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ON COMPLETIONS OF PARTIAL MONOUNARY ALGEBRAS

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Partial monounary algebras were investigated by W. Bartol [1]–[3], J. Novotný [9], M. Novotný [10]–[11], O. Kopeček [12]–[16] and the author [6]–[8]. In the papers [1]–[3], [9]–[11], [13] and [15] partial monounary algebras are called machines (because of their relations to the theory of abstract automata).

For a class $\mathscr{A}$ of partial algebras we denote by $\mathscr{A}^*$ the class of all completions of elements of $\mathscr{A}$. If $\mathscr{A} = \{A\}$ is a one-element class, then we write A^* instead of $\{A\}^*$.

H. Höft [5] proposed the question to find conditions under which $HSPA^* = (HSPA)^*$, where A is a partial algebra (the symbols H, S and P have the usual meaning). This question was solved by W. Bartol, D. Niwiński and L. Rudak [4].

We denote by $\mathscr{U}$ and $\mathscr{U}_p$ the class of all monounary algebras or the class of all partial monounary algebras, respectively. In this paper there are investigated the classes $HSP\mathscr{A}^*$ and $(HSP\mathscr{A})^*$, where $\mathscr{A} \subseteq \mathscr{U}_p$, and the relations between these classes. In particular, it will be shown that if $\mathscr{A} \subseteq \mathscr{U}_p$, $\mathscr{A} \nsubseteq \mathscr{U}$ and there is $A \in \mathscr{A}$ with card $A > 1$, then we have

$$(HSP\mathscr{A})^* = HSP\mathscr{A}^* \Leftrightarrow HSP\mathscr{A}^* = \mathscr{U}\,.$$

The author is indebted to W. Bartol for the suggestion of performing this investigation.

1. BASIC DEFINITIONS AND DENOTATIONS

1.1. Definition. By a (*partial*) *monounary* algebra we understand a pair (A, f), where A is a nonempty set and f is a (partial) mapping of A into A.

For a positive integer n the symbol $f^n(x)$ has a natural meaning; we put $f^0(x) = x$ for each $x \in A$.

1.2. Definition. Let $(A, f) \in \mathscr{U}_p$. A monounary algebra (A, g) is called a *completion of* (A, f), if $g(x) = f(x)$ whenever $f(x)$ is defined.

1.3. Definition. Let $(A, f) \in \mathscr{U}_p$, $x, y \in A$. Put $x \equiv_f y$ if and only if there are $m, n \in N \cup \{0\}$ such that $f^n(x), f^m(y)$ exist and $f^n(x) = f^m(y)$. The elements of $A/\equiv_f$ are called *connected components of* (A, f). If $A/\equiv_f$ is a one-element set, then (A, f) is called *connected*.

1.4. Definition. Let $(A, f) \in \mathscr{U}_p$. An element $x \in A$ is called *cyclic*, if there is $n \in N$ with $f^n(x) = x$. The union of all cyclic elements belonging to the same connected component of (A, f) is called a *cycle of* (A, f).

If $(A, f) \in \mathscr{U}_p$ and if no misunderstanding can occur, then we sometimes write A instead of (A, f).

All classes of partial monounary algebras are assumed to be nonempty (unless otherwise stated). If $\mathscr{A} = \{(A_i, f_i\}: i \in I\} \subseteq \mathscr{U}_p$ and if no misunderstanding can occur, we denote all partial unary operations f_i by the same symbol f.

We recall the definitions of H, S and P for partial monounary algebras.

If $(A, f), (B, g) \in \mathscr{U}_p$, then a mapping $h: A \to B$ is said to be a *homomorphism of* (A, f) *into* (B, g) if the following holds: if $x \in A$ and $f(x)$ exists, then $g(h(x))$ exists and $g(h(x)) = h(f(x))$. If such h is surjective, then (B, g) is called a *homomorphic image of* (A, f). A subalgebra of a partial monounary algebra (A, f) is any partial monounary algebra (B, g) such that $B \subseteq A$ and, for any $x \in B$, either both $f(x)$ and $g(x)$ exist and $f(x) = g(x)$, or $f(x)$ and $g(x)$ do not exist. Direct products of partial monounary algebras are defined componentwise in a natural way.

For a class $\mathscr{A} \subseteq \mathscr{U}_p$ let $H\mathscr{A}$ be the class of all homomorphic images of partial monounary algebras in $\mathscr{A}$, let $S\mathscr{A}$ be the class of all isomorphic copies of subalgebras of partial monounary algebras in $\mathscr{A}$ and let $P\mathscr{A}$ be the class of all isomorphic copies of direct products of partial monounary algebras in $\mathscr{A}$.

2. VARIETIES OF MONOUNARY ALGEBRAS

This section contains some simple auxiliary results concerning varieties of monounary algebras.

2.1. Definition. Let $n \in N$, $k \in N \cup \{0\}$. A connected monounary algebra (A, f) will be *called* (n, k)*-bounded*, if there is $n' \in N$ such that n' divides n, (A, f) contains a cycle C with card $C = n'$ and $f^k(x) \in C$ for each $x \in A$.

2.2. Definition. Let $n \in N$, $k \in N \cup \{0\}$. A monounary algebra (A, f) will be *called* (n, k)*-bounded*, if each connected component of (A, f) is (n, k)-bounded. The system of all (n, k)-bounded monounary algebras will be denoted $\mathscr{A}(n, k)$. By the symbol $\mathscr{A}_c(1, k)$ we denote the system of all connected $(1, k)$-bounded monounary algebras.

2.3. Lemma. *Let* $k \in N \cup \{0\}$, (A, f) *be a monounary algebra. Then* $f^k(x) = f^k(y)$ *for each* $x, y \in A$ *if and only if* (A, f) *is connected and* $(1, k)$*-bounded.*

Proof. It is obvious that if (A, f) is connected and $(1, k)$-bounded, the identity $f^k(x) = f^k(y)$ holds on A. Assume that $f^k(x) = f^k(y)$ for each $x, y \in A$. Then (A, f) is connected. Let $x \in A$. For $y = f(x)$ we have $f^k(x) = f^k(f(x)) = f(f^k(x))$, thus $\{f^k(x)\}$ is a cycle of (A, f) for an arbitrary $x \in A$. Therefore (A, f) is $(1, k)$-bounded.

2.4. Lemma. *Let $n \in N$, $k \in N \cup \{0\}$. Further let (A, f) be a monounary algebra. Then $f^{n+k}(x) = f^k(x)$ for each $x \in A$ if and only if (A, f) is (n, k)-bounded.*

Proof. If (A, f) is (n, k)-bounded, then evidently $f^{n+k}(x) = f^k(x)$ for each $x \in A$. Assume that $f^{n+k}(x) = f^k(x)$ holds for each $x \in A$. Since $f^n(f^k(x)) = f^k(x)$, the element $f^k(x)$ belongs to a cycle with the cardinality dividing n (for an arbitrary $x \in A$). Therefore (A, f) is (n, k)-bounded.

2.5. Lemma. *Let $n \in N$, $k \in N \cup \{0\}$ and let (A, f) be a monounary algebra. Then $f^{n+k}(x) = f^k(y)$ for each $x, y \in A$ if and only if (A, f) is connected and $(1, k)$-bounded.*

Proof. It is obvious that the identity $f^{n+k}(x) = f^k(y)$ hold in a connected and $(1, k)$-bounded monounary algebra. Let $f^{n+k}(x) = f^k(y)$ for each $x, y \in A$. Then (A, f) is connected. If $x \in A$ is an arbitrary element, we obtain

$$f^{n+k}(x) = f^k(x),$$
$$f^{n+k}(x) = f^k(f(x)),$$

from which it follows that $f^k(x) = f(f^k(x))$. This implies that (A, f) contains a cycle $\{f^k(x)\}$ for each $x \in A$, thus (A, f) is $(1, k)$-bounded.

2.6. Remark. From 2.3 and 2.5 it follows, that if $n \in N$, $k \in N \cup \{0\}$, then the identities

$$f^k(x) = f^k(y) \quad \text{for each} \quad x, y \in A,$$
$$f^{k+n}(x) = f^k(y) \quad \text{for each} \quad x, y \in A$$

are equivalent.

2.7. Lemma. *Let $\mathscr{V}$ be a variety of monounary algebras. Then onefo the following conditions is satisfied:*

(i) $\mathscr{V} = \mathscr{U}$;
(ii) $\mathscr{V} = \mathscr{A}(n, k)$ *for some* $n \in N$, $k \in N \cup \{0\}$;
(iii) $\mathscr{V} = \mathscr{A}_c(1, k)$ *for some* $k \in N \cup \{0\}$.

Proof. Let Ω be the system of all identities which hold in all algebras $(A, f) \in \mathscr{V}$. There exist only four types of identities:

$$\alpha_k\colon f^k(x) = f^k(x), \quad \text{where } k \in N \cup \{0\};$$
$$\beta_k\colon f^k(x) = f^k(y), \quad \text{where } k \in N \cup \{0\};$$
$$\gamma_{nk}\colon f^{n+k}(x) = f^k(x), \quad \text{where } n \in N, \ k \in N \cup \{0\};$$
$$\delta_{nk}\colon f^{n+k}(x) = f^k(y), \quad \text{where } n \in N, \ k \in N \cup \{0\}.$$

According to 2.6 it suffices to consider only identities of the forms α_k, β_k and γ_{nk}. There exist $K_1, K_2 \subseteq N \cup \{0\}$, $M_3 \subseteq N \times (N \cup \{0\})$ such that $\Omega = \{\alpha_k: k \in K_1\} \cup \cup \{\beta_k: k \in K_2\} \cup \{\gamma_{nk}: (n, k) \in M_3\}$. Denote $K_3 = \{k \in N \cup \{0\}$: there is $n \in N$ with $(n, k) \in M_3\}$, $N_3 = \{n \in N$: there is $k \in N \cup \{0\}$ with $(n, k) \in M_3\}$. Let $(A, f) \in \mathscr{V}$.

First let $K_2 \cup K_3 = \emptyset$. The only identities in Ω are trivial and $\mathscr{V} = \mathscr{U}$.

Now let $K_2 = \emptyset$, $K_3 \neq \emptyset$. Then

(1) $f^{n+k}(x) = f^k(x)$ for each $(n, k) \in M_3$.

According to 2.4, (1) implies

(2) (A, f) is (n, k)-bounded for each $(n, k) \in M_3$.

By the symbol m denote the least common divisor of the elements of N_3 and put $j = \min K_3$. Then (2) yields that (A, f) is (m, j)-bounded, i.e. $(A, f) \in \mathscr{A}(m, j)$, thus

(3) $\mathscr{V} \subseteq \mathscr{A}(m, j)$.

Each identity of Ω is valid in $\mathscr{A}(m, j)$ (according to 2.4), thus

(4) $\mathscr{A}(m, j) \subseteq \mathscr{V}$,

and (3) and (4) yield that $\mathscr{V} = \mathscr{A}(m, j)$.

Assume that $K_2 \neq \emptyset$. From 2.3 it follows that if $(A, f) \in \mathscr{V}$, then

(5) $(A, f) \in \mathscr{A}_c(1, k)$ for each $k \in K_2$.

Further, 2.4 implies

(6) $(A, f) \in \mathscr{A}(n, k)$ for each $(n, k) \in M_3 = N_3 \times K_3$.

According to (5) and (6) we get that (A, f) is connected and

(7) $(A, f) \in \mathscr{A}_c(1, k)$ for each $k \in K_2 \cup K_3$.

Put $l = \min(K_2 \cup K_3)$. Then (7) yields that $(A, f) \in \mathscr{A}_c(1, l)$, thus

(8) $\mathscr{V} \subseteq \mathscr{A}_c(1, l)$.

Since each identity of Ω holds in $\mathscr{A}_c(1, l)$ according to 2.3, we obtain

(9) $\mathscr{A}_c(1, l) \subseteq \mathscr{V}$,

and therefore $\mathscr{V} = \mathscr{A}_c(1, l)$.

3. *HSP*$\mathscr{A}^*$

Let $\mathscr{A}$ be a class of partial monounary algebras. Since $\mathscr{A}^*$ is the class of all completions of all partial algebras belonging to $\mathscr{A}$, we infer that $HSP\mathscr{A}^*$ is a variety of monounary algebras. All varieties of monounary algebras were described in 2.7. For each variety $\mathscr{V}$ of monounary algebras we shall give necessary and sufficient conditions (concerning $\mathscr{A}$), under which $HSP\mathscr{A}^* = \mathscr{V}$.

3.1. Lemma. *Let $k \in N \cup \{0\}$. Then $HSP\mathscr{A}^* = \mathscr{A}_c(1, k)$ if and only if $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$ and $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ for $k' < k$.*

Proof. If $HSPA^* \subseteq \mathscr{A}_c(1, k)$, then obviously $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$. If $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k')$ for $k' < k$, then $HSP\mathscr{A}^* \subseteq HSP\mathscr{A}_c(1, k') = \mathscr{A}_c(1, k') \subset \mathscr{A}_c(1, k)$, which is a contradiction.

Now let $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$, $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ for $k' < k$. Then $HSP\mathscr{A}^* \subseteq HSP\mathscr{A}_c(1, k) = \mathscr{A}_c(1, k)$. Since $HSP\mathscr{A}^*$ is a variety of monounary algebras and it is a subvariety of $\mathscr{A}_c(1, k)$, there is $k' \leqq k$ with $HSP\mathscr{A}^* = \mathscr{A}_c(1, k')$. From this it follows that $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k')$, and therefore $k' = k$, $HSP\mathscr{A}^* = \mathscr{A}_c(1, k)$.

3.2. Lemma. *Let $k \in N \cup \{0\}$. The following conditions are equivalent:*

(i) *$\mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$ and $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ for $k' < k$;*

(ii) $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$, *where* $\mathscr{A}_1 \subseteq \mathscr{A}_c(1, k)$, $\mathscr{A}_1 \nsubseteq \mathscr{A}_c(1, k')$ *for* $k' < k$ *and each element of* $\mathscr{A}_2$ *is a one-element non-complete partial monounary algebra (here* $\mathscr{A}_2$ *can be empty).*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious.

Suppose that the condition (i) is satisfied. Let $(A, f) \in \mathscr{A}$. If (A, f) is complete, then $(A, f) \in \mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$. Let (A, f) be non-complete. If (A, f) is not connected, then there is a completion (A, g) of (A, f) such that (A, g) is not connected as well. But $(A, g) \in \mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$, which is a contradiction. Hence (A, f) is connected. If (A, f) consists of more than one element, then there is a completion (A, h) of (A, f) such that (A, h) contains a cycle C with card $C \geqq 2$, a contradiction to the relation $(A, h) \in \mathscr{A}^* \subseteq \mathscr{A}_c(1, k)$. Denote $\mathscr{A}_1 = \mathscr{A} \cap \mathscr{U}$, $\mathscr{A}_2 = \mathscr{A} - \mathscr{A}_1$. Thus we have shown that $\mathscr{A}_1 \subseteq \mathscr{A}_c(1, k)$ and each algebra belonging to $\mathscr{A}_2$ is a one-element non-complete partial monounary algebra. If we suppose that $\mathscr{A}_1 \subseteq \mathscr{A}_c(1, k')$ for some $k' < k$, we get (since $\mathscr{A}_2^* \subseteq \mathscr{A}_c(1, 0)$)

$$\mathscr{A}^* = (\mathscr{A}_1 \cup \mathscr{A}_2)^* = \mathscr{A}_1^* \cup \mathscr{A}_2^* \subseteq \mathscr{A}_1 \cup \mathscr{A}_c(1, 0) \subseteq$$
$$\subseteq \mathscr{A}_c(1, k') \cup \mathscr{A}_c(1, 0) = \mathscr{A}_c(1, k'),$$

a contradiction with (i).

3.3. Lemma. *Let* $n \in N$, $n > 1$, $k \in N \cup \{0\}$. *Then* $HSP\mathscr{A}^* = \mathscr{A}(n, k)$ *if and only if* $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$ *and* $\mathscr{A}^* \nsubseteq \mathscr{A}(n', k')$ *for* $(n', k') \neq (n, k)$, $k' \leqq k$ *and* n' *dividing* n.

Proof. Let $HSP\mathscr{A}^* = \mathscr{A}(n, k)$. Then obviously $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$. Assume that $\mathscr{A}^* \subseteq \mathscr{A}(n', k')$ for some $k' \leqq k$, n' dividing n, $(n', k') \neq (n, k)$. Then $HSP\mathscr{A}^* \subseteq \subseteq HSP\mathscr{A}(n', k') = \mathscr{A}(n', k') \subset \mathscr{A}(n, k)$, which is a contradiction.

Conversely, suppose that $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$ and $\mathscr{A}^* \nsubseteq \mathscr{A}(n', k')$ for $k' \leqq k$, n' dividing n, $(n', k') \neq (n, k)$. This implies $\mathscr{A}^* \nsubseteq \mathscr{A}(1, k')$ for $k' \leqq k$ (since $n \neq 1$), and thus

(1) $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ for $k' \leqq k$.

Further, $HSP\mathscr{A}^* \subseteq HSP\mathscr{A}(n, k) = \mathscr{A}(n, k)$. Hence $HSP\mathscr{A}^*$ is a subvariety of $\mathscr{A}(n, k)$, therefore either there are n', k' such that $HSP\mathscr{A}^* = \mathscr{A}(n', k')$, $k' \leqq k$, n'/n, or there is $k' \leqq k$ with $HSP\mathscr{A}^* = \mathscr{A}_c(1, k')$. If $HSP\mathscr{A}^* = \mathscr{A}(n', k')$, then $\mathscr{A}^* \subseteq \mathscr{A}(n', k')$ and from the assumption it follows that $n' = n$, $k' = k$. If $HSP\mathscr{A}^* = \mathscr{A}_c(1, k')$ for $k' \leqq k$, then $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k')$, a contradiction to (1).

3.4. Lemma. *Let* $k \in N \cup \{0\}$. *Then* $HSP\mathscr{A}^* = \mathscr{A}(1, k)$ *if and only if* $\mathscr{A}^* \subseteq \subseteq \mathscr{A}(1, k)$, $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ *for* $k' \leqq k$ *and* $\mathscr{A}^* \nsubseteq \mathscr{A}(1, k')$ *for* $k' < k$.

Proof. Let $HSP\mathscr{A}^* = \mathscr{A}(1, k)$. Then $\mathscr{A}^* \subseteq \mathscr{A}(1, k)$. If $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k')$ for some $k' \leqq k$, then $HSP\mathscr{A}^* \subseteq HSP\mathscr{A}_c(1, k') = \mathscr{A}_c(1, k') \subset \mathscr{A}(1, k)$, which is a contradiction. If $\mathscr{A}^* \subseteq \mathscr{A}(1, k')$ for some $k' < k$, then $HSP\mathscr{A}^* \subseteq HSP\mathscr{A}(1, k') = = \mathscr{A}(1, k') \nsubseteq \mathscr{A}(1, k)$, which is a contradiction.

Conversely, let $\mathscr{A}^* \subseteq \mathscr{A}(1, k)$, $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ for $k' \leqq k$ and $A^* \nsubseteq \mathscr{A}(1, k')$ for $k' < k$. Then $HSP\mathscr{A}^* \subseteq HSP\mathscr{A}(1, k) = \mathscr{A}(1, k)$, i.e. $HSP\mathscr{A}^*$ is a subvariety of the variety $\mathscr{A}(1, k)$. Therefore either there is $k' \leqq k$ with $HSP\mathscr{A}^* = \mathscr{A}_c(1, k')$, or there is $k'' \leqq k$ with $HSP\mathscr{A}^* = \mathscr{A}(1, k'')$. If $HSP\mathscr{A}^* = \mathscr{A}_c(1, k')$, then $\mathscr{A}^* \subseteq \subseteq \mathscr{A}_c(1, k')$, a contradiction. If $HSP\mathscr{A}^* = \mathscr{A}(1, k'')$, then $\mathscr{A}^* \subseteq (1, k'')$, hence $k'' = k$.

3.5. Definition. Let $n \in N$, $k, l \in N \cup \{0\}$ and let (A, f) be a partial monounary algebra, $A = A_0 \cup A_1 \cup \ldots \cup A_l$, where either $A_0 = \emptyset$ or A_0 is complete, and $A_1, \ldots, A_l$ are distinct noncomplete connected components of (A, f) (here $A - A_0 = \emptyset$ if and only if $l = 0$). A partial monounary algebra (A, f) is said to be (n, k)-*bounded*, if there are $k_0, \ldots, k_l \in N \cup \{0\}$ such that the following conditions are satisfied:

(a) if $A_0 \neq \emptyset$, then

(i) $(A_0, f) \in \mathscr{A}(n, k_0)$ and $(A_0, f) \notin \mathscr{A}(n, k_0')$ for $k_0' < k_0$;

(ii) $k_0 + \ldots + k_l + l \leqq k$;

(b) if $A - A_0 \neq \emptyset$, then

(iii) if $i \in \{1, \ldots, l\}$, $x \in A_i$, then $f^{k_i+1}(x)$ does not exist;

(iv) l.c.m. $(1, 2, \ldots, k_1 + \ldots + k_l + l)/n$;

(c) if $A_0 = \emptyset$ and $A - A_0 \neq \emptyset$, then $k_1 + \ldots + k_l + l \leqq k + 1$. (Let us remark that this definition of (n, k)-bounded partial monounary algebra for a complete monounary algebra is in accordance with the definition 2.2.)

3.6. Definition. Let $n \in N$, $k \in N \cup \{0\}$. The class of all (n, k)-bounded partial monounary algebras (complete or non-complete) will be denoted $\mathscr{A}\not{p}(n, k)$. The class of all elements of $\mathscr{A}\not{p}(1, k)$ which are connected, is denoted by the symbol $\mathscr{A}\not{p}_c(1, k)$.

3.6.1. Corollary. $\mathscr{A}(n, k) \subseteq \mathscr{A}\not{p}(n, k)$ *for* $n \in N$, $k \in N \cup \{0\}$.

3.7. Lemma. *Let* $n \in N$, $k \in N \cup \{0\}$. *If* $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$, *then* $\mathscr{A} \subseteq \mathscr{A}\not{p}(n, k)$.

Proof. Let $\mathscr{A}^* \subseteq \mathscr{A}(n, k), (A, f) \in \mathscr{A}$. If (A, f) is complete, then $(A, f) \in \mathscr{A}\not{p}(n, k)$. Let (A, f) be non-complete. Since $(A, f)^* \subseteq \mathscr{A}^* \subseteq \mathscr{A}(n, k)$, there exist only finitely many elements x in A for which $f(x)$ is not defined (in the opposite case, after appropriate completion we could get a component without cycle, and it does not belong to $\mathscr{A}(n, k)$). Let $A = A_0 \cup A_1 \cup \ldots \cup A_l$, where either $A_0 = \emptyset$ or A_0 is complete, $l \geqq 1$ and $A_1, \ldots, A_l$ are distinct non-complete connected components of (A, f). Since $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$, each complete connected component of (A, f) is (n, k)-bounded. Hence either $A_0 = \emptyset$ or there exists $k_0 \in N \cup \{0\}$ such that A_0 is (n, k_0)-bounded and it is not (n, k_0')-bounded whenever $k_0' < k_0$. Further let $x_1 \in A_1, \ldots, x_l \in A_l$ be such that $f(x_1), \ldots, f(x_l)$ are not defined. If we define a completion $(A, g) \in (A, f)^*$ such that $g(x_1) = x_1, \ldots, g(x_l) = x_l$, from the fact that $(A, f)^* \subseteq \mathscr{A}(n, k)$ it follows that there are $k_1, \ldots, k_l$ with

(1) $g^{k_1}(x) = x_1$ for each $x \in A_1, \ldots, g^{k_l}(x) = x_l$ for each $x \in A_l$. We can suppose that k_1 (and analogously for $k_2, \ldots, k_l$) is the greatest non-negative integer such that

(2) there exists $z_1 \in A_1$ with $g^{k_1}(z_1) = x_1$, $g^i(z_1) \neq x_1$ for each $0 \leqq i < k_1$.

From this it follows

(3) $f^{k_1}(z_1) = x_1$

and

(4) if $x \in A_1$, then $f^{k_1+1}(x)$ does not exist.

Let $(A, h) \in (A, f)^*$ be such that $h(x_1) = z_2$, $h(x_2) = z_3, \ldots, h(x_l) = z_1$. Then (A, h) contains a cycle

$$\{z_1, f(z_1), \ldots, f^{k_1}(z_1) = x_1, f(z_2), \ldots, f^{k_2}(z_2) = x_2, \ldots, z_l, f(z_l), \ldots, f^{k_l}(z_l) = x_1\},$$

i.e. a cycle with $m = (k_1 + 1) + (k_2 + 1) + \ldots + (k_l + 1) = k_1 + \ldots + k_l + l$ elements. Since $(A, h) \in \mathscr{A}^* \subseteq \mathscr{A}(n, k)$, we get

(5) m/n.

Analogously as above, we can construct another completions of (A, f) which obtain cycles with $1, 2. \ldots, m - 1$ elements, hence $1/n, 2/n, \ldots, m - 1/n$, and we get

(6) l.c.m. $(1, 2, \ldots, m)/n$.

Let (A, h_1) be a completion of (A, f) such that $h_1(x_1) = z_2$, $h_1(x_2) = z_3, \ldots, h_1(x_{l-1}) = z_l$, $h_1(x_l) = x_l$. Then $(A, h_1) \in \mathscr{A}^*$, (A, h_1) contains a cycle $\{x_l\}$ and

$$(7) \qquad h_1^{m-1}(z_1) = h_1^{k_1+\ldots+k_l+l-1}(z_1) = h_1^{k_2+\ldots+k_l+l-1}(h_1^{k_1}(z_1)) =$$
$$= h_1^{k_2+\ldots+k_l+l-1}(x_1) = h_1^{k_2+\ldots+k_l+(l-2)}(z_2) = \ldots = h_1^{k_l}(z_l) = x_l,$$
$$h_1^{m-2}(z_1) \neq x_l.$$

Since $(A, h_1) \in \mathscr{A}(n, k)$, (7) yields

(8) $m - 1 \leqq k$, i.e. $m \leqq k + 1$.

If $A_0 = \emptyset$, we are ready with the proof that (A, f) is (n, k)-bounded (according to the definition 3.5). To complete the proof we ought to prove that $k_0 + k_1 + \ldots + k_l + l \leqq k$, whenever $A_0 \neq \emptyset$. Suppose that $A_0 \neq \emptyset$. From the properties of k_0 it follows that there is a complete connected component B of (A, f) such that (B, f) is (n, k_0)-bounded and (B, f) is not (n, k_0')-bounded for $k_0' < k_0$. Then there is a cycle C of B and $z_0 \in B$ with $f^{k_0}(z_0) \in C$, $f^{k_0-1}(z_0) \notin C$. Further define a completion (A, g_1) of (A, f) such that $g_1(x_1) = z_2$, $g_1(x_2) = z_3, \ldots, g_1(x_{l-1}) = z_l$, $g_1(x_l) = z_0$. Then

$$(9) \qquad g_1^{k_0+k_1+\ldots+k_l+l}(z_1) = g_1^{k_0+k_2+\ldots+k_l+l}(x_1) = g_1^{k_0+k_2+\ldots+k_l+(l-1)}(z_2) =$$
$$= g_1^{k_0+k_3+\ldots+k_l+(l-1)}(x_2) = \ldots = g_1^{k_0+1}(x_l) = g_1^{k_0}(z_0) \in C,$$

and

(10) $g_1^{k_0+\ldots+k_l+l-1}(z_1) = g_1^{k_0-1}(z_0) \notin C$.

From (9), (10) and from the relation $(A, g_1) \in \mathscr{A}(n, k)$ it follows

(11) $k_0 + \ldots + k_l + l \leqq k$.

Therefore $(A, f) \in \mathscr{A}\not{p}(n, k)$.

3.8. Lemma. *Let $n \in N$, $k \in N \cup \{0\}$. If $\mathscr{A} \subseteq \mathscr{A}\not{p}(n, k)$, then $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$.*

Proof. Assume that $\mathscr{A} \subseteq \mathscr{A}\not{p}(n, k)$. Let $B = (B, g) \in \mathscr{A}^*$, i.e. $B \in A^*$ for some $A = (A, f) \in \mathscr{A}$. Since A is (n, k)-bounded, there are $l \in N \cup \{0\}$, $k_0, \ldots, k_l \in \in N \cup \{0\}$ such that $A = A_0 \cup \ldots \cup A_l$, either $A_0 = \emptyset$ or A_0 is complete, $A_1, \ldots, A_l$ are distinct non-complete connected components of A, and (a)–(c) of 3.5 are satisfied. Consider a connected component B_1 of B. Then B_1 contains a cycle C which either was contained as a cycle in A, or has $d \leqq k_1 + \ldots + k_l + l$ elements. In the first case card C/n according to (a) (i) and in the second case card $C = d/n$ according to (b) (iv). Now let $x \in B_1$. Suppose that the first case occurs. If $f^k(x)$ exists, then $f^k(x) \in C$, since $k_0 \leqq k$ (according to (a) (ii)) and (a) (i) holds. Then

(1) $g^k(x) = f^k(x) \in C$.

If $f^k(x)$ does not exist, then $x \in A - A_0$. Since $k_0 + k_1 + \ldots + k_l + l \leqq k$ and $B \in A^*$, we obtain

(2) $g^k(x) \in C$.

Therefore (1) and (2) yield that $B_1 \in \mathscr{A}(n, k)$. The second case is analogous, the relation (2) is valid, too. Hence $B \in \mathscr{A}(n, k)$, i.e. $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$.

3.9. Lemma. *Let $n \in N$, $k \in N \cup \{0\}$. The following conditions are equivalent:*

(i) $\mathscr{A}^* \subseteq \mathscr{A}(n, k)$, $\mathscr{A}^* \nsubseteq \mathscr{A}(n', k')$ *for* $(n', k') \neq (n, k)$, n'/n, $k' \leqq k$;

(ii) $\mathscr{A} \subseteq \mathscr{A}\not{p}(n, k)$, $\mathscr{A} \nsubseteq \mathscr{A}\not{p}(n', k')$ *for* $(n', k') \neq (n, k)$, n'/n, $k' \leqq k$.

Proof. Let (i) hold. From 3.7 it follows that $\mathscr{A} \subseteq \mathscr{A}\not{p}(n, k)$. If $\mathscr{A} \subseteq \mathscr{A}\not{p}(n', k')$ for $(n', k') \neq (n, k)$, n'/n, $k' \leqq k$, then 3.8 implies that $\mathscr{A}^* \subseteq \mathscr{A}(n', k')$, a contradiction with (i). The proof of the implication (ii) $\Rightarrow$ (i) is analogous, it follows from 3.8 and 3.7.

3.10. Lemma. *Let $k \in N \cup \{0\}$. If $(A, f) \in \mathscr{A}\not{p}(1, k)$, then there are $A_0, A_1 \subseteq A$ such that $A = A_0 \cup A_1$, $A_0 \cap A_1 = \emptyset$, either $A_0 = \emptyset$ or A_0 is complete, and* card $A_1 \leqq 1$.

Proof. Let $(A, f) \in \mathscr{A}\not{p}(1, k)$. Then $A = A_0 \cup \ldots \cup A_l$ and the conditions of 3.5 are valid, where $n = 1$. If $A - A_0 \neq \emptyset$, according to (b) (iv) of 3.5 we obtain

$$\text{l.c.m.}\,(1, \ldots, k_1 + \ldots + k_l + l)/1,$$

i.e. $k_1 + \ldots + k_l + l = 1$. Since $l \geqq 1$, we get $l = 1$, $k_1 = 0$.

3.11. Corollary. *Let $k \in N \cup \{0\}$. If $(A, f) \in \mathscr{A}\not{p}_c(1, k) - \mathscr{U}$, then* card $A = 1$.

Proof. The assertion immediately follows from 3.10.

3.12. Lemma. *Let $k \in N \cup \{0\}$. Then the following conditions are equivalent:*

(i) $\mathscr{A}^* \subseteq \mathscr{A}(1, k)$, $\mathscr{A}^* \nsubseteq \mathscr{A}_c(1, k')$ *for* $k' \leqq k$ *and* $\mathscr{A}^* \nsubseteq \mathscr{A}(1, k')$ *for* $k' < k$;

(ii) $\mathscr{A} \subseteq \mathscr{A}\not{p}(1, k)$, $\mathscr{A} \nsubseteq \mathscr{A}\not{p}_c(1, k')$ *for* $k' \leqq k$ *and* $\mathscr{A} \nsubseteq \mathscr{A}\not{p}(1, k')$ *for* $k' < k$.

Proof. Let (i) hold. According to 3.7 we get $\mathscr{A} \subseteq \mathscr{A}\not{p}(1, k)$. If $\mathscr{A} \subseteq \mathscr{A}\not{p}(1, k')$ for $k' < k$, then 3.8 implies that $\mathscr{A}^* \subseteq (1, k')$, a contradiction with (i). Let $\mathscr{A} \subseteq \subseteq \mathscr{A}\not{p}_c(1, k')$ for $k' \leqq k$. Let $B \in \mathscr{A}^*$, i.e. there is $A \in \mathscr{A}$ with $B \in A^*$. If A is complete, then $B = A \in \mathscr{A}_c(1, k')$. Assume that A is non-complete. Then 3.11 implies that card $A = 1$, hence $A^* \subseteq \mathscr{A}_c(1, 0)$. Therefore $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k')$, which is a contradiction to (i).

Suppose that (ii) is valid. Then 3.8 implies that $\mathscr{A}^* \subseteq \mathscr{A}(1, k)$. If $\mathscr{A}^* \subseteq \mathscr{A}(1, k')$ for $k' < k$, then $\mathscr{A} \subseteq \mathscr{A}\mathscr{p}(1, k')$, a contradiction to (ii). Let $\mathscr{A}^* \subseteq \mathscr{A}_c(1, k')$ for $k' \leqq k$, $A \in \mathscr{A}$. Then A is connected. Since $A \in \mathscr{A}\mathscr{p}(1, k)$, then we get that $A \in \mathscr{A}\mathscr{p}_c(1, k)$, a contradiction, thus (i) is satisfied.

3.13. Denotation. Let $\mathscr{A}$ be a class of partial monounary algebras. For $k \in N \cup \{0\}$, $n \in N$, $n > 1$ let us consider the following conditions concerning $\mathscr{A}$:

(k) $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$, whete $\mathscr{A}_1 \subseteq \mathscr{A}_c(1, k)$, $\mathscr{A}_1 \nsubseteq \mathscr{A}_c(1, k')$ for $k' < k$, and each element of $\mathscr{A}_2$ is a one-element non-complete partial monounary algebra (here $\mathscr{A}_2$ can be empty);

(1, k) $\mathscr{A} \subseteq \mathscr{A}\mathscr{p}(1, k)$, $\mathscr{A} \nsubseteq \mathscr{A}\mathscr{p}(1, k')$ for $k' < k$ and $\mathscr{A} \nsubseteq \mathscr{A}\mathscr{p}_c(1, k')$ for $k' \leqq k$;

(n, k) $\mathscr{A} \subseteq \mathscr{A}\mathscr{p}(n, k)$, $\mathscr{A} \nsubseteq \mathscr{A}\mathscr{p}(n', k')$ for $(n', k') \neq (n, k)$, n'/n, $k' \leqq k$.

3.14. Theorem. *Let $\mathscr{A}$ be a class of partial monounary algebras, $n \in N$, $k \in N \cup \{0\}$. Then*

$$HSP\mathscr{A}^* = \begin{cases} \mathscr{A}_c(1, k)\,, & \text{if (k) holds;} \\ \mathscr{A}(n, k)\,, & \text{if (n, k) holds;} \\ \mathscr{U} & \text{otherwise.} \end{cases}$$

Proof. Let $\mathscr{V} = HSP\mathscr{A}^*$. Since $\mathscr{V}$ is a variety of monounary algebras, according to 2.7 we get that one of the following conditions is satisfied:

(i) $\mathscr{V} = \mathscr{A}_c(1, k)$ for some $k \in N \cup \{0\}$;
(ii) $\mathscr{V} = \mathscr{A}(1, k)$ for some $k \in N \cup \{0\}$;
(iii) $\mathscr{V} = \mathscr{A}(n, k)$ for some $n \in N$, $n > 1$, $k \in N \cup \{0\}$;
(iv) $\mathscr{V} = \mathscr{U}$.

Then 3.1, 3.2 and 3.13 imply that (k) is valid if and only if (i) holds; 3.4, 3.12 and 3.13 imply that (1, k) is valid if and only if (ii) holds; 3.3, 3.9 and 3.13 imply that (n, k) is valid if and only if (iii) holds, which completes the proof.

4. $(HSP\mathscr{A})^*$

Let $\mathscr{A}$ be a class of partial monounary algebras. If each element of $\mathscr{A}$ is complete, then obviously $HSP\mathscr{A}^* = HSP\mathscr{A} = (HSP\mathscr{A})^*$. We shall now consider the case when $\mathscr{A} \nsubseteq \mathscr{U}$.

4.1. Lemma. *Let* card $A = 1$ *for each* $A \in \mathscr{A}$. *Then*

$$(HSP\mathscr{A})^* = \mathscr{A}_c(1, 0) = HSP\mathscr{A}^*\,.$$

Proof. Let $\mathscr{S}$ be the class of all one-element partial monounary algebras. It is obvious that if $A \in \mathscr{S}$, then $H(A) \in \mathscr{S}$, $S(A) \in \mathscr{S}$. Further, if $\{A_i\}_{i \in I} \subseteq \mathscr{S}$, then

$\prod_{i \in I} A_i \in \mathscr{S}$. Since $\mathscr{A} \subseteq \mathscr{S}$, this implies

(1) $HSP\mathscr{A} \subseteq \mathscr{S}$.

The system $\mathscr{S}^*$ consists of all one-element complete monounary algebras, i.e.

(2) $\mathscr{S}^* = \mathscr{A}_c(1, 0)$.

From (1) and (2) it follows

(3) $(HSP\mathscr{A})^* \subseteq \mathscr{S}^* = \mathscr{A}_c(1, 0)$.

The relation $\mathscr{A}_c(1, 0) \subseteq HSP\mathscr{A}$ is obvious, therefore

(4) $(HSP\mathscr{A})^* = \mathscr{A}_c(1, 0)$.

According to 3.14 we have $HSP\mathscr{A}^* = \mathscr{A}_c(1, 0)$, thus $HSP\mathscr{A}^* = \mathscr{A}_c(1, 0) = (HSP\mathscr{A})^*$.

4.2. Lemma. *Let i be a cardinal number, $\mathscr{A} \nsubseteq \mathscr{U}$ and assume that there is $A \in \mathscr{A}$ with* card $A > 1$. *Then there is $B \in HSP\mathscr{A}$ such that* card $B = i$ *and each connected component of B is a one-element non-complete partial monounary algebra.*

Proof. Since $\mathscr{A} \nsubseteq \mathscr{U}$, there exists $B_1 \in \mathscr{A}$ such that B_1 is not complete. Let $b \in B_1$ such that $f(b)$ is not defined. Put $C = B_1 \times A^i$ and let p be the natural projection of C onto B_1. Denote $B_2 = \{z \in C: p(z) = b\}$. Then $f(z)$ does not exist for each $z \in B_2$ and card $B_2 \geqq$ card $A^i \geqq i$. Therefore there is $B \in S(B_2)$ with i elements. Hence $B \in HSP\mathscr{A}$ and B fulfils the assertion of the lemma.

4.3. Lemma. *Let $\mathscr{A} \nsubseteq \mathscr{U}$ and assume that there is $A \in \mathscr{A}$ with* card $A > 1$. *Then*

$$(HSP\mathscr{A})^* = \mathscr{U}\,.$$

Proof. Suppose that $C \in \mathscr{U}$. Let $i =$ card C. According to 4.2 there is $B \in HSP\mathscr{A}$ such that card $B = i$ and each connected component of B is one-element and non-complete. Therefore there is a completion (B, g) if B such that (B, g) is isomorphic to C. Since $B \in HSP\mathscr{A}$, then we have $(B, g) \in (HSP\mathscr{A})^*$, and therefore $C \in (HSP\mathscr{A})^*$.

4.4. Theorem. *Let $\mathscr{A}$ be a class of partial monounary algebras.*

(i) *If $\mathscr{A} \subseteq \mathscr{U}$, then $HSP\mathscr{A}^* = (HSP\mathscr{A})^*$.*

(ii) *If $\mathscr{A} \nsubseteq \mathscr{U}$,* card $A = 1$ *for each $A \in \mathscr{A}$, then $HSP\mathscr{A}^* = (HSP\mathscr{A})^* = \mathscr{A}_c(1, 0)$.*

(iii) *If $\mathscr{A} \nsubseteq \mathscr{U}$ and there is $A \in \mathscr{A}$ with* card $A > 1$, *then $(HSP\mathscr{A})^* = \mathscr{U}$.*

Proof. The assertion is the consequence of 4.1 – 4.3.

4.5. Corollary. *Let $\mathscr{A}$ be a class of partial monounary algebras. Then*

(i) *$HSP\mathscr{A}^* \subseteq (HSP\mathscr{A})^*$;*

(ii) *$HSP\mathscr{A}^* = (HSP\mathscr{A})^*$ if and only if $\mathscr{A} \subseteq \mathscr{U}$ or* card $A = 1$ *for each $A \in \mathscr{A}$ or $HSP\mathscr{A}^* = \mathscr{U}$.*

Proof. The assertion follows from 3.14 and 4.4.

4.6. Corollary. *There exists a partial monounary algebra (A, f) with $HSP(A, f)^* \neq (HSP(A, f))^*$.*

Proof. Let $A = \{x, y, z\}$, where $f(x) = f(y) = y$, $f(z)$ is not defined and x, y, z are distinct. Then 4.4 (iii) implies

(1) $(HSP(A,f))^* = \mathscr{U}$.

We shall show that (A,f) is $(1, 2)$-bounded (cf. Def. 3.5). If we put $l = 1$, $k_0 = 1$, $k_1 = 0$, $A_0 = \{x, y\}$, $A_1 = \{z\}$, then $A = A_0 \cup A_1$, A_1 is complete and A_0 is a non-complete connected component of A. Further

(i) (A_0, f) is $(1, 1)$-bounded and it is not $(1, 0)$-bounded;

(ii) $k_0 + k_1 + l = 1 + 0 + 1 = 2$;

(iii) $f^{k_1+1}(z) = f(z)$ does not exist;

(iv) l.c.m. $(1, \ldots, k_1 + l) =$ l.c.m. $(1) = 1/1$.

Hence 3.5 yields that (A,f) is $(1, 2)$-bounded. According to (ii) it is not $(1, 0)$-bounded or $(1, 1)$-bounded. From 3.14 we get

(2) $HSP(A,f)^* = \mathscr{A}(1, 2)$.

5. CLASSES OF PARTIAL MONOUNARY ALGEBRAS CLOSED UNDER H, S, P

In connection with the investigations performed above it seems to be natural to consider the question which classes of partial monounary algebras are closed with respect to H, S and P.

5.1. Definition. For a class $\mathscr{A}$ of partial monounary algebras denote $V\mathscr{A}$ the class of partial monounary algebras such that

(i) $\mathscr{A} \subseteq V\mathscr{A}$;

(ii) $V\mathscr{A}$ is closed under homomorphisms, subalgebras and products (H, S and P);

(iii) if $\mathscr{A} \subseteq \mathscr{V}$ and $\mathscr{V}$ is a class closed under H, S, P, then $V\mathscr{A} \subseteq \mathscr{V}$.

For completeness let us introduce the following (known) assertion:

5.2. Lemma. *If $\mathscr{A}$ is a class of complete monounary algebras, then $V\mathscr{A} = HSP\mathscr{A}$.*

5.3. Lemma. *If A is a class of partial monounary algebras, $\mathscr{A} \nsubseteq \mathscr{U}$ and* card $A = 1$ *for each $A \in \mathscr{A}$, then*

(i) *$V\mathscr{A}$ consists of all one-element partial monounary algebras;*

(ii) *$V\mathscr{A} = H\mathscr{A}$.*

Proof. Let $\mathscr{V}$ be the class consisting of all one-element partial monounary algebras. It is obvious that $\mathscr{V}$ is closed under H, S, P and $\mathscr{A} \subseteq \mathscr{V}$, hence $V\mathscr{A} \subseteq \mathscr{V}$. Since $\mathscr{A} \nsubseteq \mathscr{U}$, there is $A \in \mathscr{A}$ such that $A = \{x\}$, $f(x)$ does not exist. If $B \in \mathscr{V}$, then $B = \{y\}$ and the mapping $\varphi: x \to y$ is a homomorphism of A onto B, therefore $B \in H\mathscr{A}$. Hence $\mathscr{V} \subseteq H\mathscr{A} \subseteq V\mathscr{A}$, which completes the proof.

5.4. Lemma. *Let $\mathscr{A} \subseteq \mathscr{U}_p$, $\mathscr{A} \nsubseteq \mathscr{U}$ and assume that there is $A \in \mathscr{A} - \mathscr{U}$ with* card $A > 1$. *Then*

$$V\mathscr{A} = HSP\mathscr{A} = \mathscr{U}_p .$$

Proof. Let $C \in \mathscr{U}_p$, card $C = i =$ card I for some set of indices I. Denote $B = A^i$.

Since $A \notin \mathscr{U}$, there is $a \in A$ such that $f(a)$ does not exist. For each $j \in I$ let p_j be the natural projection of A^i onto A and denote

$$B_1 = \{x \in B: p_j(x) = a \text{ for some } j \in I\} .$$

If $x \in B_1$, then $f(x)$ does not exist. Further, card $B_1 \geqq i$. Thus there is $B \in S(A^i)$ such that card $B = i$ and $f(x)$ does not exist for each $x \in B$. Since card $B =$ card C, there is an injective mapping φ of the set B onto the set C. Obviously, φ is a homomorphism of a partial monounary algebra B onto a partial monounary algebra C, therefore $C \in H(B)$. Thus we have proved that

$$C \in H(B) \subseteq HS(A^i) \subseteq HSP\mathscr{A} ,$$

i.e. $\mathscr{U}_p \subseteq HSP\mathscr{A}$. Hence $HSP\mathscr{A} = \mathscr{U}_p$ and $HSP\mathscr{A} = V\mathscr{A}$.

5.5. Lemma. *Let $\mathscr{A} \subseteq \mathscr{U}_p$, $\mathscr{A} \nsubseteq \mathscr{U}$. Assume that there is $A \in \mathscr{A}$ with* card $A > 1$ *and that, whenever $A_1 \in \mathscr{A} - \mathscr{U}$, then* card $A_1 = 1$. *Then we have*

$$V\mathscr{A} = HSP\mathscr{A} = \mathscr{U}_p .$$

Proof. Let $C \in \mathscr{U}_p$, card $C = i =$ card I for some set of indices I. There exists a complete monounary algebra $A \in \mathscr{A}$ with card $A > 1$. Next there exists $A_1 \in \in \mathscr{A} - \mathscr{U}$ with card $A_1 = 1$. Put $B_1 = A_1 \times A^i$. Then B_1 is a partial monounary algebra with card $B_1 =$ card A^i and if $x \in B_1$, then $f(x)$ is not defined. Since $i \leqq \leqq 2^i \leqq$ card A^i, there is $B \in S(B_1)$ such that card $B = i$ and $f(x)$ does not exist whenever $x \in B$. Analogously as above, $C \in H(B)$, i.e. $HSP\mathscr{A} = \mathscr{U}_p = V\mathscr{A}$.

5.6. Theorem. *Let $\mathscr{A}$ be a class of partial monounary algebras.*

(i) *If $\mathscr{A} \subseteq \mathscr{U}$, then $V\mathscr{A} = HSP\mathscr{A}$.*

(ii) *If $\mathscr{A} \nsubseteq \mathscr{U}$ and* card $A = 1$ *for each $A \in \mathscr{A}$, then $V\mathscr{A} = H\mathscr{A}$ and $V\mathscr{A}$ consists of all one-element partial monounary algebras.*

(iii) *If $\mathscr{A} \nsubseteq \mathscr{U}$ and* card $A > 1$ *for some $A \in \mathscr{A}$, then $V\mathscr{A} = HSP\mathscr{A} = \mathscr{U}_p$.*

Proof. The assertion follows from 5.2—5.5.

5.7. Corollary. *If $\mathscr{A}$ is a class of partial monounary algebras, then $V\mathscr{A} = HSP\mathscr{A}$.*

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