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TOLERANCES AND CONGRUENCES ON IMPLICATION ALGEBRAS

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1. INTRODUCTION

The concept of an *implication algebra* was introduced by J. C. Abbott [1]. It is a groupoid $\langle I, \cdot \rangle$ with the support I and one binary operation (denoted by a dot or by a simple juxtaposition) which satisfies the following axioms:

I 1: $(ab)\,a = a$;
I 2: $(ab)\,b = (ba)\,a$;
I 3: $a(bc) = b(ac)$

(The elements a, b, c are always arbitrary elements of I.)

In [1] some fundamental properties of implication algebras are shown. We shall quote some of them.

In every implication algebra $\langle I, \cdot \rangle$ there exists an element denoted by 1 and satisfying the equality $aa = 1$ for each $a \in I$. The existence of such an element enables to introduce a partial ordering $\leqq$ on I such that $a \leqq b$ if and only if $ab = 1$. The element 1 has the property that $1a = a$, $a1 = 1$ for each $a \in I$ and it is the greatest element in the ordering $\leqq$. The set I with the ordering $\leqq$ forms a join semilattice (in general not a lattice).

Let $[a]\uparrow = \{x \in I \mid x \geqq a\}$. The set $[a]\uparrow$ is called a *principal filter* of $\langle I, \cdot \rangle$ and it is a Boolean algebra with respect to the restriction of the ordering $\leqq$ onto $[a]\uparrow$. All Boolean algebras $[a]\uparrow$ for $a \in I$ have the common greatest element 1. Hence every implication algebra is a union of Boolean algebras with a common greatest element. Then ab can be interpreted as the complement of $a \vee b$ (where $\vee$ denotes the join in $\langle I, \cdot \rangle$ as in a join semilattice) in the filter $[a]\uparrow$.

In particular, every Boolean algebra $\langle B, \vee, \wedge, ', 0, 1 \rangle$ can be considered as an implication algebra $\langle I, \cdot \rangle$, where $I = B$ and $ab = a' \vee b$ for any $a \in B$, $b \in B$. The unit element 1 of B is also the element 1 of the implication algebra. The term "implication algebra" has its origin in this fact. Namely, the statement "a implies b" is the disjunction of two statements, one of which is the negation of a and the other is b; hence it corresponds to the expression $a' \vee b$ in a Boolean algebra.

Since every implication algebra is a union of Boolean algebras, one can expect that the variety of all implication algebras would have similar properties as a variety of Boolean algebras. We shall give a list of congruence properties of this variety which show differences.

Theorem 1. *The variety of all implication algebras has the property that the congruence lattice of any of its elements is distributive.*

Proof. Put $n = 3$, $p_0(x, y, z) = x$, $p_1(x, y, z) = (y(zx))\, x$, $p_2(x, y, z) = (xy)\, z$, $p_3(x, y, z) = z$. Then we have

$$\begin{aligned} &p_0(x, y, x) = x, \\ &p_1(x, y, x) = (y(xx))\, x = (y1)\, x = 1x = x, \\ &p_2(x, y, x) = (xy)\, x = x, \\ &p_3(x, y, x) = x. \end{aligned}$$

Further

$$\begin{aligned} &p_1(x, x, z) = (x(zx))\, x = (z(xx))\, x = (z1)\, x = 1x = x = p_0(x, x, z), \\ &p_1(x, z, z) = (z(zx))\, x = (zx)\, x = (xz)\, z = p_2(x, z, z), \\ &p_2(x, x, z) = (xx)\, z = 1z = z = p_3(x, x, z). \end{aligned}$$

By the well-known Mal'cev condition (given by B. Jónsson) the assertion is prove. □

Now we shall define a tolerance relation. It can be defined for algebras in general (as in [7] and [8]) similarly as a congruence, only the requirement of transitivity is omitted. For lattices it was studied in [4], for semilattices in [9]. Every tolerance on a Boolean algebra is a congruence, i.e. it is transitive.

For implication algebras the tolerance may be defined as follows. Let $\langle I, \cdot \rangle$ be an implication algebra. A *tolerance* on $\langle I, \cdot \rangle$ is a reflexive and symmetric binary relation T on I with the property that $(x_1, y_1) \in T$, $(x_2, y_2) \in T$ imply $(x_1x_2, y_1y_2) \in T$ for any four elements x_1, y_1, x_2, y_2 of I.

A variety $\mathscr{V}$ is *congruence permutable* (*congruence* 3-*permutable*) if $\Theta_1 . \Theta_2 = \Theta_2 . \Theta_1$ ($\Theta_1 . \Theta_2 . \Theta_1 = \Theta_2 . \Theta_1 . \Theta_2$) for each two congruences $\Theta_1, \Theta_2 \in \operatorname{Con} A$ for any $A \in \mathscr{V}$.

Theorem 2. *The variety of all implication algebras is congruence 3-permutable, but not congruence permutable.*

Proof. By the result of Hagemann and Mitschke, a variety is congruence 3-permutable, if there exist ternary polynomials p_1, p_2 such that $p_1(x, x, y) = y$, $p_2(x, y, y) = x$ and $p_1(x, y, y) = p_2(x, x, y)$. In the variety $\mathscr{V}$ of all implication algebras we can put $p_1(x, y, z) = (xy)\, z$, $p_2(x, y, z) = (zy)\, x$. Then

$$\begin{aligned} &p_1(x, x, y) = (xx)\, y = 1y = y, \\ &p_2(x, y, y) = (yy)\, x = 1x = x, \\ &p_1(x, y, y) = (xy)\, y = (yx)\, x = p_2(x, x, y). \end{aligned}$$

Hence $\mathscr{V}$ is congruence 3-permutable .By [2], $\mathscr{V}$ is congruence permutable, if and only of $\mathscr{V}$ is *tolerance trivial*, i.e. if every tolerance T on every $A \in \mathscr{V}$ is congruence on A. If we consider the three-element implication algebra $\langle I, \cdot \rangle$, where $I = \{a, b, 1\}$ and the operation on this algebra is given by $aa = bb = 11 = 1$, $ab = b$, $ba = a$, $1a = a$, $1b = b$, $a1 = b1 = 1$, then the binary relation $T = \{(a, 1), (1, a), (b, 1), (1, b), (a, a), (b, b), (1, 1)\}$ is evidently a tolerance on $\langle I, \cdot \rangle$ and is not transitive (i.e. it is not a congruence). Hence $\mathscr{V}$ is not congruence permutable. $\square$

A *block* of a tolerance T on $\langle I, \cdot \rangle$ is a maximal (with respect to set inclusion) subset B of I with the property that $(x, y) \in T$ for each $x \in B$, $y \in B$.

The set of all tolerances on an algebra A forms an algebraic lattice. Hence, for any $a, b \in A$ there exists the least tolerance $\tau(a, b)$ containing the pair $\langle a, b \rangle$. An algebra A is called *principal tolerance trivial* (see [2]) if $\tau(a, b) = \Theta(a, b)$ for each $a, b \in A$. An algebra A is *tolerance trivial* if each tolerance on A is a congruence on A.

Since every implication algebra is a join semilattice with respect to the operation

$$a \vee b = (ab)\, b$$

in which every filter $[a]\uparrow$ is a Boolean algebra and since every distributive lattice is principal tolerance trivial, we have a question whether also implication algebras have this property.

Theorem 3. *The variety of all implication algebras is not principal tolerance trivial.*

Proof. Let $\langle I, \cdot \rangle$ be a free implication algebra with two free generators a, b and let T be the least (with respect to set inclusion) tolerance on $\langle I, \cdot \rangle$ which contains the pair (ab, ba). Then clearly the blocks of T are exactly those visualized in Fig. 1. Since $(a, (ab)\, b) \in T$, $((ab)\, b, b) \in T$, but $(a, b) \notin T$, the tolerance T is not a con-

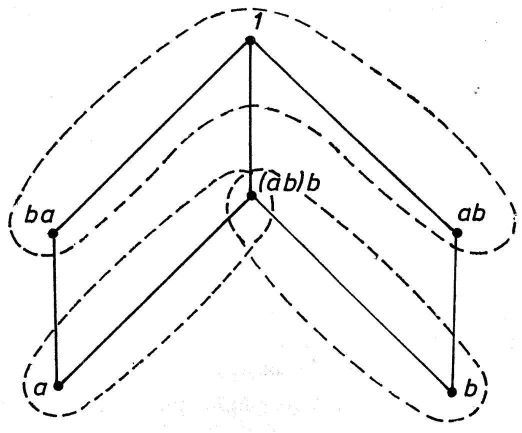

Fig. 1.

gruence and the principal congruence generated by (ab, ba) is not equal to the principal tolerance generated by (ab, ba). □

Let A be an algebra with a nullary operation c. We say that A is *weakly regular*, if $[c]_\Theta = [c]_\Phi$ implies $\Theta = \Phi$ for any two congruences Θ, Φ on A; here $[c]_\Theta$ denotes the congruence class of Θ containing c and similary $[c]_\Phi$. A variety $\mathscr{V}$ of algebras is *weakly regular*, if every $A \in \mathscr{V}$ has this property. B. Csákány [5] gave a characterization of such varieties.

Theorem 4. *The variety of all implication algebras is weakly regular.*

Proof. By [5], $\mathscr{V}$ is weakly regular (with respect to the constant c) if and only if there exist binary polynomials $b_1(x, y), \ldots, b_n(x, y)$ on an algebra $A \in \mathscr{V}$ such that $b_i(x, y) = c$ for $i = 1, \ldots, n$ if and only if $x = y$. We can put $n = 2$ and $b_1(x, y) = xy$, $b_2(x, y) = yx$. Then clearly we have simultaneously $xy = 1$ and $yx = 1$ if and only if simultaneously $x \leqq y$ and $y \leqq x$; this holds if and only if $x = y$. □

The following concepts are introduced in [3] and [6].

An algebra A with a nullary operation c is *congruence permutable at* c if $[c]_{\Theta.\Phi} = [c]_{\Phi.\Theta}$ for any two congruences Θ, Φ on A. A variety $\mathscr{V}$ is *congruence permutable at* c if every $A \in \mathscr{V}$ has this property.

Theorem 5. *The variety of all implication algebras is congruence permutable at* 1.

Proof. By Theorem 1 in [3] (see also [6]), $\mathscr{V}$ is congruence permutable at 1 if and only if there exists a binary polynomial $b(x, y)$ such that $b(x, x) = 1$ and $b(x, 1) = x$. Clearly $b(x, y) = yx$ satisfies these identities. □

Following [3], a binary relation R on an algebra A with a nullary operation c is *reflexive at* c if $(x, c) \in R$ implies $(x, x) \in R$ for each $x \in A$. The relation R is *symmetric at* c if $(x, c) \in R$ implies $(c, x) \in R$ for each $x \in A$.

Theorem 6. *Let A be an implication algebra and let R be a binary relation on A compatible with the operation of A (i.e. R is a subalgebra of the direct product $A \times A$). If R is reflexive at* 1, *then R is symmetric at* 1 *in A.* □

Proof is a direct consequence of Theorem 2 in [3]. □

3. IMPLICATION ALGEBRAS OF THE RANK 2

Here we shall study finite implication algebras. If $\langle I, \cdot \rangle$ is a finite implication algebra, then the minimum number of Boolean algebras whose union is $\langle I, \cdot \rangle$ will be called the *rank* of $\langle I, \cdot \rangle$.

Thus an implication algebra of the rank 1 is a Boolean algebra. We shall be interested in finite implication algebras of the rank 2. For the sake of simplicity we shall denote all algebras by the same symbols as their supports, i.e. we shall write only I instead of $\langle I, \cdot \rangle$. Thus let I be a finite implication algebra of the rank 2. Let it be the union of two Boolean algebras A, B. These two algebras have the common greatest element 1. Their least elements will be denoted by 0_A and 0_B respectively. The intersection $C = A \cap B$ is again a Boolean algebra. If $C = \{1\}$, the algebra I will be called *simple*. Otherwise we denote the least element of C by 0_C.

We shall prove a theorem.

Theorem 7. *Let I be a non-simple finite implication algebra of the rank 2. Then $I \cong I_0 \times I_1$, where I_0 is a simple implication algebra of the rank 2 and I_1 is is a Boolean algebra.*

Proof. Let $A, B, C, 0_A, 0_B, 0_C$ have the meaning described above. Let $d(A)$, $d(B)$, $d(C)$ be the dimensions of the algebras A, B, C respectively. Let $a_1, \ldots, a_{d(A)}$ (or $b_1, \ldots, b_{d(B)}$) be the atoms of A (or B respectively). Then the element 0_C as an element of A is the union of $d(A) - d(C)$ atoms of A and as an element of B it is the union of $d(B) - d(C)$ atoms of B. Without loss of generality let

$$0_C = \bigvee_{i=1}^{d(A)-d(C)} a_i = \bigvee_{i=1}^{d(B)-d(C)} b_i .$$

For the sake of simplicity we denote $\hat{a}_i = a_{d(A)-d(C)+i}$, $\hat{b}_i = b_{d(B)-d(C)+i}$ for $i = 1, \ldots, d(C)$. Let A_0 (or C_A) be the subalgebra of A generated by the atoms $a_1, \ldots, a_{d(A)-d(C)}$ (or $\hat{a}_1, \ldots, \hat{a}_{d(C)}$ respectively). Let B_0 (or C_B) be the subalgebra of B generated by the atoms $b_1, \ldots, b_{d(B)-d(C)}$ (or $\hat{b}_1, \ldots, \hat{b}_{d(C)}$ respectively). Consider the mapping $\varphi: C_A \to C_B$ defined so that $\varphi(x) = (x0_A)\, 0_B$ for each $x \in C_A$. Let $N_A(x)$ be the subset of the number set $\{1, \ldots, d_C\}$ consisting of all numbers i such that $a_i \leqq x$; then $x = \bigvee_{i \in N_A(x)} a_i$. Denote $\bar{N}_A(x) = \{1, \ldots, d(C)\} - N_A(x)$. As both x and 0_A belong to A, the element $x0_A$ is the complement of $x \vee 0_A = x$ in $[0_A]\uparrow = A$. Thus

$$x0_A = (\bigvee_{i=1}^{d(A)-d(C)} a_i) \vee (\bigvee_{i \in \bar{N}_A(x)} a_i) = 0_C \vee (\bigvee_{i \in \bar{N}_A(x)} a_i) .$$

The element $\bigvee_{i \in \bar{N}_A(x)} a_i$ is the complement of x in C_A; we denote it by $(x')_{C_A}$. Therefore $x0_A = 0_C \vee (x')_{C_A}$. We have $x0_A \geqq 0_C$ and thus $x0_A \in C \subseteq B$. As $x0_A \geqq 0_C$, we have $x0_A = 0_C \vee z$, where z is an element of C_B. The element $(x0_A)\, 0_B$ is the complement of $x0_A \vee 0_B = x0_A$ in $[0_B]\uparrow = B$. (In our notation the operation of the implication algebra has the priority before the join and the meet similarly as the multiplication of numbers has the priority before the addition and the subtraction.) Evidently $y = (x0_A)\, 0_B$ is the complement of z in C_B. Hence φ maps C_A onto C_B. Analogously we can prove that $(y0_B)\, 0_A = x$ and thus the mapping φ^{-1} defined so that $\varphi^{-1}(y) = (y0_B)\, 0_A$ is the inverse mapping to φ and maps C_B onto C_A. This implies that φ is a bijection of C_A onto C_B. Now let $x_1 \in C_A$, $x_2 \in C_A$, $x_1 \leqq x_2$. According to Lemma 4 in [1] the operation of the implication algebra is right antitone, hence $x_1 0_A \geqq x_2 0_A$ and $(x_1 0_A)\, 0_B \leqq (x_2 0_A)\, 0_B$. The mapping φ is isotone; as it is a bijection of one Boolean algebra onto another, it is an isomorphism.

Let $I_0 = A_0 \cup B_0$. The algebras A_0, B_0 have the unique common element which is the greatest one in both of them; namely 0_C. Hence I_0 is a simple implication algebra of the rank 2. Each element of A is the join of an element of A_0 and an element of C_A; similarly each element of B is the join of an element of B_0 and an element of C_B. Consider an element $c \in C = A \cap B$. As $c \geqq 0_C$, we have $c = 0_C \vee c_A = 0_C \vee c_B$, where $c_A \in C_A$, $c_B \in C_B$. We have $c = c_A \vee 0_C = (c'_A)c_A$ and, on the other hand, $c = c_B \vee 0_C = (c'_B)c_B$. This implies $\varphi((c'_A)c_A) = ((c'_A)c_A\, 0_A)\, 0_B = c0_B = ((c'_B)c_B\, 0_B)\, 0_B = (c'_B)c_B$. As φ is an isomorphism of C_A onto C_B, we have also $\varphi(c_A) = c_B$. Thus put $I_1 = C_A$ and consider the direct product $I_0 \times I_1$. We shall define the mapping $\psi: I_0 \times I_1 \to I$ as follows. Let $[x_0, x_1] \in I_0 \times I_1$, i.e. $x_0 \in I_0$, $x_1 \in I_1$. If $x_0 \in A_0$, then $\psi([x_0, x_1]) = x_0 \vee x_1$. If $x_0 \in B_0$, then $\psi([x_0, x_1]) = x_0 \vee \varphi(x_1)$. (If $x_0 = 0_C \in A_0 \cap B_0$, then both the ways lead to the same result; this follows from the above proved facts.) Evidently ψ is a bijection of $I_0 \times I_1$ onto I. From the definition of the implication algebra it is evident that ψ is an isomorphism. $\square$

3. TOLERANCE RELATIONS

Now we shall prove some theorems concerning tolerances and congruences on finite implication algebras of the rank 2.

Theorem 8. *Let $\mathfrak{B} = \langle B, \vee, \wedge, ', 0, 1\rangle$ be a Boolean algebra, let T be a reflexive and symmetric binary relation on B. Then T is a tolerance on $\mathfrak{B}$ as on a Boolean algebra if and only if T is a tolerance on $\mathfrak{B}$ considered as an implication algebra with $ab = a' \vee b$.*

Proof. The operation of the implication algebra was yet expressed as $ab = a' \vee b$. On the other hand, the operations of join, meet and complementation can be expressed in the following way:

$$x \vee y = (x0)\, y\,,$$
$$x \wedge y = (x(y0))\, 0\,,$$
$$x' = x0\,.$$

This implies the assertion. $\square$

Now we turn to finite implication algebras of the rank 2. First we shall investigate simple ones.

Following [1], an *upper section* of an implication algebra $\langle I, \cdot\rangle$ is a subset M of I with the property that $a \in M$, $x \geqq a$ imply $x \in M$.

If D is a congruence on $\langle I, \cdot\rangle$ and $a \in I$, then the symbol $D(a)$ denotes the class of D containing a.

Theorem 9. *Let $\langle I, \cdot\rangle$ be a simple finite implication algebra of the rank 2 being the union of Boolean algebras A and B, let T be a reflexive and symmetric binary relation on I. Then the following two assertions are equivalent:*

(i) *T is a tolerance on $\langle I, \cdot\rangle$.*

(ii) *The restriction T_A (or T_B) of T onto A (or B) is a congruence on A (or on B respectively) and $(a, b) \in T$, $a \in A$, $b \in B$ imply $a \in T_A(1)$, $b \in T_B(1)$, $(\hat{a}, \hat{b}) \in T$ for any $\hat{a} \geqq a$ and $\hat{b} \geqq b$.*

Proof. (i) $\Rightarrow$ (ii). Let T be a tolerance on I. The restriction of T onto A (or onto B) is a tolerance on A (or on B respectively); as every tolerance on a Boolean algebra is a congruence on it, this restriction is a congruence on A (or on B respectively). Now let $(a, b) \in T$, $a \in A$, $b \in B$. As T is reflexive, we have $(a, a) \in T$. From $(a, b) \in T$, $(a, a) \in T$ we have $(aa, ba) = (1, a) \in T$ and thus $a \in T_A(1)$. Analogously $b \in T_B(1)$. Let $\hat{a} \geqq a$, $\hat{b} \geqq b$. From $(\hat{a}0_A, \hat{a}0_A) \in T$, $(a, b) \in T$ we have $((\hat{a}0_A)\, a, (\hat{a}0_A)\, b) = (\hat{a}, b) \in T$. From $(\hat{b}0_B, \hat{b}0_B) \in T$, $(\hat{a}, b) \in T$ we have $((\hat{b}0_B)\, \hat{a}, (\hat{b}0_B)\, b) = (\hat{a}, \hat{b}) \in T$.

(ii) $\Rightarrow$ (i). Let (ii) be true. Let $(x_1, y_1) \in T$, $(x_2, y_2) \in T$. First suppose that $x_1 \in A$, $y_1 \in A$. If also $x_2 \in A$, $y_2 \in A$, then $(x_1, y_1) \in T_A$, $(x_2, y_2) \in T_A$, where T_A is the restriction of T onto A, i.e. a congruence on A. This implies that also $(x_1x_2, y_1y_2) \in T$. If $x_2 \in B$, $y_2 \in B$, then $(x_1x_2, y_1y_2) = (x_2, y_2) \in T$. If $x_2 \in A$, $y_2 \in B$, then $(x_1x_2, y_1y_2) = (x_1x_2, y_2)$. As $x_1x_2 = (x_10_A) \vee x_2 \geqq x_2$ and $(x_2, y_2) \in T$, we have $(x_1x_2, y_2) \in T$. Analogously for $x_2 \in B$, $y_2 \in A$. If we suppose that $x_1 \in B$, $y_2 \in B$, the proof is analogous. Now suppose that $x_1 \in A$, $y_1 \in B$. Then $x_1 \in T_A(1)$, $y_1 \in T_B(1)$. If $x_2 \in A$, $y_2 \in A$, we have $(x_1x_2, y_1y_2) = (x_1x_2, y_2)$. As $(x_1, 1) \in T$, we have $(x_1x_2, 1y_2) = (x_1x_2, y_2) \in T$. If $x_2 \in B$, $y_2 \in B$, the proof is analogous. If $x_2 \in A$, $y_2 \in B$, then $x_1x_2 \geqq x_2$, $y_1y_2 \geqq y_2$, $x_1x_2 \in A$, $y_1y_2 \in B$ and hence $(x_1x_2, y_1y_2) \in T$. If $x_2 \in B$, $y_2 \in A$, then $(x_1x_2, y_1y_2) = (x_2, y_2) \in T$. In the case when $x_1 \in B$, $y_1 \in A$ the proof is analogous. We have proved that T is a tolerance on I. □

Theorem 10. *Let $\langle I, \cdot \rangle$ be a simple finite implication algebra of the rank 2, being the union of Boolean algebras A and B, let T be an equivalence relation on I. Then the following two assertions are equivalent:*

(iii) *T is a congruence on $\langle I, \cdot \rangle$.*

(iv) *The restriction T_A (or T_B) or T onto A (or onto B) is a congruence on A (or on B respectively) and $(a, b) \in T$, $a \in A$, $b \in B$ holds if and only if $a \in T_A(1)$, $b \in T_B(1)$.*

Proof. (iii) $\Rightarrow$ (iv). Let T be a congruence on $\langle I, \cdot \rangle$. As it is a tolerance on I, the condition (ii) from Theorem 3 is satisfied. Hence T_A (or T_B) is a congruence on A (or on B respectively). Further if $(a, b) \in T$, $a \in A$, $b \in B$, then $a \in T_A(1)$, $b \in T_B(1)$. On the other hand, suppose that $a \in T_A(1)$, $b \in T_B(1)$. Then $(a, 1) \in T$, $(1, b) \in T$ and the transitivity of T yields $(a, b) \in T$.

(iv) $\Rightarrow$ (iii). Let (iv) be true. If $a \in A$, $b \in B$, $(a, b) \in T$, then $(a, 1) \in T$, $(b, 1) \in T$. The sets $T_A(1)$, $T_B(1)$ are filters of I, therefore if $\hat{a} \geqq a$, $\hat{b} \geqq b$, then also $\hat{a} \in T_A(1)$, $\hat{b} \in T_B(1)$ and $(\hat{a}, \hat{b}) \in T$. Hence (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) and T is a tolerance on I. As T is evidently transitive, it is a congruence on I. □

Now we shall use Theorem 1 and Theorem 3 to the study of tolerances on finite implication algebras of the rank 2 which are not simple. First we shall define a concept which will be important in the sequel.

Let J_1, J_2 be two implication algebras, let T_1 be a tolerance on J_1 and let T_2 be a tolerance on J_2. A tolerance T on $J_1 \times J_2$ is called the *direct product of tolerances* T_1 and T_2 and denoted by $T_1 \times T_2$, if $((u_1, u_2), (v_1, v_2)) \in T$ holds if and only if $(u_1, v_1) \in T_1$, $(u_2, v_2) \in T_2$.

Now consider a finite implication algebra I of the rank 2. According to Theorem 1 it is isomorphic to the direct product $I_0 \times I_1$, where I_0 is a simple implication algebra of the rank 2 and I_1 is a Boolean algebra. Thus we may consider I as this direct product.

The algebra I_0 will be considered as the union of two Boolean algebras A_0, B_0 whose intersection consists only of their common greatest element; we denote it by 1_0. The least elements of A_0 and B_0 will be denoted by 0_A and 0_B respectively. The greatest element of I_1 will be denoted by 1_1, its least element by 0_1. Further we denote $A = A_0 \times I_1$, $B = B_0 \times I_1$; this corresponds to the notation used above.

Before proving a theorem, we state some lemmas.

Lemma 1. *Let T be a tolerance on $I = I_0 \times I_1$. Let T_A (or T_B) be the restriction of T onto A (or B respectively). Then there exists a congruence T_A^* on A_0, a congruence T_B^* on B_0 and a congruence T^{**} on I_1 such that $T_A = T_A^* \times T^{**}$, $T_B = T_B^* \times T^{**}$.*

Proof. Consider T_A. Let $((u_1, x_1), (v_1, y_1)) \in T_A$, $((u_2, x_2), (v_2, y_2)) \in T_A$ for some elements u_1, v_1, u_2, v_2 of A_0 and x_1, y_1, x_2, y_2 of I_2. From $((1_0, 0_1), (1_0, 0_1)) \in T_A$, $((u_1, x_1), (v_1, y_1)) \in T_A$ we obtain $((u_1, 1_1), (v_1, 1_1)) \in T_A$. From $((0_A, 1_1), (0_A, 1_1)) \in T_A$, $((u_2, x_2), (v_2, y_2)) \in T_A$ we obtain $((1_0, x_2), (1_0, y_2)) \in T_A$. From this and $((0_A, 0_1), (0_A, 0_1)) \in T_A$ we obtain $((0_A, x_2 0_1), (0_A, y_2 0_1)) \in T_A$. Now from $((u_1, 1_1), (v_1, 1_1)) \in T_A$, $((0_A, x_2 0_1), (0_A, y_2 0_1)) \in T_A$ we obtain $((u_1 0_A, x_2 0_1), (v_1 0_A, y_2 0_1)) \in T_A$. From this and $((0_A, 0_1), (0_A, 0_1)) \in T_A$ we obtain $((u_1, x_2), (v_1, y_2)) \in T_A$. We may define T_A^* as the set of all pairs (u, v), where $u \in A_0$, $v \in A_0$ and $((u, 1_1), (v, 1_1)) \in T_A$. Similarly we define T^{**} as the set of all pairs (x, y), where $x \in I_1$, $y \in I_1$ and $((1_0, x), (1_0, y)) \in T_A$. As the elements $u_1, v_1, x_1, y_1, u_2, v_2, x_2, y_2$ used in the above consideration were chosen arbitrarily, we see that $T_A = T_A^* \times T^{**}$. Analogously we may prove $T_B = T_B^* \times T^{**}$; the tolerance T^{**} is the same as in the preceding case, because 1_0 is the common greatest element of both A_0 and B_0. □

In the sequel we shall use a further notation. If $x \in I_1$, $y \in I_1$, $(x, y) \in T$, then by $T(x, y)$ we denote the set of all pairs (u, v) such that $u \in I_0$, $v \in I_0$, $((u, x), (v, y)) \in T$.

Lemma 2. *Each $T(x, y)$ is a tolerance on I_0 and $T(x, y) \supseteq T_A^* \times T_B^*$.*

Proof. Let $(u_1, v_1) \in T(x, y)$, $(u_2, v_2) \in T(x, y)$, i.e. $((u_1, x), (u_2, y)) \in T$, $((u_2, x), (v_2, y)) \in T$. From $(1_0, 0_1), (1_0, 0_1)) \in T$, $((u_1, x), (v_1, y)) \in T$ we obtain $((u_1, 1_1), (v_1, 1_1)) \in T$. From this and $((u_2, x), (v_2, y)) \in T$ we obtain $((u_1 u_2, x), (v_1 v_2, y)) \in T$ and hence $(u_1 u_2, v_1 v_2) \in T(x, y)$; this proves that $T(x, y)$ is a tolerance on I_0. As $(x, y) \in T^{**}$, from Lemma 1 it follows that $((u, x), (v, y)) \in T$ for any $(u, v) \in T_A^* \cup T_B^*$ and thus $T(x, y) \supseteq T_A^* \cup T_B^*$. □

Lemma 3. *Let* $x \in I_1$, $y \in I_1$, $\hat{x} \in I_1$, $\hat{y} \in I_1$, $(x, y) \in T^{**}$, $(\hat{x}, \hat{y}) \in T^{**}$, $\hat{x} \geqq x$, $\hat{y} \geqq y$. *Then* $T(x, y) \subseteq T(\hat{x}, \hat{y})$.

Proof. Let $(u, v) \in T(x, y)$, i.e. $((u, x), (v, y)) \in T$. As $(\hat{x}, \hat{y}) \in T^{**}$, also $(\hat{x}0_1, \hat{y}0_1) \in T^{**}$ and $((1_0, \hat{x}0_1), (1_0, \hat{y}0_1)) \in T$. From this and $((u, x), (v, y)) \in T$ we obtain $((u, \hat{x}), (v, \hat{y})) \in T$, i.e. $(u, v) \in T(\hat{x}, \hat{y})$. As the pair (u, v) was chosen arbitrarily, this proves the inclusion. □

Lemma 4. *If* $((u, x), (v, y)) \in T$ *for some elements* $u \in I_0$, $v \in I_0$, $x \in I_1$, $y \in I_1$, *then* $(x, y) \in T^{**}$.

Proof. From $((u, x), (v, y)) \in T$, $((1_0, 0_1), (1_0, 0_1)) \in T$ we obtain $((1_0, x0_1), (1_0, x0_1)) \in T$. As $1_0 \in A_0 \cup B_0$, according to Lemma 1 we have $(x0_1, y0_1) \in T^{**}$ and then also $(x, y) \in T^{**}$. □

Now we can prove a theorem.

Theorem 11. *Let the symbols* $I, I_0, I_1, A_0, B_0, A, B$ *have the above described meaning. Let* T *be a reflexive and symmetric binary relation on* I, *let* T_A (*or* T_B) *be the restriction of* T *onto* A (*or* B *respectively*). *The relation* T *is a tolerance on* I *if and only if the following conditions are satisfied*:

(a) *There exists a congruence* T_A^* *on* A, *a congruence* T_B^* *on* B *and a congruence* T^{**} *on* I_1 *such that* $T_A = T_A^* \times T^{**}$, $T_B = T_B^* \times T^{**}$.

(b) *For each* $(x, y) \in T^{**}$ *the set* $T(x, y) = \{(u, v) \in I_0 \times I_0 \mid (u, x), (v, y)) \in T\}$ *is a tolerance on* I_0 *and* $T(x, y) \supseteq T_A^* \cup T_B^*$.

(c) *If* $x \in I_1$, $y \in I_1$, $\hat{x} \in I_1$, $\hat{y} \in I_1$, $(x, y) \in T^{**}$, $(\hat{x}, \hat{y}) \in T^{**}$, $\hat{x} \geqq x$, $\hat{y} \geqq y$, *then* $T(x, y) \subseteq T(\hat{x}, \hat{y})$.

(d) *If* $u \in I_0$, $v \in I_0$, $x \in I_1$, $y \in I_1$ *and* $((u, x), (v, y)) \in T$, *then* $(x, y) \in T^{**}$.

Proof. The necessity of the conditions follows from Lemmas 1, 2, 3, 4. It remains to prove their sufficiency. Let the conditions (a,) (b), (c), (d) be fulfilled. Let $((u_1, x_1), (v_1, y_1)) \in T$, $((u_2, x_2), (v_2, y_2)) \in T$. Then $(u_1, v_1) \in T(x_1, y_1)$, $(u_2, v_2) \in T(x_2, y_2)$. We have $u_1u_2 \geqq u_2$, $v_1v_2 \geqq v_2$, $x_1x_2 \geqq x_2$, $y_1y_2 \geqq y_2$. This implies $T(x_2, y_2) \subseteq T(x_1x_2, y_1y_2)$ and thus $(u_2, v_2) \in T(x_1x_2, y_1y_2)$. If $u_1u_2 \in A_0$, $v_1v_2 \in B_0$, then also $u_2 \in A_0$, $v_2 \in B_0$. As $(u_2, v_2) \in T(x_1x_2, y_1y_2)$, then, according to (c), also $(u_1u_2, v_1v_2) \in T(x_1x_2, y_1y_2) \subseteq T$, because $u_1u_2 \geqq u_2$, $v_1v_2 \geqq v_2$. In the case when $u_1u_2 \in B_0$, $v_1v_2 \in A_0$ the proof is analogous. If $u_1u_2 \in A_0$, $v_1v_2 \in A_0$, then also $u_2 \in A_0$, $v_2 \in A_0$ and, according to (a), $(u_2, v_2) \in T(0_1, 0_1) \subseteq T(x_1, y_1)$. Then $(u_1u_2, v_1v_2) \in T(x_1, y_1) \subseteq T$ according to (b). In the case when $u_1u_2 \in B_0$, $v_1v_2 \in B_0$ the proof is analogous. □

Theorem 12. *Let* $I = I_0 \times I_1$ *be a direct product of a simple finite implication algebra* I_0 *of the rank 2 and of a finite Boolean algebra* I_1. *Let* T *be an equivalence relation on* I. *The relation* T *is a congruence on* I *if and only if* $T = T_0 \times T_1$, *where* T_0 *is a congruence on* I_0 *and* T_1 *is a congruence on* I_1.

Proof. We shall use the introduced notation. Suppose that T is a congruence on I.

Then it is a tolerance on I and Theorem 5 holds for it. Evidently the tolerance $T(x, y)$ for each $(x, y) \in T$ is also a congruence. As it contains $T_A^* \cup T_B^*$ as a subset, it is uniquely determined (this follows from Theorem 3) and thus it is also the same for each $(x, y) \in T$; it contains all pairs from T_A^*, all pairs from T_B^* and all pairs, one of whose elements is in $T_A^*(1_0)$ and the other is in $T_B^*(1_0)$. We denote this tolerance by T_0. If we put $T_1 = T^{**}$, then evidently $T = T_0 \times T_1$. Conversely, a direct product of a congruence on I_0 and a congruence on I_1 is evidently a congruence on $I = I_0 \times I_1$. □

At the end we shall prove two theorems concerning tolerances of finite implication algebras in general.

Theorem 13. *Let T be a tolerance on an implication algebra I. Then each block of T is a convex subset of I.*

Proof. Let B be a block of T and let $a \in B$, $b \in B$. Suppose that $a \leqq b$ in the described ordering. Now let x, y be elements of I such that $a \leqq x \leqq b$, $a \leqq y \leqq b$. Then $x \leqq b$ implies $xb = 1$ and thus

$$(bx)\, x = (xb)\, b = 1b = b\,.$$

Then $(a, b) \in T$ and $(x, x) \in T$ imply $((ax)\, x, (bx)\, x) \in T$, i.e. $(x, b) = (x, (bx)\, x) \in T$. Analogously $(b, a) \in T$ and $(y, y) \in T$ imply $(b, y) \in T$. This yields

$$((x(ba))\, a, (b(ya))\, a) \in T\,.$$

By Theorem 5 in [1] we have

$$(x(ba))\, a = x \wedge b\,,$$
$$(b(ya))\, a = b \wedge y\,,$$

where the meets are taken in the Boolean algebra $[a]\uparrow$. Hence we have $(x \wedge b, b \wedge y) \in T$. As $x \leqq b$, $y \leqq b$, this implies $(x, y) \in T$. Thus the convexity of B is proved. □

Theorem 14. *Let I be a finite implication algebra, let T be a tolerance on I. Then for each block B of T there exists a block $\hat{B}$ of T such that $|B| \leqq |\hat{B}|$ and $1 \in \hat{B}$.*

Proof. The tolerance T is also a tolerance on I as on a semilattice, because the join can be expressed in terms of the operation of the implication algebra. Thus [5] the block B is closed under the join and has the greatest element a. Let φ be a mapping of I into itself given by $\varphi(x) = ax$. If $(x, y) \in T$, then also $(\varphi(x), \varphi(y) = (ax, ay) \in T$; hence there exists a block $\hat{B}$ of T which contains all elements $\varphi(x)$ for $x \in B$. We have $\varphi(a) = aa = 1$ and thus $1 \in \hat{B}$. Now suppose that $\varphi(x) = \varphi(y)$ for some elements x, y of B. This means $ax = ay$. As $a \vee x = a \vee y = a$, the element ax is the complement of a in $[x]\uparrow$ and ay is the complement of a in $[y]\uparrow$. Hence $ax \wedge a = x$ in $[x]\uparrow$ and $ay \wedge a = y$ in $[y]\uparrow$. But, as $ax = ay$ and a meet of given two elements is at most one, we obtain $x = y$. Hence φ is a one-to-one mapping. As φ maps B into $\hat{B}$, we have $|B| \leqq |\hat{B}|$. □

Now we generalize the notation $T(a)$ to tolerances. If I is an implication algebra, T is a tolerance on I and $a \in I$, then $T(a) = \{x \in I \mid (a, x) \in T\}$.

Theorem 15. *Let I be a finite implication algebra, let T be a tolerance on I. Then $|T(a)| \leqq |T(1)|$ for each $a \in I$.*

Proof. Also $T(a)$ is closed under join; hence the proof is analogous to the proof of Theorem 14. □

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