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RECOGNITION OF CERTAIN CLASSES OF CLOSED MAPS

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1. Introduction. The origin of this paper was the search for classes of cell-like maps which do not raise dimension. In particular, we tried to improve the following result by C. F. Zemke [10, Theorem (5.1)].

(Z) Let X be a closed convex n-cell in the Euclidean n-space. Let G be a monotone upper semicontinuous decomposition of X into convex sets such that all non-degenerate members of G lie in the interior of X. Then the quotient space X/G has dimension at most n.

In the expository article [9] J. Walsh describes an improvement of (Z) in which non-degenerate elements do not need to lie in the interior of X and where the conclusion is strengthened to the claim that the projection map $p: X \to X/G$ is approximatively right invertible.

The class of approximatively right invertible maps has been introduced by G. Kozlowski [3]. It is easy to show that these maps can not increase dimension.

Another important class of maps also defined by G. Kozlowski are hereditary shape equivalences [6]. It is well-known that a cell-like map defined on a finite-dimensional compactum does not raise dimension iff it is a hereditary shape equivalence [6]. Hence, in the result by Walsh we can even claim that the projection map is a hereditary shape equivalence.

Our approach in improving Zemke's (and Walsh's) theorem was to replace the classical notion of convexity with some of the existing axiomatic notions of convexity which have been considered recently in topology. The two most prominent are due to R. Jamison [5] and E. Michael [7].

Hence, the best description of our results would be to say that they illustrate how far one can push the arguments in [10] and [9]. In the end, we got sufficient conditions for a closed map to be approximatively right invertible (in Section 2) and to be a hereditary shape equivalence (in Section 3). Finally, in Section 4, the maps with "convex" preimages provide examples to which previous results could be applied and which represent our extensions of (Z).

This paper was written during L. Rubin's visit to the University of Zagreb on leave from the University of Oklahoma.

2. Recognition of ARI maps. In this section we prove our first two theorems which give sufficient conditions for a closed map to be approximatively right invertible.

Definition 1. Let $\mathscr{D}$ be a class of pairs of topological spaces. We shall say that a topological space X is a *$\mathscr{D}$-extensor* provided that for every $(K, K_0) \in \mathscr{D}$ and every map $f_0: K_0 \to X$ there is an extension $f: K \to X$ of f_0.

Let $\mathscr{B} = \{(B^n, S^{n-1}) \mid n = 1, 2, 3, \ldots\}$, where B^n is the n-dimensional ball and S^{n-1} is its boundary.

Definition 2. A class $\mathscr{C}$ of topological spaces is *(finite) intersections stable* ((F) IS) provided it is closed with respect to arbitrary (finite) intersections of its members.

Definition 3. Let $\mathscr{C}$ be a class of subsets of a space X. A subset A of X is $U\mathscr{C}^\infty$ provided for every neighborhood U of A in X there is a $K \in \mathscr{C}$ such that $A \subset \operatorname{int} K \subset \subset K \subset U$.

Definition 4. A map $f: X' \to X$ is *approximatively right invertible* (ARI) provided that for every normal cover $\mathscr{U}$ of X [1] there is a map $g: X \to X'$ such that $f \circ g \sim^{\mathscr{U}} \mathrm{id}_X$ (i.e., $f \circ g$ and the identity map id_X are $\mathscr{U}$-close in the sense that for each $x \in X$ there is a $U \in \mathscr{U}$ containing both $f(g(x))$ and x).

Theorem 1. *Let $\mathscr{C}$ be an IS class of $\mathscr{B}$-extensor subsets of a topological space X'. Let $f: X' \to X$ be a closed map of X' onto a paracompact Hausdorff space X such that each preimage of a point in X is a $U\mathscr{C}^\infty$ subset of X'. Then f is ARI.*

Proof. Let $\mathscr{U}$ be a normal cover of X. For each $x \in X$, choose a $U_x \in \mathscr{U}$ and a $K_x \in \mathscr{C}$ such that $x \in U_x$ and $f^{-1}(x) \subset \operatorname{int} K_x \subset K_x \subset f^{-1}(U_x)$. Let $\mathscr{V} = \{\operatorname{int} K_x \mid x \in X\}$. For each $V \in \mathscr{V}$, put $W_V = X - f(X' - V)$. Let $\mathscr{W} = \{W_V \mid V \in \mathscr{V}\}$. Observe that $\mathscr{W}$ is an open cover of X. Since X is a paracompact Hausdorff space, $\mathscr{W}$ is a normal cover [1]. Let $\mathscr{S}$ be a locally finite star refinement of $\mathscr{W}$. Let N denote the geometric realization of the nerve of $\mathscr{S}$ and let $\phi: X \to N$ be the barycentric map.

We shall now define, by induction, a map $\psi: N \to X'$ in the following way. For each i, a map ψ_i from the i-skeleton N_i of N into X' will be constructed with the property that

1) $\psi_i = \psi_{i-1}$ on N_{i-1}, and

2) If Δ is a simplex in N_i, then $\psi_i(\Delta)$ is contained in the intersection of all members of $\{K_x \mid x \in X\}$ that contain ψ_0 (vertices of Δ).

Define $\psi: N \to X'$ by $\psi \mid N_i = \psi_i$ $(i \geqq 0)$. We shall show that the composition $g = \psi \circ \phi: X \to X'$ is a map satisfying $f \circ g \sim^{\mathscr{U}} \mathrm{id}_X$.

For each $S \in \mathscr{S}$, pick an $x'_S \in f^{-1}(S)$ and define $\psi_0(S) = x'_S$. Suppose that a continuous function ψ_i has been defined on N_i satisfying 1) and 2). If $x \in N_i$, define $\psi_{i+1}(x) = \psi_i(x)$. Suppose that Δ is an $(i+1)$-simplex of N_{i+1} and that $S_0, S_1, \ldots, S_{i+1}$ are vertices of Δ. Observe that $\bigcap_{j=0}^{i+1} S_j \neq \emptyset$. Hence, there is a member W of $\mathscr{W}$

such that $\bigcup_{j=0}^{i+1} S_j \subset W$. Suppose that $W = X - f(X' - V)$, where $V = \text{int}\, K_w$ for some $w \in X$. It follows that $\{x'_{S_0}, \ldots, s'_{S_{i+1}}\} \subset K_w$. Let K be the intersection of all members of $\{K_x \mid x \in X\}$ which contain the set $\{x'_{S_0}, \ldots, x'_{S_{i+1}}\}$. Observe that K is a $\mathscr{B}$-extensor. The boundary $\partial\Delta$ of Δ is a collection of i-simplices, each of which is mapped by ψ_i into K. Therefore, $\psi_i \mid \partial\Delta$ $(=\psi_{i+1} \mid \partial\Delta)$ has an extension ψ_{i+1} to all of Δ. Since N is locally finite, taking such extensions over all $(i+1)$-simplices of N we shall get the map $\psi_{i+1}: N_{i+1} \to X'$ which satisfies 1) and 2).

It remains to see that $f \circ g \sim^{\mathscr{U}} \text{id}_X$. Suppose $x \in X$ and that $S_1, \ldots, S_n$ are precisely those members of $\mathscr{S}$ which contain x. Then $\bigcap_{i=1}^{n} S_i \neq \emptyset$. Let Δ be the simplex of N spanned by $S_1, \ldots, S_n$. There exists a $W \in \mathscr{W}$ such that $\bigcup_{i=1}^{n} S_i \subset W$. Therefore, $\{x'_{S_1}, \ldots, x'_{S_n}\} \subset f^{-1}(W)$. Since ϕ is a barycentric map, we have $\phi(x) \in \Delta$. By construction, $\psi(\Delta) \subset f^{-1}(U_w)$, where $W = X - f(X' - \text{int}\, K_w)$. Hence, $\{x\} \cup \{f(g(x))\} \subset$ $\subset U_w$.

Theorem 2. *Let $\mathscr{C}$ be an FIS class of $\mathscr{B}$-extensor subsets of a topological space X'. Let $f: X' \to X$ be a closed map of X' onto a compact Hausdorff space X such that each preimage of a point in X is a $U\mathscr{C}^\infty$ subset of X'. Then f is ARI.*

Proof. The proof of Theorem 2 is almost identical to the proof of Theorem 1. However, we must make sure that $\mathscr{S}$ is a finite cover and for each $S \in \mathscr{S}$ choose a point $x_S \in S$ and work with the finite collection $\{K_{x_S} \mid S \in \mathscr{S}\}$ instead of the collection $\{K_x \mid x \in X\}$.

3. Recognition of hereditary shape equivalences. Here we prove two theorems which give sufficient conditions for a closed map to be a hereditary shape equivalence.

For a map $f: X' \to X$ and an $A \subset X$, let $A' = f^{-1}(A)$ and let f_A denote the restriction of f onto A' considered as a map of A' onto A.

For a map $f: X' \to X$ and a space Y, let $[X, Y]$ denote the set of all homotopy classes of maps of X into Y, and let a function $f^Y: [X, Y] \to [X', Y]$ be defined by $f^Y([\alpha]) = [\alpha \circ f]$, for $[\alpha] \in [X, Y]$.

Definition 5. A map $f: X' \to X$ is a *hereditary shape equivalence* (HSE) [6] provided for every closed subset A of X and every absolute neighborhood retract Y, the function f_A^Y is a bijection.

Definition 6. A class $\mathscr{C}$ of subsets of a topological space Y has the property *contiguity implies homotopy* (CIH) provided given a space X and maps $f, g: X \to Y$ such that for every $x \in X$ there is a $K_x \in \mathscr{C}$ with $f(x), g(x) \in K_x$, then f and g are homotopic in $\bigcup\{K_x \mid x \in X\}$.

Theorem 3. *Let $\mathscr{C}$ be an IS and CIH class of $\mathscr{B}$-extensor subsets of a perfectly normal paracompact space X'. Let $f: X' \to X$ be a closed map of X' onto a space X such that each preimage of a point in X is a $U\mathscr{C}^\infty$ subset of X'. Then f is a HSE.*

Proof. Let A be a closed subset of X and let Y be an ANR (an absolute neighborhood retract for the class of all metrizable spaces). We must show that the function $f_A^Y: [A, Y] \to [A', Y]$ is a bijection.

a) In order to see that f_A^Y is an injection, suppose that $[\alpha_1], [\alpha_2] \in [A, Y]$ and that $f_A^Y([\alpha_1]) = f_A^Y([\alpha_2])$. In other words, suppose that we are given maps $\alpha_1, \alpha_2: A \to Y$ and that $\alpha_1 \circ f_A \simeq \alpha_2 \circ f_A$. Our goal is to demonstrate that $\alpha_1 \simeq \alpha_2$.

Since both paracompactness and perfect normality are preserved under closed maps [2], the space X is perfectly normal and paracompact. Since every ANR is an absolute neighborhood extensor for the class of all perfectly normal paracompact spaces [4, p. 63], there is a closed neighborhood B of A in X and extensions α_1^* and α_2^* of α_1 and α_2 to all of B such that $\alpha_1^* \circ f_B \simeq \alpha_2^* \circ f_B$. Let $\mathscr{T}$ be an open cover of Y with the property that $\mathscr{T}$-close maps into Y are homotopic. Let $\mathscr{R}$ be a common refinement of $\alpha_1^{*^{-1}}(\mathscr{T})$ and $\alpha_2^{*^{-1}}(\mathscr{T})$. Choose an open cover $\mathscr{U}$ of X such that the restriction of $\mathscr{U}$ to B refines $\mathscr{R}$ and such that the star of A with respect to $\mathscr{U}$ is contained in B.

We now use Theorem 1 to construct a map $g: X \to X'$ such that $f \circ g \sim^{\mathscr{U}} \mathrm{id}_X$. Clearly, $g(A) \subset B'$ so that we have the following chain of homotopies $\alpha_1 \simeq \alpha_1^* \circ f_B \circ \circ (g \mid A) \simeq \alpha_2^* \circ f_B \circ (g \mid A) \simeq \alpha_2$. Hence, $\alpha_1 \simeq \alpha_2$.

b) In order to prove that f_A^Y is a surjection, suppose that $[\beta] \in [A', Y]$. We must find an $[\alpha] \in [A, Y]$ such that $f_A^Y([\alpha]) = [\alpha \circ f_A] = [\beta]$. In other words, given a map $\beta: A' \to Y$ we must construct a map $\alpha: A \to Y$ such that $\beta \simeq \alpha \circ f_A$.

Let $\beta^*: B' \to Y$ be an extension of β to a neighborhood B' of A' in X'. We now use Theorem 1 to construct a map $g: X \to X'$ but this time we make sure that $K_x \subset B'$ for each $x \in A$.

Let $\alpha = \beta^* \circ (g \mid A)$. Since the class $\mathscr{C}$ has the property *CIH*, $(g \mid A) \circ f_A$ is homotopic to the inclusion $i_{A',B'}$ of A' into B'. Hence, $\alpha \circ f_A = \beta^* \circ (g \mid A) \circ f_A \simeq \simeq \beta^* \circ i_{A',B'} = \beta$.

Theorem 4. *Let $\mathscr{C}$ be an FIS and CIH class of $\mathscr{B}$-extensor subsets of a compact Hausdorff space X'. Let $f: X' \to X$ be a closed map of X' onto a space X such that each preimage of a point in X is a $U\mathscr{C}^\infty$ subset of X'. Then f is a HSE.*

Proof. Analogous to the proof of Theorem 3.

4. Maps with convex preimages. This section presents two examples of situations where the assumptions of previous theorems are satisfied. These examples concern maps with convex preimages on spaces on which a notion of convexity has been defined.

We shall consider two notions of convexity. The first was introduced by E. Michael in [7] while the second by R. Jamison in [5]. In both of these approaches the collection of all convex subsets is IS, each convex set is a $\mathscr{B}$-extensor, and each convex set is the intersection of convex neighborhoods. The class of all convex subsets in Michael's convexity has the property CIH. However, we do not know under what conditions this is true for Jamison's convexity. Hence, we have the following two results. For undefined terms see [8] for Theorem 5 and [7] for Theorem 6.

Theorem 5. *Let X' be a metrizable S_4 convex structure with compact polytopes and with connected convex sets. Let $f: X' \to X$ be a closed map of X' onto a space X such that the preimage of each point in X is a convex subset of X'. Then f is ARI.*

Theorem 6. *Let X' be a metric space with a convex structure. Let $f: X' \to X$ be a closed map of X' onto a space X such that the preimage of each point in X is a convex subset of X'. Then f is a HSE.*

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