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UNEQUIVOCAL LINEARLY ORDERED GROUPS

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1. INTRODUCTION

Radical classes of linearly ordered groups were defined and studied by C. G. Chehata and R. Wiegandt [1]. Further results in this direction were obtained in the author's paper [5].

Radical classes of abelian linearly ordered groups were investigated by B. J. Gardner [3], Pringerová [7] and the author [6]. All linearly ordered groups dealt with in the present paper are assumed to be abelian.

The notion of unequivocal ring was introduced by B. J. Gardner [4]; such rings were studied by N. Divinsky [2].

The analogous notion for linearly ordered groups can be defined in the same way as in the case of rings, namely: a linearly ordered group G will be said to be *unequivocal* if for each radical class R either G belongs to R or G is R-semisimple.

If each convex subgroup of G is unequivocal, then G will be said to be *hereditarily unequivocal.*

It is a natural question to ask whether there exists an internal characterization of unequivocality (which does not involve the class of all radical properties, or, in other words, the collection of all radical classes).

Such a question was solved positively for the case of rings (cf. Divinsky [2], Theorem 1.) A positive answer for the case of linearly ordered groups is given by the following theorems (they will be deduced from results of the author's paper [6]; cf. Section 2 below):

1.1. Theorem. *A linearly ordered group G is unequivocal if and only if for each nonzero subgroup H of G there exist a system $H_0, H_1, H_2, \ldots, H_\alpha, \ldots$ $(\alpha < \beta)$ of convex subgroups of H and a system $G_0, G_1, G_2, \ldots, G_\alpha, \ldots$ $(\alpha < \beta)$ of convex subgroups of G such that*

(i) $\{0\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots$ $(\alpha < \beta)$, $\bigcup_{\alpha<\beta} G_\alpha = G$,

(ii) *for each $\alpha < \beta$, the linearly ordered group $G_\alpha / \bigcup_{\gamma<\alpha} G_\gamma$ is isomorphic to H/H_α.*

1.2. Theorem. *Let G be a linearly ordered group. The following conditions are* *equivalent*:

(i) *G is unequivocal.*

(ii) *If H is a convex subgroup of G such that $\{0\} \neq H \neq G$, then there is a nonzero homomorphic image of H which is isomorphic to a convex subgroup of the linearly ordered group G/H.*

For a linearly ordered group G we denote by $c(G)$ the system of all convex subgroups of G; this system is partially ordered by inclusion. In fact, $c(G)$ is a linearly ordered set.

In Section 2 the proofs of Theorem 1.1 and 1.2 are given.

In the sections 3 and 4 there are investigated unequivocal linearly ordered groups G having the property that the linearly ordered set $c(G)$ has an atom or a dual atom, respectively. Section 5 deals with hereditarily unequivocal linearly ordered groups.

The author is indebted to L. A. Skornjakov for his suggestion to investigate the notion of unequivocality for linearly ordered groups.

2. UNEQUIVOCALITY

We recall the definition of radical class (as we already remarked above, all linearly ordered groups under consideration are assumed to be abelian).

We denote by $\mathscr{G}_a$ the class of all abelian linearly ordered groups. A nonempty subclass X of $\mathscr{G}_a$ is said to be a *radical class* if it satisfies the following conditions (cf. [1]):

(R1) If $A \in X$, then every nontrivial homomorphic image of A has a nontrivial convex subgroup belonging to X.

(R2) If $A \in \mathscr{G}_a$ and if every nontrivial homomorphic image of A has a nontrivial convex subgroup belonging to X, then $A \in X$.

Let β be an ordinal and for each ordinal $\alpha < \beta$ let G_α be an element of $c(G)$ such that

$$\{0\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots, \quad \bigcup_{\alpha<\beta} G_\alpha = G .$$

Then G is said to be a *transfinite extension* of the linearly ordered groups

$$H_\alpha = G_\alpha / \bigcup_{\gamma<\alpha} G_\gamma \quad (\alpha < \beta) .$$

Let X be a nonempty subclass of $\mathscr{G}_a$. We denote by Hom X the class of all homomorphic images of elements of X. Next we denote by Ext X the class of all linearly ordered groups G having the property that there exist linearly ordered groups H_α $(\alpha < \beta)$ such that each H_α is isomorphic to some element of X and that G is a transfinite extension of linearly ordered groups H_α $(\alpha < \beta)$.

We denote by $\mathscr{R}_a$ the collection of all radical classes in $\mathscr{G}_a$. The collection $\mathscr{R}_a$ is partially ordered by inclusion.

2.1. Theorem. (Cf. [6], Propos. 2.1 and Corollary 2.3.) *$\mathscr{R}_a$ is a complete lattice.*

Let A_i $(i \in I)$ be radical classes. Then

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i\,, \quad \bigvee_{i\in I} A_i = \operatorname{Ext}\left(\bigcup_{i\in I} A_i\right).$$

For $\emptyset \neq X \subseteq \mathcal{G}_a$ we denote by $T_a(X)$ the intersection of all radical classes Y with $X \subseteq Y$. Then $T_a(X)$ is said to be *the radical class generated by* X. For $X = \{G\}$ we write $T_a(X) = T_a(G)$; such a radical class is called *principal*.

2.2. Theorem. (Cf. [6], Propos. 2.2.) *Let $\emptyset \neq X \subseteq \mathcal{G}_a$. Then $T_a(X) = \operatorname{Ext} \operatorname{Hom} X$.*

For $G \in \mathcal{G}_a$ and $Y \in \mathcal{R}_a$ we denote by $Y(G)$ the join of all elements of $c(G)$ belonging to Y. Then $Y(G) \in Y$ (cf. [1], Propos. 3, or [6], Propos. 1.2). By applying the above terms, the notion of unequivocality can be defined as follows:

Let $G \in \mathcal{G}_a$. Then G is called *unequivocal* if for each $Y \in \mathcal{R}_a$ we have either $Y(G) = \{0\}$ or $Y(G) = G$.

Proof of Theorem 1.1.

Let G be unequivocal. Let H be a nonzero convex subgroup of G. Put $Y = T_a(H)$. Then we have $Y(H) = H \neq \{0\}$, hence $Y(G) = G$. Thus $G \in T_a(H)$. In view of 2.2 we obtain $G \in \operatorname{Ext} \operatorname{Hom} \{H\}$. Therefore the conditions (i) and (ii) from 1.1 are valid.

Conversely, assume that G is a linearly ordered group such that for each nonzero convex subgroup H of G the conditions (i) and (ii) from 1.1 are satisfied. Let $Y \in \mathcal{R}_a$ and suppose that $Y(G) \neq \{0\}$. Denote $Y(G) = H$ and $T_a(H) = Y_1$. In view of the conditions (i) and (ii), and by applying 2.2 we obtain $G \in Y_1$, hence $G \in Y$ and therefore $Y(G) = G$. Thus G is unequivocal.

2.3. Corollary. *Let $G \in \mathcal{G}_a$. Then the following conditions are equivalent:*

(i) *G is unequivocal.*

(ii) *For each nonzero convex subgroup H of G we have $X(G) = G$, where $X = T_a(H)$.*

Proof of Theorem 1.2.

The implication (i) $\Rightarrow$ (ii) is a consequence of 1.1. Suppose that (ii) is valid. By way of contradiction, assume that G fails to be unequivocal. Thus in view of 2.3 there exists a nonzero convex subgroup H_1 of G such that $G \notin T_a(H_1)$. Put $Y = T_a(H_1)$. Hence we have $H_1 \subseteq Y(G) \neq G$.

Denote $Y(G) = H$. According to (ii) there exists a nonzero homomorphic image H' of H which is isomorphic to a convex subgroup H'' of the linearly ordered group G/H. Let H^* be the set of all $g \in G$ such that $g + H$ belongs to H''. Then H^* is a convex subgroup of G, $H \subset H^*$ and H'' is isomorphic to H^*/H.

We have $H' \in T_a(H)$, since $T_a(H)$ is closed with respect to homomorphism. Also, $T_a(H) \subseteq T_a(H_1)$, since $H \in T_a(H_1)$. Thus $H' \in T_a(H_1)$. Hence $H'' \in T_a(H_1)$. Now from the relations

$$H \in T_a(H_1)\,, \quad H^*/H \in T_a(H_1)$$

and from the fact $T_a(H_1)$ is closed with respect to extensions we infer that H^* belongs to $T_a(H_1)$. Hence $H^* \subseteq Y(G) = H$, which is a contradiction.

The following proposition and its corollaries show a method of constructing new unequivocal linearly ordered groups from a given one.

2.4. Proposition. *Let $G \in \mathscr{G}_a$. Let $G_0, G_1, G_2, \ldots, G_n \in c(G)$ such that $\{0\} = G_0 \subset \subset G_1 \subset \ldots \subset G_n = G$. Assume that all linearly ordered groups G_i/G_{i-1} $(i = 1, 2, \ldots, n)$ are unequivocal and that $T_a(G_1) = T_a(G_i/G_{i-1})$ is valid for $i = 1, 2, \ldots, n$. Then G is unequivocal as well.*

Proof. Let H be a nonzero convex subgroup of G, $H \neq G$. Put $X = T_a(H)$. In view of 2.3 we have to verify that the relation $X(G) = G$ is valid.

There exists $m \in \{0, 1, 2, \ldots, n-1\}$ such that $G_m \subset H \subseteq G_{m+1}$. Hence there is $H_1 \in \operatorname{Hom} H$, $H_1 \neq \{0\}$ such that the linearly ordered group H_1 is isomorphic to some convex subgroup H_1' of G_{m+1}/G_m. Thus we have $H_1 \in X$ and $H_1' \in X$. Because G_{m+1}/G_m is unequivocal and $H_1' \neq \{0\}$ we infer that G_{m+1}/G_m belongs to X. Since $T_a(G_{m+1}/G_m) = T_a(G_1)$ and $T_a(G_{m+1}/G_m) \subseteq X$, we obtain $G_1 \in X$. Thus $G_i/G_{i-1} \in X$ for $i = 1, 2, \ldots, n$. Hence $G \in \operatorname{Ext} X = X$.

2.5. Corollary. *Let $G \in \mathscr{G}_a$. Let $G_0, G_1, G_2, \ldots, G_n \in c(G)$ such that $\{0\} = G_0 \subset \subset G_1 \subset G_2 \subset \ldots \subset G_n = G$. Assume that G_1 is unequivocal and that G_i/G_{i-1} is isomorphic to G_1 for $i = 2, 3, \ldots, n$. Then G is unequivocal.*

2.6. Corollary. *Let G_1 be a nonzero unequivocal linearly ordered group, $I = \{1, 2, \ldots, n\}$. For each $i \in I$ let A_i be a linearly ordered group isomorphic to G_1. Then the lexicographic product $G = A_1 \circ A_2 \circ \ldots \circ A_n$ is unequivocal.*

2.7. Proposition. *Let $G \in \mathscr{G}_a$. Let $G_0, G_1, G_2, \ldots$ be elements of $c(G)$ such that $\{0\} = G_0 \subset G_1 \subset G_2 \subset \ldots$ and $\bigcup_{n=1,2,\ldots} G_n = G$. Assume that all linearly ordered groups G_n/G_{n-1} are unequivocal and that $T_a(G_1) = T_a(G_n/G_{n-1})$ for each positive integer n. Then G is unequivocal.*

The proof is the same as in 2.4.

2.8. Corollary. *Let G_1 be a nonzero unequivocal linearly ordered group. Let I be the set of all positive integers and for each $i \in I$ let A_i be a linearly ordered group isomorphic to G_1. Then the lexicographic product $G = \Gamma_{i \in I} A_i$ is unequivocal as well.*

The following proposition shows that neither 2.7 nor 2.8 holds, in general, for the case of ordinals larger than ω.

Let α be an infinite cardinal and let $\omega(\alpha)$ be the first ordinal having the property that the power of the set of all ordinals less than $\omega(\alpha)$ is equal to α.

2.9. Proposition. *Let G_1 be a nonzero unequivocal linearly ordered group. Let α be a cardinal, $\alpha >$ card G_1. Let I be the set of all ordinals β with $\beta \leqq \omega(\alpha)$. For each $i \in I$ let A_i be a linearly ordered group isomorphic to G_1. Then the lexicographic product $G = \Gamma_{i \in I} A_i$ is not unequivocal.*

Proof. Let H be the set of all elements $f \in G$ such that $f(\omega(\alpha)) = 0$. Then H is a convex subgroup of G. If $\{0\} \neq H_1 \in \operatorname{Hom} \{H\}$, then card $H_1 \geqq \alpha$. Similarly, if

$\{0\} \neq H_2 \in \operatorname{Ext} \operatorname{Hom} \{H\}$, then card $H_2 \geqq \alpha$. Hence G_1 does not belong to $\operatorname{Ext} \operatorname{Hom} \{H\}$. On the other hand, $G_1 \in \operatorname{Hom} \{G\}$, thus G does not belong to $T_a(H) = \operatorname{Ext} \operatorname{Hom} \{H\}$. Therefore in view of 2.3, G is not unequivocal.

3. THE CASE WHEN $c(G)$ HAS AN ATOM

Let L be a lattice, x and y be elements of L with $x < y$. If card $[x, y] = 2$, then we write $x \prec y$. We also say that y *covers* x.

If $G \in \mathscr{G}_a$ and card $c(G) \leqq 2$, then G is said to be a *simple linearly ordered group*.

3.1. Lemma. *Assume that G is unequivocal. Let $c(G)$ have an atom. Then the following condition is satisfied:*

(a) *$c(G)$ is well-ordered. If $H \in c(G)$, $H \neq \{0\}$ qnd if $\{H_i\}_{i \in I}$ is the set of all elements of $c(G)$ with $H_i < H$, then either $H / \bigcup_{i \in I} H_i = \{0\}$ or $H / \bigcup_{i \in I} H_i$ is isomorphic to G_0, where G_0 is the atom in $c(G)$.*

Proof. In view of 2.3 we have $X(G) = G$, where $X = T_a(G_0)$, G_0 being the atom of $c(G)$. Moreover, G_0 is simple and thus $\operatorname{Hom} \{G_0\} = \{\{0\}, G_0\}$. Thus in view of 2.2 we obtain $G \in \operatorname{Ext} \{\{0\}, G_0\}$. In the case $G = G_0$ the condition (a) is obviously satisfied. Assume that $G_0 \subset G$. Hence there is an ordinal β and for each $\alpha < \beta$ there is $G_\alpha \in c(G)$ such that

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \beta)$$

where $\bigcup_{\alpha < \beta} G_\alpha = G$ and for each $\alpha < \beta$, the linearly ordered group $G_\alpha / \bigcup_{\gamma < \alpha} G_\gamma$ is isomorphic either to $\{0\}$ or to G_0.

Let $H \in c(G)$, $\{0\} \neq H \neq G$. There exists $\alpha < \beta$ such that G_α fails to be a subset of H; let α be the first ordinal having this property. Then

$$G_\alpha \subset H \subseteq G_{\alpha+1} .$$

Hence $G_{\alpha+1}/G_\alpha \neq \{0\}$ and thus $G_{\alpha+1}/G_\alpha$ is isomorphic to G_0. Because G_0 is simple, we must have $H = G_{\alpha+1}$. Thus

$$c(G) = \{\{0\}, G_0, G_1, G_2, \ldots, G_\alpha, \ldots, G\} .$$

We infer that the condition (a) is satisfied.

3.2. Lemma. *Assume that $c(G)$ is finite, $c(G) = \{\{0\}, G_0, G_1, G_2, \ldots, G_n\}$, $n \geqq 1$, where $\{0\} \prec G_0 \prec G_1 \prec G_2 \prec \ldots \prec G_n = G$. Then the following conditions are equivalent:*

(i) *G is unequivocal.*

(ii) *If $i \in \{0, 1, \ldots, n-1\}$, then the linearly ordered group G_{i+1}/G_i is isomorphic to G_0.*

Proof. The implication (ii) $\Rightarrow$ (i) follows from 2.4. In view of 3.1, the relation (i) $\Rightarrow$ (ii) holds.

3.3. Lemma. *Assume that $c(G)$ is a well-ordered set isomorphic to $\omega + 1$. Then G fails to be unequivocal.*

Proof. By way of contradiction, assume that G is unequivocal. The nonzero convex subgroups of G distinct from G can be indexed by the ordinals α with $\alpha \leqq \omega$ such that $\alpha(1) < \alpha(2)$ implies $G_{\alpha(1)} \subset G_{\alpha(2)}$. Put $H = G_\omega$. Then for no homomorphic image H' of H, $c(H')$ possesses a dual atom. On the other hand, $c(G)$ has a dual atom. Therefore G does not belong to Ext Hom $\{H\}$. According to 2.2, $G \notin T_a(H)$. Thus in view of 2.3, G is not unequivocal.

3.4. Theorem. *Assume that $c(G)$ has an atom. Then the following conditions are equivalent:*

(i) *G is hereditarily unequivocal.*

(ii) *either $c(G)$ is finite, $c(G) = \{\{0\}, G_0, G_1, \ldots, G_n\}$ with $\{0\} \prec G_0 \prec G_1 \prec$ $\prec G_2 \prec \ldots \prec G_n$, where G_{i+1}/G_i is isomorphic to G_0 for each $i = n - 1$,*

or $c(G) = \{\{0\}, G_0, G_1, G_2, \ldots\}$ with $\{0\} \prec G_0 \prec G_1 \prec G_2 \prec \ldots$, where G_{i+1}/G_i is isomorphic to G_0 for each positive integer i.

Proof. The implication (i) $\Rightarrow$ (ii) follows from 3.1 and 3.3. In view of 2.2 and 2.3, the relation (ii) $\Rightarrow$ (i) is valid.

Let $\mathscr{U}$ be the class of all nonisomorphic types of unequivocal linearly ordered groups. Further let $\mathscr{U}_1$ be a class containing exactly one linearly ordered group from each element of $\mathscr{U}$. We denote by $\mathscr{U}_{1a}$ the class of all linearly ordered groups $G \in \mathscr{U}_1$ such that $c(G)$ has an atom. Let $\mathscr{U}_{1ah}$ be the class of all elements of $\mathscr{U}_{1a}$ which are hereditarily unequivocal.

It will be shown that the class $\mathscr{U}_{1a}$ is rather large (it is a proper class in the sense that there exists an injective mapping of the class of all infinite cardinals into $\mathscr{U}_{1a}$) on the other hand, $\mathscr{U}_{1ah}$ fails to be a proper class.

Let α be an infinite cardinal. Let $\omega(\alpha)$ be as in Section 2. Let I be a linearly ordered set isomorphic to $\omega(\alpha)$. Let K be a nonzero archimedean linearly ordered group. For each $i \in I$ let A_i be a linearly ordered group isomorphic to K.

Consider the lexicographic product

$$G(K, \alpha) = \Gamma_{i \in I} A_i \,.$$

The following result is easy to verify.

3.5. Proposition. *The linearly ordered group $G(K, \alpha)$ is hereditarily unequivocal. If α and β are distinct infinite cardinals, then $G(K, \alpha)$ is not isomorphic to $G(K, \beta)$. The linearly ordered set $c(G(K, \alpha))$ has an atom.*

For an infinite cardinal α we denote by $f(\alpha)$ the linearly ordered group which belongs to $\mathscr{U}_{1a}$ and is isomorphic to $G(K, \alpha)$. Then in view of 3.5, f is an injective mapping of the class of all infinite cardinals into the class $\mathscr{U}_{1a}$.

Now let $G \in \mathscr{U}_{1ah}$. Then the condition (ii) from 3.4 is satisfied. The linearly ordered group G_0 is simple and thus it is archimedean. Hence G_0 is isomorphic to a subgroup of the linearly ordered group of all reals. Therefore card $G_0 \leqq c$ (the power of the continuum). Thus in view of 3.4 we obtain:

3.6. Proposition. *Let G be a hereditarily unequivocal linearly ordered group. Then* card $G \leqq c^{\aleph_0} = c$.

From 3.6 we infer that $\mathscr{U}_{1ah}$ fails to be a proper class.

4. EXISTENCE OF A DUAL ATOM IN $c(G)$

In this section we deal with linearly ordered groups G such that $c(G)$ has a dual atom.

4.1. Lemma. *Let G^0 be a dual atom of $c(G)$. Assume that G is unequivocal. Then*

(i) *$c(G)$ is dually well-ordered;*

(ii) *if H_1 and H_2 are elements of $c(G)$ such that $H_1 \prec H_2$, then the linearly ordered group H_2/H_1 is isomorphic to G/G^0.*

Proof. By way of contradiction, suppose that (i) does not hold. Then there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of elements of $c(G)$ such that $A_n \subset A_{n+1} \neq G$ is valid for each positive integer n. Put $A = \bigcup_{n=1,2,\ldots} A_n$. We have $A \in c(G)$ and $A \neq G$. If A' is a homomorphic image of A with $A' \neq \{0\}$, then $c(A')$ has no dual atom. Hence if A'' is a nonzero element of Ext Hom $\{A\}$, then $c(A'')$ has no dual atom. Therefore G does not belong to Ext Hom $\{A\}$. Thus in view of 2.2 and 2.3, G fails to be unequivocal, which is a contradiction.

Let H_1 and H_2 be elements of $c(G)$ such that $H_1 \prec H_2$. Because G is unequivocal, we have $G \in T_a(H_2) =$ Ext Hom $\{H_2\}$. Therefore, in view of the fact that $c(G)$ has a dual atom, there exist $K \in c(G)$ and $K_1 \in c(H_2)$ such that $K \neq G$, $K_1 \neq H_2$ and G/K is isomorphic to H_2/K_1. Under such an isomorphism, the linearly ordered group G^0/K corresponds to H_1/K_1. Hence the linearly ordered groups

$$(G/K)/(G^0/K) \quad \text{and} \quad (H_2/K_1)/(H_1/K_1)$$

are isomorphic. Therefore the linearly ordered groups G/G^0 and H_2/H_1 are isomorphic as well.

A convex subgroup H of G is said to be an *upper limit element of* $c(G)$ if there are $H_n \in C(G)$ $(n = 1, 2, \ldots)$ such that $A_1 \subset A_2 \subset \ldots$ and $\bigcup_{n=1} A_n = H$.

4.2. Theorem. *Assume that $c(G)$ has a dual atom G_0. Then the following conditions are equivalent:*

(a) *The conditions* (i) *and* (ii) *from* 4.1 *are valid. If H is a nonzero upper limit element of $c(G)$, then the condition from* 1.2, (ii) *holds for H.*

(b) *G is unequivocal.*

Proof. Let (b) be valid. Then in view of 4.1 and 1.2 the condition (a) is satisfied. Conversely, assume that (a) holds. In view of 1.2 it suffices to verify that whenever H

is a nonzero convex subgroup of G such that H is not an upper limit element of $c(G)$ and $H \neq G$, then the condition from 1.2, (ii) is valid for H.

Thus let $\{0\} \neq H \in c(H)$ and assume that H fails to be an upper limit element of G, $H \neq G$. Hence there exists $H_1 \in c(G)$ with $H \prec H_1$. In view of 4.1, (i), there is $H_2 \in c(G)$ such that $H_2 \in H$. According to 4.1 (ii), the linearly ordered group H_1/H and H/H_2 are isomorphic, which completes the proof.

4.3. Corollary. *Let I be a dually well-ordered set,* card $I \geqq 2$. *Assume that I has the least element. There exists a hereditarily unequivocal linearly ordered group G such that the linearly ordered set $c(G)$ is isomorphic to I.*

Proof. Let A be a nonzero archimedean linearly ordered set. For each $i \in I$ let G_i be a linearly ordered group isomorphic to I. Put

$$G = \Gamma_{i \in I}\, G_i\,.$$

Then we have $c(G) \simeq I$. According to 4.2, G is hereditarily unequivocal.

From 4.2 and 3.1 we obtain:

4.4. Corollary. *Let G be a linearly ordered group, $G \neq \{0\}$. Then the following conditions are equivalent:*

(i) *G is unequivocal and $c(G)$ has an atom and a dual atom.*

(ii) *$c(G)$ is finite and whenever H_i and H_j are elements of $c(G)$ such that $H_i \prec H_j$, then the linearly ordered group H_j/H_i is isomorphic to G_0, where G_0 is the atom of $c(G)$.*

4.5. Corollary. *Let G be a linearly ordered group satisfying the condition* (ii) *from* 4.4. *Then* card $G \leqq c$.

Let us remark that there exists a nonzero unequivocal linearly ordered group G such that $c(G)$ has neither an atom nor a dual atom.

4.6. Example. Let R be the additive group of all reals with the natural linear order. Let I be a linearly ordered set isomorphic to R and for each $i \in I$ let A_i be a linearly ordered group which is isomorphic to the linearly ordered group R. Put $G = \Gamma_{i \in I} A_i$. Then G is unequivocal, neither an atom nor a dual atom does exist in $c(G)$.

Let G be a linearly ordered group and let $H_1, H_2 \in c(G)$ such that $H_1 \prec H_2$. Then the linearly ordered group H_2/H_1 will be said to be *p-factor of* G.

From 3.1 and 4.1 we infer:

4.7. Proposition. *Let G be a linearly ordered group such that either an atom or a dual atom does exist in $c(G)$. Then any two p-factors of G are isomorphic.*

If neither an atom nor a dual atom does exist in G, then the assertion of the above proposition need not hold.

4.8. Example. Let R be as in 4.6. Let R_0 be the subgroup of R consisting of all rational numbers. Let I be as in 4.6 and for each $i \in I$ let A_i be a linearly ordered group such that A_i is isomorphic to R_0 if i is rational and A_i is isomorphic to R otherwise. Put $G = \Gamma_{i \in I} A_i$. Then there exist p-factors B_1 and B_2 in G such that B_1 is isomorphic to R_0 and B_2 is isomorphic to R. In fact, each p-factor of G is isomorphic either to R_0 or to R. The linearly ordered group G is unequivocal.

By a method similar to that applied in 4.8 we can construct a linearly ordered group G which is unequivocal and possesses an infinite number of mutually nonisomorphic p-factors.

Let G_1 and G_2 be unequivocal linearly ordered groups such that $c(G_1)$ has a dual atom, $c(G_2)$ is isomorphic to $c(G)$ and that, if C_1 is a p-factor of G_1 and C_2 is a p-factor of G_2, then C_1 is isomorphic to C_2. The linearly ordered groups G_1 and G_2 need not be isomorphic.

4.9. Example. Let R be as in 4.6. Let α be an infinite cardinal and let I be a linearly ordered set dually isomorphic to $\omega(\alpha)$. For each $i \in I$ let A_i be a linearly ordered group isomorphic to R. Put $G_1 = \Gamma_{i \in I} A_i$. Let G_2 be the subgroup of G_1 consisting of all elements of G_1 with finite support. Then both G_1 and G_2 are unequivocal and $c(G_1) \simeq c(G_2)$. G_1 is not isomorphic to G_2. Each p-factor of G_i $(i = 1, 2)$ is isomorphic to R.

5. HEREDITARILY UNEQUIVOCAL LINEARLY ORDERED GROUPS

In this section the structure of a hereditarily unequivocal linearly ordered group G will be investigated (without assuming that $c(G)$ has an atom or a dual atom).

5.1. Lemma. *Let G be a hereditarily unequivocal linearly ordered group. Let $H \in c(G)$ such that $\{0\} \neq H \neq G$. Then there exists $H_1 \in c(G)$ such that $H_1 \prec H$.*

Proof. By way of contradiction, suppose that there does not exist any H_1 with the mentioned property. Then for each $H_2 \in c(H)$ with $H_2 \neq H$ the interval $[H_2, H]$ of $c(H)$ is infinite.

Since $H \neq G$, there exist H_3 and H_4 in $c(G)$ such that $H \leqq H_3 \prec H_4$ is valid in $c(G)$. Since G is hereditarily unequivocal, H_4 must be unequivocal and thus $H_4 \in T_a(H) = \text{Ext Hom}\,\{H\}$. Because $c(H_4)$ has a dual atom there must exist a homomorphic image $K \neq \{0\}$ of H and a homomorphic image K' of H_4 such that K is isomorphic to K'. But $c(K)$ has no dual atom and $c(K')$ possesses a dual atom, which is a contradiction.

5.2. Corollary. *Let G and H be as in* 5.1. *Then the interval $[H, G]$ of $c(G)$ is either finite or is isomorphic to the linearly ordered set of all ordinals α with $\alpha \leqq \omega$.*

From 5.1 and 5.2 we obtain:

5.3. Theorem. *Let $G \neq \{0\}$ be a linearly ordered group. Assume that G is hereditarily unequivocal. Then one of the following conditions is valid:*

(i) *The linearly ordered set $c(G)$ is dually well-ordered.*

(ii) *There exist linearly ordered sets* L_1 *and* L_2 *such that*

a) L_1 *is dually well-ordered;*

b) L_2 *is isomorphic to the set of all ordinals equal or less than* ω*;*

c) $c(G)$ *is isomorphic to* $L_1 \oplus L_2$.

Moreover, any two p-factors of G *are isomorphic.*

5.4. Proposition. *Let* L_1 *and* L_2 *be linearly ordered sets such that the conditions* a) *and* b) *from* 5.3 *are satisfied. Then there exists a linearly ordered group* G *such that*

(i) G *is hereditarily unequivocal,*

(ii) *the condition* c) *from* 5.3 *holds.*

Proof. Let $I = L_1 \oplus L_2$ and for each $i \in I$ let A_i be a linearly ordered group isomorphic to R. Put $G = \Gamma_{i \in I} A_i$. By applying 1.2 we easily obtain that G is hereditarily unequivocal. The validity of (ii) is obvious.

Let $\mathscr{H}$ be the collection of all radical classes $T_a(G)$, where G is a hereditarily unequivalent linearly ordered group. The collection $\mathscr{H}$ is partially ordered by inclusion. Put $0^- = T_a(\{0\})$. Then 0^- is the least element of $\mathscr{H}$.

5.5. Proposition. *The partially ordered collection* $\mathscr{H}$ *has no maximal element.*

Proof. Let $T_a(G) \in H$, $G \neq \{0\}$. Let α be a cardinal, $\alpha > c(G)$. Let I be a linearly ordered set which is dually isomorphic to $\omega(\alpha)$. For each $i \in I$ let A_i be a linearly ordered group isomorphic to R_1, where R_1 is a p-factor of G. Put $G_1 = \Gamma_{i \in I} A_i$. If H is a nonzero convex subgroup of G_1, then card $c(H) = \alpha$. Thus no homomorphic image of G is isomorphic to H. Hence G_1 does not belong to Ext Hom $\{G\} = T_a(G)$.

Denote $G_2 = G_0 \circ G_1$. Then $G \in \text{Hom}\{G_2\}$, whence $T_a(G) \leqq T_a(G_2)$. In view of 5.4 and 2.4, G_2 is hereditarily unequivocal. We have $G_2 \notin T_a(G)$, whence $T_a(G) < < T_a(G_2)$.

Since the cardinal α in the above proof can be chosen as to be arbitrarily large, Proposition 5.5 can be sharpened as follows.

5.5.1. Proposition. *For each element* $T_a(G)$ *of* $\mathscr{H}$ *there exists* $\mathscr{H}_1 \subset \mathscr{H}$ *such that*

(i) $\mathscr{H}_1$ *is a proper collection and* $\mathscr{H}_1$ *is linearly ordered;*

(ii) $T_a(G)$ *is the least element of* $\mathscr{H}_1$.

The proof will be omitted.

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