H_0 and a sequence $\{Y_k(0): k \in N^+\}$ of disjoint subsets in $\Sigma_0 = \Sigma(B_0, \bar{e})$ be chosen satisfying the conditions (1) and (2) of Lemma 4.1, where $T_\alpha = T$ for each $\alpha \in B_0$. Assume that for some $n \in N$ there are defined a set $B_n \supseteq B_0$, a group $H_n = \Pi\{T_\alpha: \alpha \in B_n\}$ and a sequence $\{Y_k(n): k \in N^+\}$ of disjoint subsets in $\Sigma_n = \Sigma(B_n, \bar{e}_n)$ satisfying the following conditions:

(1) $\pi^n_M(Y_k(n)) = H_M$ for any $k > n$ and any countable subset $M \subseteq B$, where π^n_M is the projection of H_n onto H_M;

(2) $\pi^n_{\alpha(n)}(g) \neq e$ for any element $g \in F_n \setminus \{\bar{e}_n\}$, where $\alpha(n)$ is the fixed index of B_n and F_n is the subgroup of H_n generated by $Z_n = \bigcup\{Y_i(n): i \leqq n\}$;

(3) the projection p_n of H_n onto H_0 is a bijection of $Y_k(n)$ onto $Y_k(0)$ for each $k \in N^+$.

Now let A_{n+1} be a set of cardinality $\boldsymbol{c}$, $A_{n+1} \cap B_n = \emptyset$. For the set $B_{n+1} = B_n \cup A_{n+1}$ we define the group $H_{n+1} = \Pi\{T_\alpha: \alpha \in B_{n+1}\}$, where $T_\alpha = T$ for any $\alpha \in B_{n+1}$, and an index $\alpha_{n+1} \in A_{n+1}$. A routine recursive construction enables us to define a sequence $\{Y_k(n+1): k \in N^+\}$ of disjoint subsets in Σ_{n+1}, satisfying the conditions (1)–(3) at $(n+1)$-th step. We omit the construction which involves a few enumerations of points of Σ_{n+1}.

Now let $B = \bigcup\{B_n: n \in N\}$, $H = \Pi\{T_\alpha: \alpha \in B\}$, pr_n the projection of H onto H_n and p^{n+1}_n the projection of H_{n+1} onto H_n, $n \in N$. The condition (3) of the construction implies that p^{n+1}_n is a one-to-one mapping of X_{n+1} onto X_n, where $X_k = \bigcup\{Y_m(k): m \in N^+\}$ for each $k \in N$. Consider the set $X = \{x \in H: \mathrm{pr}_n(x) \in X_n \text{ for each } n \in N\}$. It is clear that pr_n is a one-to-ine mapping of X onto X_n. Let $G_\omega = \mathrm{gp}_H(X)$ and $G_n = \mathrm{gp}_{H_n}(X_n)$, $n \in N$. For each $n \in N$ the restriction $\pi_n = \mathrm{pr}_n|_{G\omega}$ is a one-to-one continuous homomorphism of G_ω onto G_n. As G_0 is isomorphic to a free abelian group, so the same is true for G_n $(n \in N)$ and G_ω. The condition (1) implies that $\pi^n_M(G_n) \supseteq \pi^n_M(X_n) = H_M$ for each countable subset $M \subseteq B_n$, hence G_n is a dense pseudocompact subgroup of H_n (see Lemma of [23]). Consequently G_ω is a dense subgroup of H and $\hat{G}_\omega = T^c$. Let $\mathscr{T}_n$ be the topology of the group G_n. Then the topology $\mathscr{T}$ of the group G_ω is the upper bound of the pseudocompact topological group topologies $\pi_n^{-1}(\mathscr{T}_n)$, $n \in N$, for $B_0 \subseteq B_1 \subseteq \ldots$ and $B = \bigcup\{B_n : n \in N\}$. As $X_n \subseteq \Sigma_n$ for each $n \in N$, so $X \subseteq \Sigma = \Sigma(B, \bar{e})$, where $\bar{e}$ is the identity of H. Consequently $G_\omega \subseteq \Sigma$. All factors T_α, $\alpha \in B$, are metrizable hence the spaces Σ and G_ω have the Frechét-Urysohn property (see [18]). It remains to varify that $\psi(G_\omega) \leqq \aleph_0$. Consider the set $\mathscr{D} = \{\alpha_n : n \in N\}$. Then the condition (2) implies that $\pi_D(g) \neq \bar{e}_D$ for each $g \in G_\omega \setminus \{\bar{e}\}$, where $\pi_D \colon H \to H_D$ and $\bar{e}_D$ is the identity of H_D. Consequently $\pi_D^{-1}(\bar{e}_D) \cap G_\omega = \{\bar{e}\}$ whence it follows that $\psi(G_\omega) \leqq \aleph_0$. In turn it implies that the group G_ω is not pseudocompact. Indeed, a pseudocompact group of countable pseudocharacter has a countable character, hence it is metrizable (Lemma 3.1 of [9]). However there are no dense metrizable subspaces of T^c. ∎

The following example shows that the limit group of a countable spectrum with pseudocompact groups and open epimorphisms need not be pseudocompact.

4.3. Example. There exists a countable inverse spectrum $\mathscr{X} = \{G_n, p_{n,m}\}_{m,n \in N}$ with pseudocompact abelian groups G_n and continuous open epimorphisms $p_{n,m}$ such that

(i) the limit group $G = \varprojlim \mathscr{X}$ is not pseudocompact and $\psi(G) \leqq \aleph_0$;

(ii) the group G has the Frechét-Urysohn property;

(iii) the completion $\hat{G}$ of G is topologically isomorphic to T^c.

Construction. Let B_0, H_0 and a sequence $\{Y_k(0) : k \in N\}$ be chosen as in the previous construction. For each $n \in N$ one can define a set B_n of cardinality $\boldsymbol{c}$, the group $H_n = \Pi\{T_\alpha : \alpha \in B_n\}$ and a sequence $\{Y_k(n) : k \in N\}$ of disjoint subsets of $\Sigma_n = \Sigma(B_n, \bar{e}_n)$, where $\bar{e}_n$ is the identity of H_n, satisfying the following conditions:

(1) $B_n \subseteq B_{n+1}$;

(2) $\pi_{n,m}(Y_k(n)) = Y_k(m)$ for each $k \in N$ and $m \leqq n$, where $\pi_{n,m}$ is the projection of H_n onto H_m;

(3) the group $G_n \subseteq H_n$ generated by the set $X_n = \bigcup\{Y_k(n) : k \in N\}$ is isomorphic to a free abelian group on X_n;

(4) $\pi^n_M(Y_k(n)) = \Pi\{T_\alpha : \alpha \in M\} =^{\text{def}} H_M$ for each $k > n$ and each countable subset $M \subseteq B_n$, where π^n_M is the projection of H_n onto H_M;

(5) there exists a countable subset $C(n) \subseteq B_n \setminus B_{n-1}$ such that $\pi_{C(n)}(g) \neq \bar{e}_{C(n)}$ for any $g \in G^*_n \setminus \{\bar{e}_n\}$, where $\bar{e}_{C(n)}$ is the identity of the group $H_{C(n)}$ and G^*_n is the subgroup of H_n generated by the set $\bigcup\{Y_i(n) : i \leqq n\}$, $n > 0$;

(6) for any $g \in X_{n-1}$, finite set $P \subseteq A_n$ and $\bar{t} \in H_P$ there exists a point $h \in X_n$ with $\pi_{n,n-1}(h) = g$ and $\pi_P(h) = \bar{t}$, where $A_n = B_n \setminus B_{n-1}$ and $X_{n-1} = \bigcup\{Y_k(n-1) : k \in N\}$, $n > 0$.

The conditions (1)–(5) are similar to the corresponding conditions of the previous construction. The condition (6) is of the other type, it implies that the projection

$\pi_{n,n-1}: X_n \to X_{n-1}$ is open. In its turn it implies that the epimorphism $p_{n,n-1} = \pi_{n,n-1}|_{G_n}$ is open, where G_n is the subgroup of H_n generated by X_n (use the divisibility of the group T).

Now let all be defined at n-th step. We choose a set A_{n+1} with $A_{n+1} \cap B_n = \emptyset$, $|A_{n+1}| = \boldsymbol{c}$ and define $B_{n+1} = B_n \cup A_{n+1}$, $H_{n+1} = \Pi\{T_\alpha: \alpha \in B_{n+1}\}$, $\Sigma_{n+1} = \Sigma(B_{n+1}, \bar{e}_{n+1})$. Let also C_{n+1} be a subset of A_{n+1} with $|C_{n+1}| = \aleph_0$, $Z = \bigcup\{Y_k(n): k \leqq n + 1\}$ and $\sigma = \{x \in H_{A_{n+1}}: |k(x)| < \aleph_0\}$ an usual σ-product. Consider the enumeration $Z \times \sigma = \{y_\alpha: \alpha < \boldsymbol{c}\}$. Evidently there exists a subset $I = \{t_\alpha: \alpha < \boldsymbol{c}\}$ of the group T such that the subgroup of T generated by I is isomorphic to a free abelian group on I and $t_\alpha \neq t_\beta$ if $\alpha \neq \beta$. For each $\alpha < \boldsymbol{c}$ let $\bar{y}_\alpha$ be a point of H_{n+1} with $\pi_\beta(\bar{y}_\alpha) = t_\alpha$ if $\beta \in C_{n+1} \setminus k(y_\alpha)$, and $\pi_\beta(\bar{y}_\alpha) = \pi_\beta(y_\alpha)$ if $\beta \in B_{n+1} \setminus (C_{n+1} \setminus k(y_\alpha))$. We define $\bar{Z} = \{\bar{y}_\alpha: \alpha < \boldsymbol{c}\}$ and $\bar{Y}_k(n) = \{\bar{y}_\alpha: \alpha < \boldsymbol{c}, y_\alpha \in Y_k(n)\}$, where $k \leqq n$. It is clear that $\bar{Z} = \bigcup\{\bar{Y}_k(n): k \leqq n\}$.

Let $\{x_\alpha: \alpha < \boldsymbol{c}\}$ be an enumeration of the set $\bar{Z} \cup ((X_n \setminus Z) \times \Sigma')$, where $\Sigma' = \Sigma(A_{n+1}, \tilde{e})$ and $\tilde{e}$ is the identity of $H_{A_{n+1}}$. As the set $\bar{A} = A_{n+1} \setminus C_{n+1}$ is of cardinality $\boldsymbol{c}$, there exists a one-to-one mapping φ of $\boldsymbol{c}$ into $\bar{A}$ such that $\varphi(\alpha) \notin \bigcup\{k(x_\beta): \beta \leqq \alpha\}$ for each $\alpha < \boldsymbol{c}$. Pick an element t^* of the infinite order in T. For each $\alpha < \boldsymbol{c}$ we define the point $\bar{x}_\alpha \in H_{n+1}$ by $\pi_{\varphi(\alpha)}(\bar{x}_\alpha) = t^*$ and $\pi_M(\bar{x}_\alpha) = \pi_M(x_\alpha)$, where $M = B_{n+1} \setminus \{\varphi(\alpha)\}$. Finally, we define $Y_k(n + 1) = \{\bar{x}_\alpha: \alpha < \boldsymbol{c}$ and $x_\alpha \in Y_k(n)\}$ for each $k > n$, $Y_k(n + 1) = \{\bar{x}_\alpha: \alpha < \boldsymbol{c}$ and $x_\alpha \in \bar{Y}_k(n)\}$ for each $k \leqq n$, and $X_{n+1} = \bigcup\{Y_k(n + 1): k \in \boldsymbol{N}^+\}$. It is easy to see that $X_{n+1} \subseteq \Sigma_{n+1}$ and our construction at $(n + 1)$-th step is complete. We omit the proof of the fact that the conditions (1)–(6) are satisfied.

Let $B = \bigcup\{B_n: n \in \boldsymbol{N}\}$, $H = \Pi\{T_\alpha: \alpha \in B\}$ and $G = \{x \in H: \pi_n(x) \in G_n$ for each $n \in \boldsymbol{N}\}$, where $\pi_n: H \to H_n$. Then G is a dense subgroup of H and $p_n = \pi_n|_G$, $p_{n,m} = \pi_{n,m}|_{G_n}$ are open epimorphisms of G to G_n and G_n to G_m, resp. It is clear that there exists a topological isomorphism of the limit group of the spectrum $\mathscr{X} = \{G_n, p_{n,m}\}_{m,n \in \boldsymbol{N}}$ onto G. Epimorphisms $p_{n,m}$ are open and the groups G_n are pseudocompact (see the condition (2)). The inclusions $G_n \subseteq \Sigma_n$ imply that $G \subseteq \Sigma(B, \bar{e})$, where $\bar{e}$ is the identity in H. The condition (5) implies that $\pi_C^{-1}(\bar{e}_C) \cap G = \{\bar{e}\}$, where $\pi_C: H \to H_C$ and $\bar{e}_C$ is the identity of H_C. Consequently $\psi(G) \leqq \aleph_0$. The other properties of the group G may be proved so as in Example 4.2. ∎

Our last example makes clear the distance between pseudocompact and countably compact groups. We need a primary definition.

4.4. Definition. A subgroup H of a topological group G is said to be *h-embedded into* G provided that any homomorphism φ of H to an arbitrary compact group K is extendable to a continuous homomorphism $\tilde{\varphi}: G \to K$.

Note that if H is an h-embedded subgroup of G then any homomorphism of H to a compact group is continuous.

4.5. Example. There exists a pseudocompact abelian group G with $|G| = w(G) = \boldsymbol{c}$ any countable subgroup of which is closed and h-embedded into G. Moreover, the group G is zero-dimensional and contains no non-trivial convergent sequences.

4.6. Remark. The first example of infinite countably compact topological group without non-trivial convergent sequences was created by A. Hajnal and I. Juhász [14]. Their group is constructed under CH and is hereditary separable. The other countably compact group with no non-trivial convergent sequences was constructed by E. K. van Douwen ubder MA [11]. Our group G is of completely other type. An absence of convergent sequences in G is an easy corollary of the facts that each countable subgroup of G is closed in G and the group G is boolean, i.e. $x + x = 0_G$ for each $x \in G$ (check it).

4.7. Some words on construction. The boolean groups have a few important properties.

Fact 1. A boolean group is abelian and for each boolean groups H, K the equality $|H| = |K|$ implies that $H \cong K$.

Fact 2. Each homomorphism defined on a subgroup of boolean group H extends to a homomorphism of H, see [11].

Fact 3. Each compact boolean group H of weight $\tau \geqq \aleph_0$ is isomorphic continuously to the group $\boldsymbol{Z}(2)^\tau$ [15, Theorem 25.9].

The group G in Example 4.5 should be realized as a dense pseudocompact subgroup of $\boldsymbol{Z}(2)^{\boldsymbol{c}} \times \boldsymbol{Z}(2)^{\boldsymbol{c}}$. There are only $\boldsymbol{c}$ countable subgroups lying in $\Sigma = \Sigma(\boldsymbol{c}, \bar{0}) \subseteq \boldsymbol{Z}(2)^{\boldsymbol{c}}$, where $\bar{0}$ is the neutral element of $\boldsymbol{Z}(2)^{\boldsymbol{c}}$. For each countable subgroup $S \subseteq \Sigma$ there are at most $\boldsymbol{c}$ homomorphisms of S to $\boldsymbol{Z}(2)$. We enumerate the points of Σ, countable subgroups of Σ and homomorphisms of these groups to $\boldsymbol{Z}(2)$, say $\Sigma = \{x_\alpha : \alpha < \boldsymbol{c}\}$, $\mathscr{M} = \{S_\eta : \eta < \boldsymbol{c}\}$ and $\mathscr{H} = \{h_\xi : \xi < \boldsymbol{c}\}$. By induction on $\alpha < \boldsymbol{c}$ one can define an extension of each h_ξ to a homomorphisms $\hat{h}_\xi : \Sigma \to \boldsymbol{Z}(2)$ (Fact 2 is used here). At α-th step of construction it is necessary to define homomorphisms $h_{\alpha,\xi}$ on the corresponding subgroups $E_{\alpha,\xi} \subseteq \Sigma$, $\xi < \boldsymbol{c}$, satisfying the conditions

(1) $E_{\beta,\xi} \subseteq E_{\alpha,\xi}$ and $h_{\alpha,\xi}|_{E_{\beta,\xi}} = h_{\beta,\xi}$ whenever $\xi < \boldsymbol{c}$ and $\beta < \alpha$;

(2) $S_{\eta(\xi)} \subseteq E_{\alpha,\xi}$ and $|E_{\alpha,\xi}| \leqq |\alpha| \,.\, \aleph_0$ whenever $\alpha, \xi < \boldsymbol{c}$, where $S_{\eta(\xi)} = \operatorname{dom}(h_\xi)$.

Now let we have defined the homomorphisms $\hat{h}_\xi = \bigcup_{\alpha < \boldsymbol{c}} h_{\alpha,\xi}$, $\xi < \boldsymbol{c}$.

Let id be the identity mapping of Σ onto itself and p the diagonal product of homomorphisms id and $\hat{h}_\xi$, $\xi < \boldsymbol{c}$. Then $p : \Sigma \to \Sigma \times \boldsymbol{Z}(2)^{\boldsymbol{c}}$ and we put $G = p(\Sigma)$. With the help of enumeration $\Sigma = \{x_\alpha : \alpha < \boldsymbol{c}\}$ the extensions $\hat{h}_\xi$ may be chosen such that projections of G fill up all the countable subproducts of $\Sigma \times \boldsymbol{Z}(2)^{\boldsymbol{c}} \subseteq \boldsymbol{Z}(2)^{\boldsymbol{c}} \times \boldsymbol{Z}(2)^{\boldsymbol{c}}$. It implies that G is a dense pseudocompact subgroup of $H^* = \boldsymbol{Z}(2)^{\boldsymbol{c}} \times \boldsymbol{Z}(2)^{\boldsymbol{c}}$. Each homomorphism $\hat{h}_\xi$ is identified with the restriction of projection $\mathrm{pr}_\xi : H^* \to (\boldsymbol{Z}(2))_\xi$ to G, hence all countable subgroups of G are h-embedded into G (apply Fact 3). It remains to note that if all countable subgroups of a boolean group are h-embedded then these subgroups are closed. ■

4.8. Remark. It was noted earlier that there exists an infinite pseudocompact space X each countable subset of which is closed in X. However any pseudocompact group with this property is finite. Indeed, a pseudocompact group is totally bounded [7] and each subgroup of totally bounded group inherits the last property. Consequently any infinite subgroup of totally bounded (pseudocompact) group is not discrete.

It is easy to see that each countable subgroup of G inherits from G the finest totally bounded topological group topology. Therefore all infinite countable subgroups of G are topologically isomorphic.

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