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ON VERONESE SURFACES

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0. The following result is known, see [1]: Let (M, g) be a closed, connected 2-dimensional manifold with curvature K. Let (i) $\frac{1}{3} \leqq K \leqq 1$ or (ii) $\frac{1}{6} \leqq K \leqq \frac{1}{3}$ resp. If $\sigma: M \to S^N(1)$ is a minimal isometric immersion then $K = 1$ or $K = \frac{1}{3}$ in the case (i) and $K = \frac{1}{3}$ or $K = \frac{1}{6}$ in the case (ii). For $K = \frac{1}{3}$, $\sigma(M) \subset S^4(1) \subset S^N(1)$ is a Veronese surface.

Here, I study minimal immersions $\sigma: M \to S^4(1)$. To each such immersion, I associate a normal vector bundle of $\sigma(M)$ and its curvature k. If K and k satisfy certain inequalities, $\sigma(M)$ is a Veronese surface as well.

1. Let M be a 2-dimensional manifold, $\sigma: M \to S^4(1)$ an immersion into the 4-dimensional unit sphere of the real Euclidean space $\mathbb{R}^5$. To each point $m_0 \in M$, let us associate an orthonormal frame $\{m; v_1, \ldots, v_5\}$ of $\mathbb{R}^5$ such that $m = \sigma(m_0)$; $v_1, v_2 \in T_m(\sigma(M))$; $m + v_5 =$ the center of $S^4(1)$. Then we have the fundamental equations of our moving frames

$$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2 , \tag{1.1}$$

$$\mathrm{d}v_1 = \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 + \omega^1 v_5 , \quad \mathrm{d}v_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 + \omega^2 v_5 ,$$

$$\mathrm{d}v_3 = -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 , \quad \mathrm{d}v_4 = -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_4 ,$$

$$\mathrm{d}v_5 = -\omega^1 v_1 - \omega^2 v_2$$

with the integrability conditions $(\omega_i^j = -\omega_j^i)$

$$\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i , \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j . \tag{1.2}$$

From

$$\omega^3 = \omega^4 = \omega^5 = 0 , \tag{1.3}$$

we get

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0 \tag{1.4}$$

and the existence of the functions $a_1, \ldots, b_3$ such that

$$\omega_1^3 = a_1\omega^1 + a_2\omega^2 , \quad \omega_1^4 = b_1\omega^1 + b_2\omega^2 ,$$
$$\omega_2^3 = a_2\omega^1 + a_3\omega^2 , \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 . \tag{1.5}$$

From (1.1) and (1.5), we get

(1.6) $$\Delta m = (a_1 + a_3) v_3 + (b_1 + b_3) v_4 + 2v_5 ,$$

Δ being the Laplace operator. The mapping σ is called a *minimal immersion* if the vector Δm is a multiple of v_5, i.e., if

(1.7) $$a_1 + a_3 = b_1 + b_3 = 0 .$$

In what follows, let us restrict ourselves to minimal immersions.

Around the point m, consider a field of tangent unit vectors

(1.8) $$t = xv_1 + yv_2 ; \quad x^2 + y^2 = 1 .$$

By ∇, we denote the symbol of the covariant differentiation associated to the induced metric

(1.9) $$ds^2 = (\omega^1)^2 + (\omega^2)^2 .$$

Then it is easy to see that

(1.10) $$\nabla_t t = (\cdot) v_1 + (\cdot) v_2 + (\cdot) v_5 + \{a_1(x^2 - y^2) + 2a_2 xy\} v_3 + \\ + \{b_1(x^2 - y^2) + 2b_2 xy\} v_4 .$$

Each unit vector t (1.8) is thus mapped into the point

(1.11) $$m + \xi v_3 + \eta v_4 ; \quad \xi = a_1(x^2 - y^2) + 2a_2 xy , \quad \eta = b_1(x^2 - y^2) + 2b_2 xy$$

of the plane $v_m = \{m; v_3, v_4\}$ of the *normal bundle* v of $\sigma(M)$. The points (1.11) form, for m fixed and all t's, the ellipse

(1.12) $$(b_1^2 + b_2^2) \xi^2 - 2(a_1 b_1 + a_2 b_2) \xi\eta + (a_1^2 + a_2^2) \eta^2 = (a_1 b_2 - a_2 b_1)^2 ,$$

the so-called *indicatrix of normal curvature.*

The *Gauss curvature* K and the *curvature of the normal bundle* k are defined by

(1.13) $$d\omega_1^2 = -K\omega^1 \wedge \omega^2 , \quad d\omega_3^4 = -k\omega^1 \wedge \omega^2$$

resp; we get

(1.14) $$K = 1 - a_1^2 - a_2^2 - b_1^2 - b_2^2 , \quad k = 2(a_1 b_2 - a_2 b_1) .$$

The *Veronese surface* is defined as follows: In the Euclidean 3-space $\mathbb{R}^3$, consider orthonormal coordinates (x, y, z) and the mapping $S^2(\sqrt{3}) \to S^4(1)$ given by

(1.15) $$u_1 = \tfrac{1}{3}\sqrt{3} \, . \, yz , \quad u_2 = \tfrac{1}{3}\sqrt{3} \, . \, xz , \quad u_3 = \tfrac{1}{3}\sqrt{3} \, . \, xy , \\ u_4 = \tfrac{1}{6}\sqrt{3} \, . \, (x^2 - y^2) , \quad u_5 = \tfrac{1}{6}(x^2 + y^2 - 2z^2) ,$$

$(u_1, \ldots, u_5)$ being orthonormal coordinates in $\mathbb{R}^5$. To each point of the Veronese surface, we may associate orthonormal frames such that we get (1.1) with

(1.16) $$\omega_1^3 = -\omega_2^4 = \tfrac{1}{3}\sqrt{3} \, . \, \omega^2 , \quad \omega_1^4 = \omega_2^3 = \tfrac{1}{3}\sqrt{3} \, . \, \omega^1 , \quad \omega_3^4 = -2\omega_1^2 ;$$

see [2]. For our Veronese surface, we get

(1.17) $$K = \tfrac{1}{3} , \quad k = -\tfrac{2}{3} .$$

2. We are going to prove the following (auxiliary)

Theorem 1. *Let* $\sigma\colon M \to S^4(1)$, $\dim M = 2$, *be a minimal immersion; let M be compact. If*

(2.1) $$2K > k$$

on M, the indicatrices of normal curvature are circles.

Proof. Let us start with the equations (1.5) + (1.7). The differential consequences being

(2.2) $$\begin{aligned}
&(\mathrm{d}a_1 - 2a_2\omega_1^2 - b_1\omega_3^4) \wedge \omega^1 + (\mathrm{d}a_2 + 2a_1\omega_1^2 - b_2\omega_3^4) \wedge \omega^2 = 0\,,\\
&(\mathrm{d}a_2 + 2a_1\omega_1^2 - b_2\omega_3^4) \wedge \omega^1 - (\mathrm{d}a_1 - 2a_2\omega_1^2 - b_1\omega_3^4) \wedge \omega^2 = 0\,,\\
&(\mathrm{d}b_1 - 2b_2\omega_1^2 + a_1\omega_3^4) \wedge \omega^1 + (\mathrm{d}b_2 + 2b_1\omega_1^2 + a_2\omega_3^4) \wedge \omega^2 = 0\,,\\
&(\mathrm{d}b_2 + 2b_1\omega_1^2 + a_2\omega_3^4) \wedge \omega^1 - (\mathrm{d}b_1 - 2b_2\omega_1^2 + a_1\omega_3^4) \wedge \omega^2 = 0\,,
\end{aligned}$$

we get the existence of functions $\alpha_1, \ldots, \beta_2$ such that

(2.3) $$\begin{aligned}
\mathrm{d}a_1 - 2a_2\omega_1^2 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2\,,\\
\mathrm{d}b_1 - 2b_2\omega_1^2 + a_1\omega_3^4 &= \beta_1\omega^1 + \beta_2\omega^2\,,\\
\mathrm{d}a_2 + 2a_1\omega_1^2 - b_2\omega_3^4 &= \alpha_2\omega^1 - \alpha_1\omega^2\,,\\
\mathrm{d}b_2 + 2b_1\omega_1^2 + a_2\omega_3^4 &= \beta_2\omega^1 - \beta_1\omega^2\,.
\end{aligned}$$

From this,

(2.4) $$\begin{aligned}
&\mathrm{d}(a_1 + b_2) - (a_2 - b_1)(2\omega_1^2 - \omega_3^4) = A_1\omega^1 + A_2\omega^2\,,\\
&\mathrm{d}(a_2 - b_1) + (a_1 + b_2)(2\omega_1^2 - \omega_3^4) = A_2\omega^1 - A_1\omega^2\,;\\
&A_1 := \alpha_1 + \beta_2\,,\quad A_2 := \alpha_2 - \beta_1\,.
\end{aligned}$$

The exterior differentiation of (2.4) yields

(2.5) $$\begin{aligned}
&\{\mathrm{d}A_1 - A_2(3\omega_1^2 - \omega_4^3)\} \wedge \omega^1 + \{\mathrm{d}A_2 + A_1(3\omega_1^2 - \omega_3^4)\} \wedge \omega^2 =\\
&\quad = (2K - k)(a_2 - b_1)\,\omega^1 \wedge \omega^2\,,\\
&\{\mathrm{d}A_2 + A_1(3\omega_1^2 - \omega_3^4)\} \wedge \omega^1 - \{\mathrm{d}A_1 - A_2(3\omega_1^2 - \omega_3^4)\} \wedge \omega^2 =\\
&\quad = (k - 2K)(a_1 + b_2)\,\omega^1 \wedge \omega^2\,.
\end{aligned}$$

The function f being defined by

(2.6) $$2f = (a_1 + b_2)^2 + (a_2 - b_1)^2\,,$$

we have

$$\mathrm{d}f = \{(a_1 + b_2)A_1 + (a_2 - b_1)A_2\}\,\omega^1 + \{(a_1 + b_2)A_2 - (a_2 - b_1)A_1\}\,\omega^2$$

and

(2.7) $$\mathrm{d} * \mathrm{d}f = 2\{A_1^2 + A_2^2 + (2K - k)f\}\,\omega^1 \wedge \omega^2\,.$$

The supposition (2.1) and the Stokes theorem (or the maximum principle) imply $f \equiv 0$, i.e.,

(2.8) $$b_1 = a_2\,,\quad b_2 = -a_1\,.$$

Now, look at (1.12). QED.

3. Let us prove our main Theorems.

Theorem 2. *Let* $\sigma: M \to S^4(1)$, $\dim M = 2$, *be a minimal immersion*; *let* M *be compact. If*

$$2K > k \geqq -2K \tag{3.1}$$

on M, *there are just two cases possible*: (i) $K = 1$, $k = 0$, *and* $\sigma(M)$ *is a great sphere*; (*ii*) $K = \frac{1}{3}$, $k = -\frac{2}{3}$, *and* $\sigma(M)$ *is the Veronese surface.*

Proof. Theorem 1 implies (2.8), and the equations (2.3) reduce to

$$\begin{aligned} da_1 - a_2(2\omega_1^2 + \omega_3^4) &= \alpha_1\omega^1 + \alpha_2\omega^2 , \\ da_2 + a_1(2\omega_1^2 + \omega_3^4) &= \alpha_2\omega^1 - \alpha_1\omega^2 . \end{aligned} \tag{3.2}$$

The differential consequences are

$$\begin{aligned} &\{d\alpha_1 - \alpha_2(3\omega_1^2 + \omega_3^4)\} \wedge \omega^1 + \{d\alpha_2 + \alpha_1(3\omega_1^2 + \omega_3^4)\} \wedge \omega^2 = \\ &\quad = (2K + k)\, a_2\omega^1 \wedge \omega^2 , \\ &\{d\alpha_2 + \alpha_1(3\omega_1^2 + \omega_3^4)\} \wedge \omega^1 - \{d\alpha_1 - \alpha_2(3\omega_1^2 + \omega_3^4)\} \wedge \omega^2 = \\ &\quad = -(2K + k)\, a_1\omega^1 \wedge \omega^2 , \end{aligned} \tag{3.3}$$

and we get the existence of functions α_{ij} such that

$$\begin{aligned} d\alpha_1 - \alpha_2(3\omega_1^2 + \omega_3^4) &= \alpha_{11}\omega^1 + \alpha_{12}\omega^2 , \\ d\alpha_2 + \alpha_1(3\omega_1^2 + \omega_3^4) &= \alpha_{21}\omega^1 + \alpha_{22}\omega^2 ; \\ \alpha_{21} - \alpha_{12} = (2K + k)\, a_2 , \quad &\alpha_{11} + \alpha_{22} = (2K + k)\, a_1 . \end{aligned} \tag{3.4}$$

For the function g defined by

$$2g = a_1^2 + a_2^2 , \tag{3.5}$$

we get

$$\begin{aligned} dg &= (a_1\alpha_1 + a_2\alpha_2)\, \omega^1 + (a_1\alpha_2 - a_2\alpha_1)\, \omega^2 \\ d * dg &= \{2(\alpha_1^2 + \alpha_2^2) + (2K + k)(a_1^2 + a_2^2)\}\, \omega^1 \wedge \omega^2 . \end{aligned} \tag{3.6}$$

The supposition $2K + k \geqq 0$ and the Stokes theorem (or the maximum principle as well) imply $\alpha_1 = \alpha_2 = 0$ and $a_1^2 + a_2^2 = \text{const.}$

First of all, let $a_1 = a_2 = 0$. Then the equations (1.1) reduce to

$$\begin{aligned} &dm = \omega^1 v_1 + \omega^2 v_2 , \quad dv_1 = \omega_1^2 v_2 + \omega^1 v_5 , \quad dv_2 = -\omega_1^2 v_1 + \omega^2 v_5 , \\ &dv_3 = \omega_3^4 v_4 , \quad dv_4 = -\omega_3^4 v_3 , \quad dv_5 = -\omega^1 v_1 - \omega^2 v_2 , \end{aligned} \tag{3.7}$$

and $\sigma(M)$ is the sphere $S^2(1)$ in the (fixed) space $\mathbb{R}^3$ through the center of $S^4(1)$ spanned by the vectors v_1, v_2, v_5.

Now, let $a_1^2 + a_2^2 \neq 0$. Then

$$2K + k = 0 \tag{3.8}$$

from the integral formula based on (3.6). From (2.8) and (1.14),

$$K = 1 - 2(a_1^2 + a_2^2) , \quad k = -2(a_1^2 + a_2^2) . \tag{3.9}$$

Inserting this into (3.8), we get $a_1^2 + a_2^2 = \frac{1}{3}$ and $K = \frac{1}{3}$, $k = -\frac{2}{3}$. To the points of our surface $\sigma(M)$, let us associate the frames $\{m; v_1, v_2, v_3^*, v_4^*, v_5\}$ with

$$(3.10) \qquad v_3^* = \sqrt{3}\,.\,(a_2 v_3 - a_1 v_4)\,, \quad v_4^* = \sqrt{3}\,.\,(a_1 v_3 + a_2 v_4)\,.$$

By a direct calculation, we get the fundamental equations (1.1) with (1.16), we have just to replace v_3, v_4 by v_3^*, v_4^* resp. Thus $\sigma(M)$ is the Veronese surface. QED.

Theorem 3. *Let $\sigma: M \to S^4(1)$, $\dim M = 2$, be a minimal immersion; let M be compact. Suppose, on M, $K > 0$ and*

$$(3.11) \qquad -\tfrac{7}{3}\min_M K < \min_M k \leqq \max_M k \leqq -2\max_M K\,.$$

Then $\sigma(M)$ is the Veronese surface.

Proof. From $K > 0$ and (3.11), we get (2.1), and we may use the equations (3.2) and (3.4). The prolongation of $(3.4_{1,2})$ yields

$$(3.12) \qquad \begin{aligned} &\{d\alpha_{11} - (\alpha_{12} + 3\alpha_{21})\,\omega_1^2 - \alpha_{21}\omega_3^4\} \wedge \omega^1 + \\ &+ \{d\alpha_{12} + (\alpha_{11} - 3\alpha_{22})\,\omega_1^2 - \alpha_{22}\omega_3^4\} \wedge \omega^2 = (3K + k)\,\alpha_2\omega^1 \wedge \omega^2\,, \\ &\{d\alpha_{21} + (3\alpha_{11} - \alpha_{22})\,\omega_1^2 + \alpha_{11}\omega_3^4\} \wedge \omega^1 + \\ &+ \{d\alpha_{22} + (\alpha_{21} + 3\alpha_{12})\,\omega_1^2 + \alpha_{12}\omega_3^4\} \wedge \omega^2 = -(3K + k)\,\alpha_1\omega^1 \wedge \omega^2 \end{aligned}$$

and the existence of functions α_{ijk} such that

$$(3.13) \qquad \begin{aligned} d\alpha_{11} - (\alpha_{12} + 3\alpha_{21})\,\omega_1^2 - \alpha_{21}\omega_3^4 &= \alpha_{111}\omega^1 + \alpha_{112}\omega^2\,, \\ d\alpha_{12} + (\alpha_{11} - 3\alpha_{22})\,\omega_1^2 - \alpha_{22}\omega_3^4 &= \alpha_{121}\omega^1 + \alpha_{122}\omega^2\,, \\ d\alpha_{21} + (3\alpha_{11} - \alpha_{22})\,\omega_1^2 + \alpha_{11}\omega_3^4 &= \alpha_{211}\omega^1 + \alpha_{212}\omega^2\,, \\ d\alpha_{22} + (\alpha_{21} + 3\alpha_{12})\,\omega_1^2 + \alpha_{12}\omega_3^4 &= \alpha_{221}\omega^1 + \alpha_{222}\omega^2\,; \end{aligned}$$

$$(3.14) \qquad \alpha_{121} - \alpha_{112} = (3K + k)\,\alpha_2\,, \quad \alpha_{212} - \alpha_{221} = (3K + k)\,\alpha_1\,.$$

The differential consequences of $(3.4_{3,4})$ are then

$$(3.15) \qquad \begin{aligned} \alpha_{211} - \alpha_{121} &= (2K_1 + k_1)\,a_2 + (2K + k)\,\alpha_2\,, \\ \alpha_{212} - \alpha_{122} &= (2K_2 + k_2)\,a_2 - (2K + k)\,\alpha_1\,, \\ \alpha_{111} + \alpha_{221} &= (2K_1 + k_1)\,a_1 + (2K + k)\,\alpha_1\,, \\ \alpha_{112} + \alpha_{222} &= (2K_2 + k_2)\,a_1 + (2K + k)\,\alpha_2\,, \end{aligned}$$

the first covariant derivatives of K and k being defined by

$$(3.16) \qquad dK = K_1\omega^1 + K_2\omega^2\,, \quad dk = k_1\omega^1 + k_2\omega^2$$

resp.

By a direct calculation, we get, for each $r \in \mathbb{R}$,

$$(3.17) \qquad \begin{aligned} &\tfrac{1}{2}d * \{d(\alpha_1^2 + \alpha_2^2) - (2K + k + r)\,d(a_1^2 + a_2^2)\} = \\ &= \{\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{12}^2 + \alpha_{22}^2 + (3K + k - 2r)(\alpha_1^2 + \alpha_2^2) - \\ &- (2K + k)(2K + k + r)(a_1^2 + a_2^2)\}\,\omega^1 \wedge \omega^2\,, \end{aligned}$$

and the corresponding integral formula (in the case of M being non-orientable, we pass to its universal covering $M^* \to M$). Let us take

$$r = -\tfrac{1}{4}(\min_M K + \min_M k) . \tag{3.18}$$

Then using (3.11),

$$\begin{aligned} 2K + k &\leqq 2 \max_M K + \max_M k \leqq 0 , \\ 3K + k - 2r &= 3(K - \min_M K) + (k - \min_M k) + \\ &\quad + \tfrac{3}{2}(\min_M k + \tfrac{7}{3} \min_M K) > 0 , \\ 2K + k + r &= 2(K - \min_M K) + (k - \min_M k) + \\ &\quad + \tfrac{3}{4}(\min_M k + \tfrac{7}{3} \min_M K) > 0 . \end{aligned} \tag{3.19}$$

Because of this, our integral formula yields $\alpha_{ij} = 0$ and

$$\alpha_1 = \alpha_2 = 0 , \tag{3.20}$$

$$(2K + k)(a_1^2 + a_2^2) = 0 . \tag{3.21}$$

Thus $a_1^2 + a_2^2 =$ const. In the case $a_1 = a_2 = 0$, we get $K = 1$, $k = 0$, a contradiction to (3.11). Thus $a_1^2 + a_2^2 \neq 0$, and (3.21) implies (3.8). Now follow the proof of the preceding Theorem. QED.

References

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