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A GLOBAL CONTINUATION THEOREM FOR OBTAINING EIGENVALUES AND BIFURCATION POINTS

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0. INTRODUCTION

Let us consider the equation of the type

$$(0.1) \qquad u - T(\lambda)\, u + H_\tau(\lambda, u) = 0$$

in a real Banach space $\mathbb{V}$, and the norm condition

$$(0.2') \qquad \|u\|^2 = \delta\tau\,,$$

where $\lambda \in J$ is a bifurcation parameter (J is an open interval), $\tau \in \langle 0, 1)$ is an additional parameter, $\delta > 0$ is fixed. We shall suppose (with the exception of Section 3) that

(0.3) *for any $\lambda \in J$, $T(\lambda)$ is a linear completely continuous operator in $\mathbb{V}$; the mapping $\lambda \to T(\lambda)$ of J into the space of linear continuous operators in $\mathbb{V}$ is continuous;*

(0.4′) *for any $\tau \in \langle 0, 1)$, H_τ: $J \times \mathbb{V} \to \mathbb{V}$ is a continuous mapping; the mapping $R(\lambda, u, \tau) = T(\lambda)\, u - H_\tau(\lambda, u)$ of $J \times \mathbb{V} \times \langle 0, 1)$ into $\mathbb{V}$ is completely continuous, $H_0(\lambda, u) = 0$;*

(0.5) $\displaystyle\lim_{\substack{\|u\|+\tau\to 0\\ u\in\mathbb{V},\tau\in\langle 0,1)}} \frac{\|H_\tau(\lambda, u)\|}{\|u\| + \tau} = 0$ *uniformly on compact λ – subintervals of J.*

We shall consider a simple critical point λ_0 of T, i.e. λ_0 such that the solutions of

$$(0.6) \qquad u - T(\lambda)\, u = 0$$

for $\lambda = \lambda_0$ form a one-dimensional subspace in $\mathbb{V}$. Our aim is to show, under certain assumptions, the existence of a connected branch of solutions $[\lambda, u, \tau]$ of (0.1), (0.2′) containing at least one $[\lambda_\tau, u_\tau, \tau]$ for any $\tau \in \langle 0, 1)$, starting at $[\lambda_0, 0, 0]$ and lying in a suitable subinterval J_0 of J in the variable λ. If H_τ is defined (and continuous) for all $\tau \in \langle 0, 1\rangle$ then it implies the existence of a solution λ_1, u_1 of

$$(0.7) \qquad u - T(\lambda)\, u + H_1(\lambda, u) = 0$$

with $\|u_1\| = \delta$, $\lambda_1 \in \bar{J}_0$. In general, H_1 need not be defined but in some applications the limiting process $\tau \to 1$ in (0.1), (0.2') yields a solution of a problem which cannot be written as an equation of the type (0.7) in $\mathbb{V}$, e.g. a variational inequality (see Example 1.2, cf. also [4], [3]). If our assumptions are fulfilled for any $\delta \in (0, \delta_0)$ (with J_0 fixed) then it follows that there exists a bifurcation point $[\lambda_1^b, 0]$ of (0.7) (or of the corresponding limit problem if H_1 is not defined) with respect to the line of trivial solutions, $\lambda_1^b \in \bar{J}_0$. In some cases it is possible to show that also $\lambda_1^b \in J_0$ (see e.g. [5], [2]). Roughly speaking, (0.1) can be understood as a homotopy joining (0.6) to (0.7) (or to the corresponding limit problem) and we transfer the information concerning the existence of critical points of (0.6) to that concerning the existence of critical or bifurcation points of (0.7). Notice that H_1 need not be o $(\|u\|)$ at $u = 0$ under our assumptions, i.e. the linearization

$$u - T_1(\lambda)\, u = 0 \tag{0.8}$$

of (0.7) (where $T_1(\lambda)$ is the Frechet differential of $T(\lambda) - H_1(\lambda, \cdot)$ at $u = 0$) can differ from (0.6).

From the point of view of the proof, it is more natural to transform our problem in such a way that H_τ is defined for all $\tau \in \langle 0, +\infty)$ (instead of $\langle 0, 1)$), that means, to replace (0.2') by

$$\|u\|^2 = \frac{\delta\tau}{1 + \tau} \tag{0.2}$$

and (0.4') by the assumption that

(0.4) *for any* $\tau \in \langle 0, +\infty)$, H_τ: $J \times \mathbb{V} \to \mathbb{V}$ *is a continuous mapping*; *the mapping* $R(\lambda, u, \tau) = T(\lambda)\, u - H_\tau(\lambda, u)$ *of* $J \times \mathbb{V} \times \mathbb{R}^+$ *into* $\mathbb{V}$ *is completely continuous*, $H_0(\lambda, u) = 0$.

Further, we shall consider our problem in this form (with the exception of Theorem 1.3, where the closed interval $\langle 0, 1\rangle$ is considered).

The main results are formulated and illustrated by examples in Section 1. First, we describe all possibilities of the behaviour of the branch of nontrivial solutions of (0.1), (0.2) starting at $[\lambda_0, 0, 0]$ in the direction of a given eigenvector w_0 of $T(\lambda_0)$ (Theorem 1.1). The continuation theorem mentioned above (Theorem 1.2) is an easy consequence. The principle of the proof is the same as in [2–6] where it was used for some particular situations. The system (0.1), (0.2) is transformed to the bifurcation equation of a similar type as that studied in [1] and then the proof consists of Dancer's considerations combined with the investigation of the properties of the set of solutions of (0.1), (0.2) under our special assumptions. The detailed proof (including a repetition of considerations from [1]) is presented in Section 2. The relations to [1] are mentioned in Remarks 1.4, 2.7. A generalization to the case of nonlinear operators $T(\lambda)$ is given in Appendix.

Throughout the paper, we shall use the following notation:

$\mathbb{R}$, $\mathbb{R}^+$ – set of all reals and of all nonnegative reals, respectively,

$\mathbb{V}$ — real Banach space,
$X = \mathbb{V} \times \mathbb{R}$,
$\|\cdot\|$, $|||\cdot|||$ — norm in $\mathbb{V}$ and in X (i.e. $|||x|||^2 = \|u\|^2 + \tau^2$, $x = [u, \tau]$),
$\to$, $\rightharpoonup$ — strong convergence and weak convergence,
$\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_X$ — duality between $\mathbb{V}$ and $\mathbb{V}^*$ (the dual space) and between X and X^*,
δ — fixed positive number,
J — open interval in $\mathbb{R}$,
M — set closed in $J \times \mathbb{V} \times \mathbb{R}^+$ or in $J \times \mathbb{V} \times \langle 0, 1)$,
$E_T(\lambda)$, $E_L(\lambda)$ — set of all solutions of (0.6) and of (2.5), respectively (with a given λ),
$C = \mathrm{Cl}\,\{[\lambda, u, \tau] \in J \times \mathbb{V} \times \mathbb{R}^+;\ \tau \neq 0,\ (0.1), (0.2) \text{ is fulfilled}\}$ (closure in $\mathbb{R} \times \mathbb{V} \times \mathbb{R}$),
$\Lambda = \{\lambda \in J;\ E_T(\lambda) \neq \{0\}\}$ $(= \{\lambda \in J;\ E_L(\lambda) \neq \{0\}\}$ — see Remark 2.1) — set of all critical points of our problem in J,
λ_0, w_0 — given critical point and the corresponding solution from $E_T(\lambda_0)$ (see Theorem 1.1),
$x_0 = [w_0, 0]$,
y_0^* — fixed element of X^* such that $\langle y_0^*, x_0 \rangle_X = 1 = \sup\limits_{|||x||| \leqq 1} \langle y_0^*, x \rangle_X$,
$K_\eta = \{[\lambda, x] \in J \times X;\ |\langle y_0^*, x \rangle_X| > \eta |||x|||\}$, $K_\eta^+ = \{[\lambda, x] \in K_\eta;\ \langle y_0^*, x \rangle_X > 0\}$, $K_\eta^- = K_\eta \setminus K_\eta^-$ ($\eta \in (0, 1)$ will be fixed),
$B_r(z)$ — open ball in the space considered (usually in $\mathbb{R} \times X$ or X) with the radius r centered at z (sometimes we write $B_r^{\mathbf{V}}(z)$, $B_r^X(z)$ etc. in order to indicate the space),
M^0, ∂M — interior and boundary of M, respectively.

Further, the following symbols will be introduced in the text:

C_ε — see Remark 2.4,
L, G, $\Phi(\lambda)$ — see Remark 2.1,
S — see Remark 2.3,
δ_1, δ_2 — see Remark 2.6.

1. FORMULATION AND ILLUSTRATION OF MAIN RESULTS

We shall always suppose that

(1.1) $\quad\quad \Lambda$ *is nowhere dense*

and consider a critical point $\lambda_0 \in \Lambda$ satisfying $E_T(\lambda_0) = \mathrm{Lin}\,\{w_0\}$ and the following condition:

(1.2) *for any* $\gamma_0 > 0$ *there exist* γ_+, γ_- *such that* $0 < \gamma_- < \gamma_+ < \gamma_0$, $\lambda_0 \pm \gamma_\pm \notin \Lambda$, $\theta(\lambda_0 + \gamma_+) - \theta(\lambda_0 - \gamma_-)$ *is odd,*

where $\theta(\lambda)$ denotes the sum of all algebraic multiplicites of all positive eigenvalues of the operator $T(\lambda) - I$.

Remark 1.1. Recall that if $\lambda \notin \Lambda$ then $\deg(I - T(\lambda), 0, B_\sigma(0)) = (-1)^{\beta(\lambda)}$ for any $\sigma > 0$. (This holds for a general linear completely continuous operator – see e.g. [10].) Hence, (1.2) ensures that the number $\deg(I - T(\lambda_0 + \gamma_+), 0, B_\sigma(0)) - \deg(I - T(\lambda_0 - \gamma_-), 0, B_\sigma(0))$ is even, different from zero for $\sigma > 0$ small enough.

Further, an open bounded interval J_0, a set M closed in $J \times \mathbb{V} \times \mathbb{R}^+$ such that $\lambda_0 \in \bar{J}_0$, $J_0 \cup \{\lambda_0\} \subset J$, and the following assumptions about the structure of the set of solutions of (0.1), (0.2) in a neighbourhood of $[\lambda_0, 0, 0]$ will play a basic role:

$$[\lambda_n, u_n, \tau_n] \in C\,, \quad [\lambda_n, u_n, \tau_n] \to [\lambda_0, 0, 0]\,, \quad u_n/\|u_n\| \to w_0 \Rightarrow \lambda_n \in J_0\,, \quad [\lambda_n, u_n, \tau_n] \notin M \quad for \quad n \geqq n_0\,, \tag{1.3}$$

$$[\lambda_n, u_n, \tau_n] \in C \setminus M\,, \quad \lambda_n \in J_0\,, \quad [\lambda_n, u_n, \tau_n] \to [\lambda_0, 0, 0] \Rightarrow u_n/\|u_n\| \to w_0\,. \tag{1.4}$$

Note that in examples, M is usually of the form $M = J \times K \times \mathbb{R}^+$, where K is a closed cone in $\mathbb{V}$ with its vertex at the origin.

Remark 1.2. If we know that the implication

$$[\lambda_n, u_n, \tau_n] \in C\,, \quad [\lambda_n, u_n, \tau_n] \to [\lambda_0, 0, 0]\,, \quad u_n/\|u_n\| \to w_0 \Rightarrow \lambda_n > \lambda_0 \quad for \quad n \geqq n_0 \tag{1.5}$$

holds then we try to consider an interval $J_0 = (\lambda_0, \lambda^m)$ with some $\lambda^m > \lambda_0$. See Examples 1.1, 1.2 below. Analogously for $\lambda_n < \lambda_0$, see Example 1.3.

Remark 1.3. Suppose that $M = J \times K \times \mathbb{R}^+$ where K is a closed cone in $\mathbb{V}$ with its vertex at the origin. Then it is easy to see (precisely, see the considerations in Remark 2.2) that the following assertions hold. If $w_0 \notin K$ and (1.5) holds then (1.3) is satisfied with an arbitrary $J_0 = (\lambda_0, \lambda^m)$, $\lambda^m > \lambda_0$. If $-w_0 \in K^0$ then (1.4) is fulfilled. In general, if $J_0 = (\lambda_0, \lambda^m)$, then (1.4) is equivalnt to the implication

$$[\lambda_n, u_n, \tau_n] \in C\,, \quad u_n \notin K\,, \quad [\lambda_n, u_n, \tau_n] \to [\lambda_0, 0, 0]\,, \quad u_n/\|u_n\| \to -w_0 \Rightarrow \lambda_n \leqq \lambda_0 \quad for \quad n \geqq n_0\,.$$

Lemma 1.1. *Let (0.3)–(0.5), (1.1) be fulfilled and let $\lambda_0 \in \Lambda$ satisfy (1.2), $E_T(\lambda_0) = \mathrm{Lin}\{w_0\}$. Suppose that M is a set closed in $J \times \mathbb{V} \times \mathbb{R}^+$, J_0 is an open bounded interval such that $J_0 \cup \{\lambda_0\} \subset J$, $\lambda_0 \in \bar{J}_0$ and (1.3) is satisfied. Then there exist a closed connected set $C_0^+ \subset C$ containing $[\lambda_0, 0, 0]$, and $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0)$ there is at least one couple λ_τ, u_τ satisfying $[\lambda_\tau, u_\tau, \tau] \in C_0^+ \setminus M$, $\lambda_\tau \in J_0$.*

Remark 1.4. In the sequel, we shall study the global behaviour of the branch C_0^+ of solutions to (0.1), (0.2) starting at $[\lambda_0, 0, 0]$ in the direction w_0, i.e. containing a sequence $[\lambda_n, u_n, \tau_n]$ from (1.3). (For the definition of C_0^+ see Remark 2.4.) Roughly speaking, it can be shown that C_0^+ either returns to a neighbourhood of $[\lambda_0, 0, 0]$ in the opposite direction $-w_0$ or it is not compact (see Lemmas 2.4, 2.5). This idea is taken from E. N. Dancer [1] where it is used in another situation (see also Remark 2.7). The condition (1.3) ensures that C_0^+ lies in J_0 (in λ) and outside of M at first (Lemma 1.1). Condition (1.4) says that C_0^+ cannot return immediately to $[\lambda_0, 0, 0]$

in the direction $-w_0$ from J_0 and from outside of M. But C_0^+ can possibly return to $[\lambda_0, 0, 0]$ in the direction $-w_0$ either if it crosses ∂M with some $\lambda \in J_0$, $\tau > 0$ (the condition (a) in Theorem 1.1 below) or if it crosses ∂M at $\lambda \in \Lambda \cap J_0$, $\tau = 0$ and continues outside of M (the condition (b)) or if it reaches ∂J_0 from J_0 and outside of M at a point different from $[\lambda_0, 0, 0]$ (the condition (c)). If none of these situations occurs then C_0^+ remains in J_0 and outside of M and therefore it cannot return to a neighbourhood of $[\lambda_0, 0, 0]$ in the direction $-w_0$. Hence, it is noncompact, i.e. unbounded in τ in view of the boundedness of J_0, local compactness of C (following from (0.4)) and (0.2) (the condition (d) in Theorem 1.1). The last case is interesting for our purposes.

Theorem 1.1. *Let* (0.3)–(0.5), (1.1) *be fulfilled, let* $\lambda_0 \in \Lambda$ *satisfy* (1.2), $E_T(\lambda_0) = \operatorname{Lin}\{w_0\}$. *Consider a set* M *closed in* $J \times \mathbb{V} \times \mathbb{R}^+$ *and an open bounded interval* J_0 *such that* $\lambda_0 \in \bar{J}_0$, $J_0 \cup \{\lambda_0\} \subset J$, (1.3), (1.4) *hold. Then there exists a closed connected set* $C_0^+ \subset C$ *satisfying at least one of the following conditions*:

(a) *there exists* $[\lambda, u, \tau] \in C_0^+ \cap \partial M$ *with* $\lambda \in J_0$, $\tau \neq 0$;

(b) *there exist* $\tilde{\lambda} \in \Lambda \cap J_0$ *and* $[\lambda_n^1, u_n^1, \tau_n^1] \in C_0^+ \setminus M$, $[\lambda_n^2, u_n^2, \tau_n^2] \in C_0^+ \cap M$ *such that* $\tau_n^i \neq 0$, $[\lambda_n^i, u_n^i, \tau_n^i] \to [\tilde{\lambda}, 0, 0]$ (*for* $n \to +\infty$, $i = 1, 2$);

(c) *there exist* $[\lambda_n, u_n, \tau_n] \in C_0^+ \setminus M$, $\lambda_n \in J_0$ *such that* $[\lambda_n, u_n, \tau_n] \to [\tilde{\lambda}, \tilde{u}, \tilde{\tau}] \neq [\lambda_0, 0, 0]$, $\tilde{\lambda} \in \partial J_0$;

(d) *for any* $\tau \in \mathbb{R}^+$ *there is at least one* λ_τ, u_τ *with* $[\lambda_\tau, u_\tau, \tau] \in C_0^+$; *further,* $\lambda \in J_0$ *for any* $[\lambda, u, \tau] \in C_0^+$, $[\lambda, u, \tau] \neq [\lambda_0, 0, 0]$, *and* $C_0^+ \cap M$ *contains only points of the type* $[\lambda, 0, 0]$.

Remark 1.5. In applications we desire to choose J_0 and M for (a), (b), (c) to be excluded in Theorem 1.1, i.e. for (d) to be ensured. Hence, Theorem 1.1 can be understood as a continuation theorem in the sense mentioned in Introduction. For instance, it is easy to see that the condition (1.7) below ensures (1.3), (1.4) and excludes (c), while the conditions (1.8) and (1.9) exclude (a) and (b), respectively:

$$(1.7) \qquad \textit{if} \quad [\lambda_n, u_n, \tau_n] \in C, \quad \lambda_n \to \lambda \in \partial J_0, \quad \tau_n \to \tau \quad \textit{then}$$
$$[\lambda = \lambda_0,\ \tau = 0,\ u_n/\|u_n\| \to w_0 \Leftrightarrow \lambda_n \in J_0,\ [\lambda_n, u_n, \tau_n] \notin M \ \textit{for}\ n \geqq n_0];$$

$$(1.8) \qquad [\lambda, u, \tau] \notin \partial M \quad \textit{for any} \quad [\lambda, u, \tau] \in C, \quad \lambda \in J_0, \quad \|u\| \neq 0;$$

$$(1.9) \qquad \textit{if} \quad \tilde{\lambda} \in J_0 \cap \Lambda \quad \textit{then either} \quad C \cap B_r(\tilde{\lambda}, 0, 0) \subset M$$
$$\textit{or} \quad C \cap B_r(\tilde{\lambda}, 0, 0) \cap M = \emptyset \quad \textit{for} \quad r \quad \textit{small enough}.$$

The last condition is ensured e.g. if

$$(1.9') \qquad (\{\tilde{\lambda}\} \times E_T(\tilde{\lambda}) \times \{0\}) \cap M = \{[\tilde{\lambda}, 0, 0]\} \quad \textit{for any} \quad \tilde{\lambda} \in J_0 \cap \Lambda,$$

i.e.

$$(1.9'') \qquad E_T(\tilde{\lambda}) \cap K = \{0\} \quad \textit{for any} \quad \tilde{\lambda} \in J_0 \cap \Lambda$$

in the case $M = J \times K \times \mathbb{R}^+$. (Precisely, see the considerations in Remark 2.2.) Hence, the following continuation theorem is a consequence of Theorem 1.1.

Theorem 1.2. *Let* (0.3)–(0.5), (1.1) *be fulfilled, let* λ_0 *satisfy* (1.2), $E_T(\lambda_0) = \text{Lin}\{w_0\}$. *Consider a set* M *closed in* $J \times \mathbb{V} \times \mathbb{R}^+$ *and an open bounded interval* J_0 *such that* $\lambda_0 \in \bar{J}_0$, $J_0 \cup \{\lambda_0\} \subset J$, (1.7)–(1.9) *hold. Then there exists a closed connected set* $C_0^+ \subset C$ *containing* $[\lambda_0, 0, 0]$ *and satisfying* (d) *from Theorem* 1.1.

Remark 1.6. Let $M = J \times K \times \mathbb{R}^+$ where K is a closed cone in $\mathbb{V}$ with its vertex at the origin, $w_0 \notin K$, $-w_0 \in K$. Suppose that the conditions (1.5) and

$$(1.10) \qquad \lambda < \lambda^m \quad \textit{for any} \quad [\lambda, u, \tau] \in C \quad (\textit{with some } \lambda^m > \lambda_0 \textit{ fixed}),$$

$$(1.11) \qquad \lambda \neq \lambda_0 \quad \textit{for any} \quad [\lambda, u, \tau] \in C \quad \textit{with} \quad u \notin K$$

hold. (See Examples 1.1, 1.2.) Set $J_0 = (\lambda_0, \lambda_m)$. Let us show that then (1.7) is ensured if

$$(1.12) \quad [\lambda_n, u_n, \tau_n] \in C\,, \quad u_n \notin K\,, \quad \lambda_n \to \lambda_0\,, \quad \tau_n \to \tau\,, \quad u_n/\|u_n\| \to w \in K \Rightarrow \lambda_n \leqq \lambda_0 \quad \text{for} \quad n \geqq n_0\,.$$

Indeed, the implication $\Rightarrow$ in (1.7) follows from the first part of Remark 1.3. Further, let $[\lambda_n, u_n, \tau_n] \in C \setminus M$, $\lambda_n \in (\lambda_0, \lambda^m)$, $\lambda_n \to \lambda \in \partial J_0$, $\tau_n \to \tau$. The compactness argument yields $u_n/\|u_n\| \to w$ at least for some subsequence (denoted again by $[\lambda_n, u_n, \tau_n]$). Precisely, see Remark 2.2. The condition (0.2) implies $u_n \to u = (\delta\tau/(1 + \tau))^{1/2}\, w$. Consequently, $[\lambda, u, \tau] \in C$, i.e. $\lambda = \lambda_0$ by (1.10), and $w \notin K$ by (1.12). The case $\tau > 0$ is impossible by (1.11). Hence $\tau = 0$ and $w \in E_T(\lambda_0)$ follows (precisely, see Remark 2.2), i.e. $w = w_0$ (because $E_T(\lambda_0) = \text{Lin}\{w_0\}$, $-w_0 \in K$). This holds for any convergent subsequence of $u_n/\|u_n\|$ and it follows that $\Leftarrow$ in (1.7) is true.

Note that a special version of our continuation theorem based on the assumptions (1.10)–(1.12) is given in [3].

Theorem 1.3. *The assertions of Theorems* 1.1, 1.2 *remain valid if we replace* $\mathbb{R}^+$ *by* $\langle 0, 1)$ *or* $\langle 0, 1\rangle$ *in all assumptions* (*including the definition of* C), *and* (0.2) *by* (0.2'). *Particularly, under the assumptions of Theorem* 1.2 *modified in this way for* $\langle 0, 1\rangle$, *there is at least one couple* λ_1, u_1 *satisfying* (0.1), $\|u_1\| = \delta$, $\lambda_1 \in J_0$, $[\lambda_1, u_1, 1] \notin M$.

Proof for the case $\langle 0, 1)$ follows directly from Theorems 1.1, 1.2 by substitution $\tau' = \tau/(1 + \tau)$. The case $\langle 0, 1\rangle$ follows by the limiting process $\tau \to 1-$. The assertion that $\lambda_1 \in J_0$, $[\lambda_1, u_1, 1] \notin M$ is a consequence of the assumptions (1.7), (1.8) where also $\tau = 1$ is admissible in the case considered. (If we had $\lambda_1 \in \partial J_0$, $[\lambda_1, u_1, 1] \in C_0^+$ then $[\lambda_n, u_n, \tau_n] \in C_0^+ \setminus M$ would exist such that $\lambda_n \in J_0$, $[\lambda_n, u_n, \tau_n] \to [\lambda_1, u_1, 1]$ by the properties of C_0^+ on $\langle 0, 1)$, and this would contradict (1.7).)

Remark 1.7. In the following examples, we shall suppose that $\mathbb{V}$ is a Hilbert space, A is a linear completely continuous operator in $\mathbb{V}$, K_I is a closed convex cone in $\mathbb{V}$ with its vertex at the origin. We shall consider an eigenvalue problem for the inequality

$$(1.13) \qquad u \in K_I\,,$$
$$\langle \lambda u - Au, v - u\rangle \geqq 0 \quad \textit{for all} \quad v \in K_I\,.$$

Note that (1.13) is equivalent to the equation

$$\lambda u - PAu = 0 \tag{1.14}$$

where P is the projection onto K_I, i.e. $\|Pu - u\| = \min_{v \in K_I} \|v - u\|$ for any $u \in K_I$ (cf. [7], for the properties of projections on convex sets see [15]). Recall that P is lipschitzian, positively homogeneous (i.e. $P(tu) = tPu$ for $t \geqq 0$, $u \in \mathbb{V}$), and

$$(I - P)\, v = 0\,, \quad \langle (I - P)\, u, u\rangle > 0\,, \tag{1.15}$$
$$\langle (I - P)\, u, v\rangle \leqq 0 \quad \text{for all} \quad v \in K_I\,, \quad u \notin K_I\,.$$

Set

$$K_I^P = \{v \in K;\ \langle (I - P)\, u, v\rangle < 0 \text{ for all } u \notin K_I\}\,.$$

It is easy to see that $K_I^P \supset K_I^0$. (In the opposite case $\langle (I - P)\, u, v\rangle = 0$ for some $u \notin K_I$, $v \in K_I^0$ in accordance with (1.15); there is $w \in \mathbb{V}$ such that $\langle (I - P)\, u, w\rangle > 0$ and $v + w \in K_I^0$, i.e. $\langle (I - P)\, u, v + w\rangle > 0$, which contradicts (1.15).) As an illustration consider $\mathbb{V} = W_2^1(\Omega)$, where Ω is a domain in $\mathbb{R}^n$ with a lipschitzian boundary $\partial\Omega$,

$$K_I = \{u \in W_2^1(\Omega);\ u \geqq 0 \text{ on } \Gamma \text{ in the sense of traces}\}\,, \tag{1.16}$$

where $\Gamma \subset \partial\Omega$.

Then $K_I^P \supset \{v \in \mathbb{V};\ v \geqq \varepsilon \text{ on } \Gamma \text{ for some } \varepsilon > 0\}$. (This can be shown analogously as the inclusion $K_I^P \supset K_I^0$.) Note that $\{v \in \mathbb{V};\ v \geqq \varepsilon \text{ on } \Gamma \text{ for some } \varepsilon > 0\} = K_I^0$ in in the case $n = 1$, $\Omega = (0, 1)$, but $K_I^0 = \emptyset$ in the case $n > 1$.

Example 1.1. Let $\lambda_0 > 0$ be a simple eigenvalue of A with the corresponding eigenvector w_0. Let w_0^* be the corresponding eigenvector of the adjoint operator A^*, i.e.

$$\lambda_0 w_0^* - A^* w_0^* = 0\,. \tag{1.17}$$

We shall use the notation from Remark 1.7. Suppose that

$$w_0 \notin K_I\,, \quad -w_0 \in K_I\,, \quad w_0^* \in K_I^P\,, \quad \langle w_0, w_0^*\rangle > 0\,, \tag{1.18}$$

(1.19) *all the eigenvectors corresponding to the tentative eigenvalues of A greater then λ_0 lie outside of K_I.*

(The last condition is automatically fulfilled if λ_0 is the greatest real eigenvalue of A.) Set $\delta = 1$, $J = R^+$, $M = J \times K \times \langle 0, 1\rangle$, $T(\lambda)\, u = (1/\lambda)\, Au$, $H_\tau(\lambda, u) = (\tau/\lambda)\,.\,(I - P)\, Au$ (for $\tau \in \langle 0, 1\rangle$), where $K = \{u \in \mathbb{V};\ Au \in K_I\}$. Setting $\tau = 0$ and $\tau = 1$ in (0.1), we receive $\lambda u - Au = 0$ and (1.14), respectively, i.e., (0.1) joins the eigenvalue problem for A to that for the inequality (1.13) (see Remark 1.7). We shall show that the assumptions of Theorem 1.2 modified for $\langle 0, 1\rangle$ (see Theorem 1.3) are fulfilled for $J_0 = (\lambda_0, \lambda^m)$ with λ^m large enough. Verification of (0.3)–(0.5), (1.1) is trivial, (1.2) follows from the simplicity of λ_0.

Verification of (1.7): In accordance with Remark 1.6, it is sufficient to prove (1.5), (1.10)–(1.12). (Note that if $-w_0 \in K_I^0$ were supposed in addition to (1.18) then (1.12) would be unnecessary – cf. Remark 1.3.) First, realize that $[\lambda_n, u_n, \tau_n] \in C$ means that

$$(1.20) \qquad \lambda_n u_n - Au_n + \tau_n(I - P)\, Au_n = 0$$

and (0.2) holds.

Proof of (1.5). Multiply (1.20) by w_0^*, (1.17) by u_n and subtract the resulting expressions. We get

$$(1.21) \qquad (\lambda_n - \lambda_0)\, \langle u_n, w_0^* \rangle + \tau_n \langle (I - P)\, Au_n, w_0^* \rangle = 0\,.$$

If $u_n/\|u_n\| \to w_0 = Aw_0/\lambda_0$ then (1.18) implies $\langle u_n, w_0^* \rangle > 0$, $Au_n \notin K_I$, $\langle (I - P)\, Au_n, w_0^* \rangle < 0$ for $n \geqq n_0$ and $\lambda_n > \lambda_0$ follows.

Proof of (1.10). If there were no such λ^m then $[\lambda_n, u_n, \tau_n]$ satisfying (1.20), $\lambda_n \to +\infty$, $\|u_n\| \neq 0$ would exist. Multiply (1.20) by u_n. The resulting left-hand side should be positive due to the boundedness of A and P for $n \geqq n_0$, which is a contradiction.

Set $J_0 = (\lambda_0, \lambda^m)$.

Proof of (1.11). If this were not true then

$$(1.22) \qquad \lambda u - Au + \tau(I - P)\, Au = 0$$

would hold with $\lambda = \lambda_0$, $u \notin K$, i.e. $Au \notin K_I$, $\tau \neq 0$. Multiply (1.22) by w_0^*, (1.17) by u and subtract. We receive $\langle (I - P)\, Au, w_0^* \rangle = 0$ which contradicts (1.18) and the definition of K_I^P.

Proof of (1.12). Let (1.20) hold with $u_n \notin K$, $\lambda_n \to \lambda_0$, $\tau_n \to \tau$, $u_n/\|u_n\| \to w \in K$. Dividing it by $\|u_n\|$, letting $n \to \infty$ and using (1.15), we receive $\lambda_0 w - Aw = 0$, i.e. either $w = w_0$ or $w = -w_0$. The first case is excluded by (1.18) because $Aw = \lambda_0 w \in K$, i.e. $w \in K_I$. Now, (1.21) implies $\lambda_n \leqq \lambda_0$ analogously as in the proof of (1.5). (Note that $\langle u_n, w_0^* \rangle$ now has the opposite sign.)

Verification of (1.8): (1.22) with $Au \in \partial K_I$, $\|u\| \neq 0$ means by (1.15) that λ is an eigenvalue of A corresponding to the eigenvector $u = (1/\lambda)\, Au \in \partial K_I$. This is excluded for $\lambda > \lambda_0$ by (1.19).

Verification of (1.9) : (1.19) implies (1.9″) and it is sufficient to use the end of Remark 1.5.

Now, Theorem 1.3 together with Remark 1.7 ensures the existence of an eigenvalue $\lambda_I \in (\lambda_0, \lambda^m)$ of (1.13) with the corresponding eigenvector $u_1 \in \partial K_I$. The last inclusion follows from the fact that $[\lambda_\tau, u_\tau, \tau] \notin M$ (i.e. $Au_\tau \notin K_I$) for any $[\lambda_\tau, u_\tau, \tau] \in C_0^+$ with $\tau \in (0, 1\rangle$ (see (d)), and from the fact that

$$\lambda_\tau u_\tau = (1 - \tau)\, Au_\tau + \tau PAu_\tau \notin K_I \quad \text{if} \quad Au_\tau \notin K_I\,, \quad \tau \text{ is close to } 1\,,$$

and that $\lambda_1 u_1 = PAu_1 \in K_I$.

Remark 1.8. Consider a completely continuous operator β in $\mathbb{V}$ such that

$$(1.23) \qquad \beta v = 0\,, \quad \langle \beta u, u \rangle > 0\,, \quad \langle \beta u, v \rangle \leqq 0 \quad \textit{for all} \quad u \notin K_I\,, \quad v \in K_I$$

(a penalty operator corresponding to K_I). Let β be positively $(1 + \alpha)$-homogeneous with some $\alpha \geqq 0$ (i.e. $\beta(tu) = t^{1+\alpha}\beta u$ for $t > 0$, $u \in \mathbb{V}$). Set $K_I^\beta = \{v \in K;\ \langle \beta u, v \rangle < 0$ for all $u \notin K_I\}$. Analogously as for K_I^P (see Remark 1.7) we receive $K_I^\beta \supset K_I^0$. For instance, in the case of the cone (1.16) we can consider the operator defined by

$$\langle \beta u, v \rangle = -\int_\Gamma (u^-)^{1+\alpha} v dS \quad \text{for all} \quad u, v \in \mathbb{V}$$

with an arbitrary fixed $\alpha \geqq 0$. Then $K_I^\beta = \{v \in \mathbb{V}; v > 0 \text{ on } \Gamma \text{ in the sense of traces}\}$. This set coincides with K_I^0 in the case $n = 1$, $\Omega = (0, 1)$, while $K_I^0 = \emptyset$ in the case $n > 1$. In general, it is known that if $\lambda_n u_n - Au_n + \tau_n \beta u_n = 0$, $\lambda_n \to \lambda_I$, $u_n \notin K_I$, $u_n \rightharpoonup u_I$, $\tau_n \to +\infty$ then $u_n \to u_I$ and λ_I, u_I satisfy (1.13), $u_I \in \partial K_I$ (see [4, proof of Lemma 2.4] or [5, proof of Lemma 3.3]).

Example 1.2. Consider the same situation as in Example 1.1 but replace K_I^P by K_I^β (Remark 1.8) and set $J = \mathbb{R}^+$, $K = K_I$, $M = J \times K \times \mathbb{R}^+$, $T(\lambda) u = (1/\lambda) Au$, $H_\tau(\lambda, u) = (\tau/\lambda) \beta u$ (for $\tau \in \mathbb{R}^+$). Then (0.1) joins the eigenvalue problem for A ($\tau = 0$) to that for the inequality (1.13) ("$\tau = +\infty$") again (see Remark 1.8). Since the properties of the penalty operator β are analogous to those of $(I - P) A$ (cf. (1.23), (1.15)), the assumptions of Theorem 1.2 can be verified similarly as in Example 1.1. (In the case $\alpha > 0$ it is necessary in the proof of (1.12) to use the fact that $\beta u/\|u\| \to 0$ if $\|u\| \to 0$. Theorem 1.2 together with Remark 1.8 yields the existence of an eigenvalue $\lambda_\infty \in \langle \lambda_0, \lambda^m \rangle$ of (1.13) with the corresponding eigenvector $u_\infty \in \partial K_I$. In fact, in reasonable cases (e.g. in the case of the cone (1.16)) it is possible to show $\lambda_\infty > \lambda_0$ (see e.g. [5, Lemma 3.3], [3, proof of Theorem 1.1]).

Example 1.3. Let $\lambda_0, \tilde{\lambda}_0$ be two simple eigenvalues of A with the corresponding eigenvectors $w_0, \tilde{w}_0$, $\lambda_0 > \tilde{\lambda}_0 > 0$. Denote by w_0^* and $\tilde{w}_0^*$ the eigenvectors of A^* corresponding to λ_0 and λ_0^*, respectively, and suppose that

$$w_0 \notin K_I\,, \quad \tilde{w}_0 \notin K_I\,, \quad w_0^* \in K_I^0\,, \quad \tilde{w}_0^* \in K_I^0\,, \quad \langle w_0, w_0^* \rangle < 0\,, \quad \langle \tilde{w}_0, \tilde{w}_0^* \rangle < 0\,, \tag{1.24}$$

$$-\tilde{w}_0 \in K_I^0 \tag{1.25}$$

(see Remark 1.9 below). Let all the eigenvectors corresponding to the tentative eigenvalues $\lambda \in (\tilde{\lambda}_0, \lambda_0)$ of A lie outside of K_I. Consider δ, J, M, K, $T(\lambda)$, H_τ ($\tau \in \langle 0, 1 \rangle$) from Example 1.1, $J_0 = (\tilde{\lambda}_0, \lambda_0)$. Analogously as in Example 1.1 we can show that (1.5), (1.11) and (1.12) with $>$ replaced by $<$ and $\leqq$ replaced by $\geqq$ hold. It follows that the equivalence in (1.7) is true for $\lambda_n \to \lambda_0$ (cf. the considerations in Remark 1.6). Analogously, it can be shown that (1.5), (1.11) with $\lambda_0, w_0, >$ replaced by $\tilde{\lambda}_0, \tilde{w}_0, <$ hold. Simultaneously, the assumption $-\tilde{w}_0 = -(1/\tilde{\lambda}_0) A\tilde{w}_0 \in K_I^0$ excludes the existence of a sequence $\{u_n\}$ satisfying $u_n \notin K$, $u_n/\|u_n\| \rightharpoonup -\tilde{w}_0$. (Here we really need K_I^0, which could be replaced by K_I^P in all other considerations, cf. Example 1.1.) It follows that there is no $[\lambda, u, \tau] \in B_\sigma(\tilde{\lambda}_0, 0, 0) \cap C$ with $\lambda > \tilde{\lambda}_0$, $u \notin K$ if $\sigma > 0$ is small enough (precisely, see Remark 2.2). Now, it is easy to see that for the proof of (1.7) it is sufficient to show that there is no sequence satisfying

$$[\lambda_n, u_n, \tau_n] \in C\,, \quad u_n \notin K\,, \quad \lambda_n \in (\tilde{\lambda}_0, \lambda_0)\,, \quad \lambda_n \to \tilde{\lambda}_0\,, \quad \tau_n \to \tau \neq 0\,, \quad u_n \to u \in K\,. \tag{1.26}$$

If (1.26) were true then we would receive $\tilde{\lambda}_0 u - Au = 0$ by the limiting process in (1.20) and (1.15). Hence, $u = \pm\tilde{w}_0$, $u = (1/\tilde{\lambda}_0) Au \in \partial K_I$. This would contradict the assumptions $\tilde{w}_0 \notin K$, $-\tilde{w}_0 \in K_I^0$. The assumptions (1.8), (1.9) can be verified analogously as in Example 1.1. Now, Theorem 1.3 (the case of the closed interval $\langle 0, 1\rangle$) together with Remark 1.7 ensure the existence of an eigenvalue $\lambda_1 \in (\tilde{\lambda}_0, \lambda_0)$ of the inequality (1.13) with the corresponding eigenvector $u_1 \in \partial K_I$ similarly as in Example 1.1.

Remark 1.9. If A is selfadjoint then (1.24) means $w_0^* = -w_0 \in K_I^0$, $\tilde{w}_0^* = -\tilde{w}_0 \in \in K_I^0$ and the inequalities demanded are fulfilled automatically.

Remark 1.10. The aim of Examples 1.1–1.3 is only to illustrate our abstract theorems. In fact, Theorems 1.2, 1.3 can be used to prove more general results concerning the existence of eigenvalues and bifurcations of inequalities lying above the greatest eigenvalue of the operator ([2], [3]), or between given eigenvalues of the operator ([4], [5]). Also multiple eigenvalues can be considered ([6]). In the papers mentioned above, no general continuation theorem is used but in fact its assertion is always proved on the basis of Dancer's global bifurcation result [1] for the particular situation. A special version of Theorem 1.2 (formulated for $M = \mathbb{R} \times K \times \mathbb{R}^+$ by using the assumptions (1.5), (1.10), (1.11), (1.12)) is used and briefly proved in [3].

The results concerning eigenvalues and bifurcations of inequalities of the type mentioned can be proved also by a method developed by P. Quittner [12], [13]. It is based on a direct application of the Leray-Schauder degree and is simpler than the proof of Theorems 1.1, 1.2. However, this method can give only existence results while Theorem 1.2 simultaneously gives approximations of the eigenvalues (or bifurcation points) and eigenvectors of the inequality by those of the equation with the penalty. Speaking about higher eigenvalues and bifurcations of variational inequalities we must mention also the first results in this direction given by E. Miersemann [8], [9], where the potential case is considered.

2. PROOF OF ABSTRACT RESULTS

Remark 2.1. The system (0.1), (0.2) can be written in the form of the single equation

$$x - L(\lambda)\, x + G(\lambda, x) = 0 \tag{2.1}$$

in the space $X = \mathbb{V} \times \mathbb{R}$, or simply

$$\Phi(\lambda)\,(x) = 0\,, \tag{2.1'}$$

where

$$\begin{aligned} L(\lambda)\, x &= [T(\lambda)\, u, 0]\,, \\ G(\lambda, x) &= [H_\tau(\lambda, u), -((1+\tau)/\delta)\, \|u\|^2] \quad \text{for all} \quad x = [u, \tau] \in X\,, \quad \lambda \in J\,, \\ \Phi(\lambda)\,(x) &= x - L(\lambda)\, x + G(\lambda, x)\,. \end{aligned}$$

The following conditions are satisfied under the assumptions (0.3), (0.4), (0.5):

(2.2) *for any $\lambda \in J$, $L(\lambda)$ is a linear continuous operator in X; the mapping $\lambda \to L(\lambda)$ of J into the space of linear continuous operators in X is continuous;*

(2.3) *the mapping $M(\lambda, x) = L(\lambda)\,x - G(\lambda, x)$ of $J \times X$ into X is completely continuous, $G(\lambda, 0) = 0$;*

(2.4) $\lim_{|||x||| \to 0} \dfrac{G(\lambda, x)}{|||x|||} = 0$ *uniformly on compact λ-intervals.*

Hence, the corresponding linearized equation to (2.1) is

$$x - L(\lambda)\,x = 0\,. \tag{2.5}$$

Further, $x \in E_L(\lambda)$ if and only if $x = [u, 0]$, $u \in E_T(\lambda)$. We shall set $x_0 = [w_0, 0]$. Hence, $E_L(\lambda_0) = \mathrm{Lin}\,\{x_0\}$ under our assumptions.

Remark 2.2. Let $[\lambda_n, x_n] = [\lambda_n, u_n, \tau_n] \in C$, $\tau_n \neq 0$ (i.e. $\|u_n\| \neq 0$), $[\lambda_n, x_n] \to [\lambda, 0]$. It follows from (2.1) for λ_n, x_n (see Remark 2.1) divided by $|||x_n|||$ and from (2.2), (2.4) that there exists a subsequence such that $x_{k_n}/|||x_{k_n}||| \to y$ for some $y \in E_L(\lambda)$, $|||y||| = 1$. According to (0.2) and Remark 2.1, that means also $u_{k_n}/\|u_{k_n}\| \to w \in E_T(\lambda)$, $\|w\| = 1$, where $y = [w, 0]$. Particularly, $\lambda \in \Lambda$ for any $[\lambda, 0, 0] \in C$. Further, if $[\lambda_n, x_n] = [\lambda_n, u_n, \tau_n] \in C \cap K_\eta^+$ (see list of notation), $[\lambda_n, x_n] \to [\lambda_0, 0]$, then $x_n/|||x_n||| \to x_0$, i.e. $u_n/\|u_n\| \to w_0$. Indeed, in the opposite case the considerations mentioned above would imply the existence of a subsequence such that $x_{k_n}/|||x_{k_n}||| \to y \in E_L(\lambda_0) \cap \overline{K_\eta^+}$, $y \neq x_0$. But this is impossible because x_0 is the only normed element in $E_L(\lambda_0) \cap \overline{K_\eta^+}$ under our assumptions. Analogously for K_η^-, $-x_0$. Of course, all these assertions could be shown directly on the basis of the equation (0.1) and the assumptions (0.3), (0.5), without Remark 2.1.

Remark 2.3. For any $\eta \in (0, 1)$ fixed there is $S > 0$ such that $(C \setminus \{[\lambda_0, 0]\} \cap B_S(\lambda_0, 0) \subset K_\eta$ (cf. [1], [11]). In the opposite case $\eta \in (0, 1)$ and a sequence $\{[\lambda_n, x_n]\} \subset C$ would exist such that $[\lambda_n, x_n] \to [\lambda_0, 0]$, $|\langle y_0^*, x_n\rangle_X| < \eta$. According to Remark 2.2, we can suppose $x_n/|||x_n||| \to y \in E_L(\lambda_0)$, i.e. $y = \pm x_0$. Simultaneously $|\langle y_0^*, y\rangle| < \eta$ which contradicts the definition of y_0^* (see list of notation). Moreover, it follows from Remark 2.2 that under the assumption (1.3) S can be chosen so that $K_\eta^+ \cap B_S(\lambda_0, 0) \cap \{[\lambda, u, \tau] \in C;\ \lambda \notin J_0\} = \emptyset$.

It is easy to see that if we consider a sequence of mappings G_m satisfying (2.3), (2.4) uniformly with respect to m then S with the properties mentioned (for G_m instead of G) can be chosen indepedently of m (cf. [11]).

Remark 2.4. According to Remark 2.1 we have

$$C = \mathrm{cl}\,\{[\lambda, x] \in J \times X;\ |||x||| \neq 0,\ (2.1)\ \text{is fulfilled}\}\,,$$

Note that C is locally compact by (2.3). Choose $\eta \in (0, 1)$ fixed. For any $\varepsilon > 0$, we denote by C_ε the component of $C \setminus B_\varepsilon(\lambda_0, 0) \cap K_\eta^-$ containing $[\lambda_0, 0]$. It follows from Remarks 2.2, 2.3 that if $[\lambda_n, x_n] \in C_\varepsilon$, $[\lambda_n, x_n] \to [\lambda_0, 0]$ then $x_n/|||x_n||| \to x_0$.

That means $[\lambda, x] \notin M$, $\lambda \in J_0$ for any $[\lambda, x] \in C_\varepsilon \cap B_\sigma(\lambda_0, 0) \setminus \{[\lambda_0, 0]\}$ if $\sigma > 0$ is small enough under the assumption (1.3). Further, we shall show that C_ε has all properties of C_0^+ from Theorem 1.1 if $\varepsilon > 0$ is small enough.

Lemma 2.1. *Let none of the conditions* (a), (b), (c) *from Theorem* 1.1 *be satisfied for* $C_0^+ = C_\varepsilon$ *with some* $\varepsilon > 0$ *fixed. Then*

$$[\lambda, u, \tau] \notin M \quad \textit{for any} \quad [\lambda, u, \tau] \in C_\varepsilon \quad \textit{with} \quad \tau \neq 0\,, \tag{2.6}$$

$$\lambda \in J_0 \quad \textit{for any} \quad [\lambda, u, \tau] \in C_\varepsilon \setminus \{[\lambda_0, 0, 0]\}\,. \tag{2.7}$$

Proof. The assumption (1.3) together with Remark 2.4 imply that our assertion holds in a neighbourhood of $[\lambda_0, 0, 0]$, i.e. (2.6), (2.7) with C_ε replaced by $C_\varepsilon \cap B_\sigma(\lambda_0, 0, 0)$ hold for $\sigma > 0$ small enough. It follows by using the connectedness of C_ε and Remark 2.2 that if (2.6) or (2.7) were not true then at least one of the conditions (a), (b), (c) for $C_0^+ = C_\varepsilon$ would be fulfilled.

Remark 2.5. Note that
$\deg(\Phi(\lambda), 0, B_\sigma^X(0)) = \deg(I - L(\lambda), 0, B_\sigma^X(0)) = \deg(I - T(\lambda), 0, B_\sigma^V(0)) = (-1)^{\beta(\lambda)}$ for any $\lambda \in J \setminus \Lambda$ and $\sigma > 0$ small enough. This follows from Remarks 1.1, 2.1 by using the homotopy invariance and elementary properties of the Leray-Schauder degree.

Remark 2.6. Let us consider that there exists $\delta_1 > 0$ such that

$$\Phi(\lambda_0)(x) \neq 0 \quad \text{if} \quad 0 < |||x||| \leqq \delta_1 \tag{2.8}$$

(i.e., $x = 0$ is an isolated solution of (0.1), (0.2) with $\lambda = \lambda_0$). In this case, (2.3) implies also the existence of $\delta_2 > 0$ such that

$$\Phi(\lambda)(x) \neq 0 \quad \text{if} \quad |||x||| = \delta_1\,, \quad |\lambda - \lambda_0| \leqq \delta_2\,. \tag{2.9}$$

Lemma 2.2. (cf. [1, Lemma 1]). *Let* (2.8), (2.9) *be fulfilled with* $\delta_1, \delta_2 > 0$, $\delta_1 + \delta_2 < S$. *If* $0 < \gamma_+ < \delta_2$, $0 < \gamma_- < \delta_2$, $\lambda_0 \pm \gamma_\pm \notin \Lambda$, $\gamma_\pm$ *satisfy the oddness condition from* (1.2) *then*

$$\deg(\Phi(\lambda_0 + \gamma_+), 0, W_{\sigma,\delta_1}^+) - \deg(\Phi(\lambda_0 - \gamma_-), 0, W_{\sigma,\delta_1}^+) \tag{2.10}$$

is odd for any $\sigma > 0$ *small enough, where* $W_{\sigma,\delta_1}^+ = \{x \in X;\, [\lambda, x] \in K_\eta^+,\, \sigma < |||x||| < \delta_1\}$.

Proof. There exists a mapping $\tilde{G}\colon J \times X \to X$ such that $\tilde{G}(\lambda, x) = G(\lambda, x)$ for $[\lambda, x] \in K_\eta^+$ (where G is from Remark 2.1), $\tilde{G}(\lambda, x) = -\tilde{G}(\lambda, -x)$ for all $\lambda \in J$, $x \in X$, $\tilde{G}$ satisfies (2.3), (2.4) again (with $\tilde{G}$ instead of G), and

$$(\tilde{C} \setminus \{[\lambda_0, 0]\}) \cap B_S(\lambda_0, 0) \subset K_\eta \tag{2.11}$$

where $\tilde{C} = \operatorname{cl}\{[\lambda, x] \in J \times X;\, \tilde{\Phi}(\lambda)(x) = 0\}$, $\tilde{\Phi}(\lambda)(x) = x - L(\lambda)x + \tilde{G}(\lambda, x)$.

Indeed, we can define

$$\tilde{G}(\lambda, x) = G(\lambda, x) \quad \text{for all} \quad [\lambda, x] \in K_\eta^+\,,$$

$$\tilde{G}(\lambda, x) = \frac{\langle x_0^*, x\rangle_X}{\eta|||x|||} G(\lambda, \eta|||x||| x_0 + y) \quad \text{for}$$

$$x = \langle x_0^*, x\rangle x_0 + y, \quad \eta|||x||| \geqq \langle x_0^*, x\rangle_X \geqq 0,$$

$$\tilde{G}(\lambda, -x) = -\tilde{G}(\lambda, x) \quad \text{for} \quad \langle x_0^*, x\rangle \leqq 0$$

(cf. the proof of Theorem 1.25 in [11]). It follows from (2.11) and the oddness of $\tilde{\Phi}(\lambda)$ that

$$2 \deg (\Phi(\lambda_0 \pm \gamma_\pm), 0, W_{\sigma,\delta_1}^+) = \deg (\tilde{\Phi}(\lambda_0 \pm \gamma_\pm), 0, W_{\sigma,\delta_1}^+) + \\ + \deg (\tilde{\Phi}(\lambda_0 \pm \gamma_\pm), 0, W_{\sigma,\delta_1}^-) = \deg (\tilde{\Phi}(\lambda_0 \pm \gamma_\pm), 0, B_{\delta_1}(0)) - \\ - \deg (\tilde{\Phi}(\lambda_0 \pm \gamma_\pm), 0, B_\sigma(0)),$$

where

$$W_{\sigma,\delta_1}^- = \{x \in X;\ [\lambda_0, x] \in K_\eta^-,\ \sigma < |||x||| < \delta_1\}.$$

Subtract these equations (for + and −). Our assertion is an easy consequence because $\deg (\tilde{\Phi}(\lambda_0 + \gamma_+), 0, B_{\delta_1}(0)) = \deg (\tilde{\Phi}(\lambda_0 - \gamma_-), 0, B_{\delta_1}(0))$ by the homotopy invariance and (2.9), $\deg (\tilde{\Phi}(\lambda_0 + \gamma_+), 0, B_\sigma(0)) - \deg (\tilde{\Phi}(\lambda_0 - \gamma_-), 0, B_\sigma(0)) = \pm 2$ for σ small by the assumption (1.2) (see Remark 2.5).

Analogously as in [11] and [1], we shall use the following assertion on continua.

Lemma 2.3 (see [14]). *Let K be a compact metric space, A, B disjoint closed subsets of K. Then either there is a closed connected subset of K meeting both A and B or $K = K_A \cup K_B$, where K_A, K_B are disjoint compact subsets of K, $A \subset K_A$, $B \subset K_B$.*

Lemma 2.4 (cf. [1, Lemma 2]). *If (2.8) holds and $0 < \varepsilon < S$, $C_\varepsilon \cap \partial B_\varepsilon(\lambda_0, 0) \cap K_\eta^- = \emptyset$ then C_ε is noncompact.*

Proof. We can suppose $\delta_1 < \varepsilon$, (2.9) holds with $\delta_2 < \varepsilon - \delta_1$. Assume that C_ε is compact and $C_\varepsilon \cap \partial B_\varepsilon(\lambda_0, 0) \cap K_\eta^- = \emptyset$ (i.e. also $C_\varepsilon \cap \partial B_\varepsilon(\lambda_0, 0) \cap \overline{K_\eta^-} = \emptyset$ by Remark 2.3). Let W be an ε_0-neighbourhood of C_ε, $\varepsilon_0 < \varepsilon$, $\varepsilon_0 < \operatorname{dist}(C_\varepsilon, \partial B_\varepsilon(\lambda_0, 0) \cap \overline{K_\eta^-})$, $\overline{W} \subset J \times X$. The set $(\overline{W} \cap C) \setminus (B_\varepsilon(\lambda_0, 0) \cap K_\eta^-)$ forms a compact metric space in view of (2.3), C_ε and $(C \cap \partial W) \setminus (B_\varepsilon(\lambda_0, 0) \cap K_\eta^-)$ are its closed disjoint subsets. The first possibility in Lemma 2.3 is excluded with respect to the definition of C_ε and therefore there exist disjoint compact sets K_1, K_2 such that

$$(\overline{W} \cap C) \setminus (B_\varepsilon(\lambda_0, 0) \cap K_\eta^-) = K_1 \cup K_2, \tag{2.12}$$

$$C_\varepsilon \subset K_1, \quad (C \cap \partial W) \setminus (B_\varepsilon(\lambda_0, 0) \cap K_\eta^-) \subset K_2. \tag{2.13}$$

We can choose γ_+, γ_- satisfying the oddness condition from (1.2), such that

$$0 < \gamma_- \leqq \gamma_+ < \inf(\delta_1, \delta_2), \quad \lambda_0 \pm \gamma_\pm \notin \Lambda, \tag{2.14}$$

$$|\lambda - \lambda_0| > \gamma_+ \quad \text{for any} \quad [\lambda, x] \in K_2, \quad |||x||| \leqq \delta_1. \tag{2.15}$$

Further, there exists an open set U in X such that

$$\overline{U} \subset W, \quad K_1 \subset U, \quad K_2 \cap \overline{U} = \emptyset, \quad [\lambda_0 \pm \gamma_\pm, 0] \notin \overline{U}, \tag{2.16}$$

$$\overline{U} \cap B_\varepsilon(\lambda_0, 0) \cap K_\eta^- \subset B_{\gamma_-}(\lambda_0, 0). \tag{2.17}$$

Particularly,

$$\partial U \cap C \subset B_{\gamma_-}(\lambda_0, 0)\,. \tag{2.18}$$

Now, we shall show that

$$\deg\,(\Phi(\lambda_0 \pm \gamma_\pm), 0, U_{\lambda_0 \pm \gamma_\pm}) \quad \text{are even}\,, \tag{2.19}$$

where we denote $U_\lambda = \{x \in X;\ [\lambda, x] \in U\}$.

First, $\deg\,(\Phi(\lambda), 0, U_\lambda) = 0$ for λ sufficiently close to ∂J because $U_\lambda = \emptyset$ for such λ. It follows from (2.16) that for any $[\tilde{\lambda}, 0] \in K_2$ there is $\theta > 0$ such that $0 \notin \overline{U}_\lambda$ if $|\lambda - \tilde{\lambda}| \leqq \theta$. Thus, $\deg\,(\Phi(\lambda), 0, U_\lambda)$ is constant for $|\lambda - \tilde{\lambda}| \leqq \theta$ by the homotopy invariance of the degree. (Realize that if $\Phi(\lambda)(x) = 0$, $[\lambda, x] \notin C$ then $x = 0$.) Hence, for the proof of (2.19) it is sufficient to show that if $\lambda_1, \lambda_2 \in J \setminus \Lambda$, $\lambda_2 > \lambda_1 \geqq \lambda_0 + \gamma_+$ or $\lambda_1 < \lambda_2 \leqq \lambda_0 - \gamma_-$, $0 \notin U_{\lambda_1} \cup U_{\lambda_2}$, $[\lambda, 0] \notin K_2$ for all $\lambda \in \langle \lambda_1, \lambda_2 \rangle$, then $\deg\,(\Phi(\lambda_1), 0, U_{\lambda_1}) - \deg\,(\Phi(\lambda_2), 0, U_{\lambda_2})$ is even. For any such couple λ_1, λ_2 there is a sufficiently small $\sigma > 0$ such that $U_{\lambda_1} \cap B_\sigma(0) = U_{\lambda_2} \cap B_\sigma(0) = \emptyset$ and $[\lambda, x] \notin K_2$ for any $\lambda \in \langle \lambda_1, \lambda_2 \rangle$, $|||x||| = \sigma$. This together with (2.12), (2.18) yields that $\deg\,(\Phi(\lambda), 0, U_\lambda \cup B_\sigma(0))$ is constant for $\lambda \in \langle \lambda_1, \lambda_2 \rangle$. Thus,

$$\begin{aligned} &\deg\,(\Phi(\lambda_1), 0, U_{\lambda_1}) - \deg\,(\Phi(\lambda_2), 0, U_{\lambda_2}) = \\ &= \deg\,(\Phi(\lambda_2), 0, B_\sigma(0)) - \deg\,(\Phi(\lambda_1), 0, B_\sigma(0))\,. \end{aligned}$$

The right-hand side is even for σ small by Remark 2.5 and (2.19) is proved. Further,

$$\deg\,(\Phi(\lambda_0 + \gamma_+), 0, U_{\lambda_0 + \gamma_+} \setminus B_{\delta_1}(0)) = \deg\,(\Phi(\lambda_0 - \gamma_-), 0, U_{\lambda_0 - \gamma_-} \setminus B_{\delta_1}(0)) = 0$$

by (2.9), (2.14), (2.18). Hence, (2.19) means that also

$$\deg\,(\Phi(\lambda_0 + \gamma_+), 0, U_{\lambda_0 + \gamma_+} \cap B_{\delta_1}(0)) - \deg\,(\Phi(\lambda_0 - \gamma_-), 0, U_{\lambda_0 - \gamma_-} \cap B_{\delta_1}(0)) \tag{2.20}$$

is even.

There is $\sigma \in (0, \delta_1)$ such that $\Phi(\lambda_0 \pm \gamma_\pm)(x) \neq 0$ if $0 < |||x||| < \sigma$ (see (2.14) and Remark 2.2). It follows from Remark 2.3, (2.17), (2.12), (2.15), (2.16) that if $\Phi(\lambda_0 \pm \gamma_\pm)(x) = 0$, $|||x||| < \delta_1$, then $x \in U_{\lambda_0 \pm \gamma_\pm}$ if and only if $[\lambda_0 \pm \gamma_\pm, x] \in K_\eta^+$, $\sigma < |||x||| < \delta_1$.

That means

$$\deg\,(\Phi(\lambda_0 \pm \gamma_\pm), 0, U_{\lambda_0 \pm \gamma_\pm} \cap B_{\delta_1}(0)) = \deg\,(\Phi(\lambda_0 \pm \gamma_\pm), 0, W^+_{\sigma,\delta_1})\,,$$

where W^+_{σ,δ_1} was introduced in Lemma 2.2. This together with (2.20) contradicts Lemma 2.2.

Lemma 2.5 (cf. [1, Lemma 3]). *The assertion of Lemma 2.4 holds without the assumption* (2.8).

Proof. Let x_0^* be a nontrivial solution of $y - L(\lambda_0)^*\,y = 0$ where $L(\lambda_0)^*$ is the adjoint operator to $L(\lambda_0)$. Let $\tilde{x}_0 \in X$ be such that $\langle x_0^*, \tilde{x}_0 \rangle_X = 1$. According to (2.4) there exist $c > 0$ and a continuous function ϱ on $\mathbb{R}^+$ such that

$$\varrho(0) = 0\,, \quad \varrho(t) > \sup_{|||x||| = t} \frac{\langle x_0^*, G(\lambda_0, x) \rangle_X}{|||x|||} \quad \text{for} \quad 0 < t \leqq c\,.$$

Consider continuous functions $f_n\colon \mathbb{R}^+ \to \langle 0, 1\rangle$ such that $f_n(t) = t$ for $0 \leqq t \leqq c/2n$, $f_n(t) = 0$ for $t \geqq c/n$ $(n = 1, 2, \ldots)$. Introduce the mappings $G_n\colon J \times X \to X$ by $G_n(\lambda, x) = G(\lambda, x) + f_n(|||x|||)\, \varrho(|||x|||)\, \tilde{x}_0$. These mappings satisfy (2.3), (2.4) uniformly with respect to n, and (2.8) holds for any Φ_n (with δ_1 depending on n, $n = 1, 2, \ldots$), $\Phi_n(\lambda)(x) = x - L(\lambda)(x) + G_n(\lambda, x)$. Remark 2.4 implies that there is $S > 0$ such that $(C_n \setminus \{[\lambda_0, 0]\}) \cap B_S(\lambda_0, 0) \subset K_\eta$, where

$$C_n = \mathrm{Cl}\,\{[\lambda, x] \in J \times X;\ \Phi_n(\lambda)(x) = 0,\ |||x||| \neq 0\}\,.$$

Let C^n_ε be the component of $C_n \setminus (B_\varepsilon(\lambda_0, 0) \cap K^-_\eta)$ containing $[\lambda_0, 0]$. Suppose that the assertion of Lemma 2.4 is not true, i.e. C_ε is compact and $C_\varepsilon \cap \partial B_\varepsilon(\lambda_0, 0) \cap \cap K^-_\eta = \emptyset$. Analogously as in the proof of Lemma 2.4 there exists a bounded open set U in $J \times X$ such that $[\lambda_0, 0] \in U$, $[\partial U \cap (C \cup \Lambda)] \subset (B_\varepsilon(\lambda_0, 0) \cap K^-_\eta)$, $\bar{U} \cap \partial B_\varepsilon(\lambda_0, 0) \cap K^-_\eta = \emptyset$, $\bar{U} \subset J \times X$. Lemma 2.4, the connectedness of C^n_ε imply the existence of $[\lambda_n, x_n] \in \partial U \cap C^n_\varepsilon$ $(n = 1, 2, \ldots)$. We can assume $[\lambda_n, x_n] \to \to [\lambda, x]$ using the compactness argument. It follows $[\lambda, x] \in [\partial U \cap (C \cup \Lambda)] \setminus \setminus (B_\varepsilon(\lambda_0, 0) \cap K^-_\eta)$ which contradicts the properties of U.

Proof of Lemma 1.1. Lemmas 2.4, 2.5 ensure that $C_\varepsilon \neq \{[\lambda_0, 0]\}$ for any $\varepsilon > 0$. Hence, the assertion of Lemma 1.1 follows from Remark 2.4.

Proof of Theorem 1.1. If the set $C^+_0 = C_\varepsilon$ satisfies at least one of the conditions (a), (b), (c) for some $\varepsilon > 0$ then the assertion of Theorem 1.1 is true. Suppose that all the conditions (a), (b), (c) are excluded for $C^+_0 = C_\varepsilon$ with any $\varepsilon > 0$. Let us prove that then

$$C_\varepsilon \cap \partial B_\varepsilon(\lambda_0, 0) \cap K^-_\eta = \emptyset \quad \text{for} \quad \varepsilon > 0 \quad \text{small enough}\,. \tag{2.21}$$

If (2.21) were not true then we would receive $[\lambda_n, x_n] \in C_{\varepsilon_n} \cap \partial B_{\varepsilon_n}(\lambda_0, 0) \cap K^-_\eta$, $\varepsilon_n \to 0$. Remark 2.2 gives $x_n/|||x_n||| \to -x_0$, but this contradicts the assumption (1.4) because $[\lambda_n, x_n] \notin M$, $\lambda_n \in J_0$ by Lemma 2.1. Hence, (2.21) is proved. Fix $\varepsilon \in (0, S)$ such that (2.21) holds and set $C^+_0 = C_\varepsilon$. Lemmas 2.4, 2.5 ensure that C^+_0 is non-compact. It follows from (0.2) (which holds for all elements of C^+_0), local compactness of C_ε, Lemma 2.1 and the boundedness of J_0 that C^+_0 is unbounded in τ. Moreover, $\tau \geqq 0$ for all $[\lambda, u, \tau] \in C^+_0$ because $[\lambda_0, 0, 0] \in C^+_0$, C^+_0 is connected and (0.2) cannot be fulfilled with $\tau \in (-1, 0)$. The assertion (d) is now a consequence of Lemma 2.1.

Remark 2.7. Lemmas 2.2, 2.4, 2.5 are only modifications of Lemmas 1–3 from [1], where the case $J = \mathbb{R}$, $T(\lambda) = \lambda T$ with a linear completely continuous operator in a real Banach space is considered. (Note that our Lemmas are proved in fact also for a general real Banach space X.) In this special case, the assumption (1.1) is fulfilled automatically and (1.2) is a consequence of the algebraic simplicity of the characteristic value λ_0 which is supposed in [1]. The proofs of Lemmas 2.2, 2.4, 2.5 are almost the same as in [1].

Remark 3.1. The linearity of the operators $T(\lambda)$ (for $\lambda \in J$ fixed), the simplicity of the critical point λ_0 and the assumption (1.2) were not used directly in the proof of Theorem 1.1. Only the following conditions were essential:

(3.1) *for any couple* $\lambda_1, \lambda_2 \in J \setminus \Lambda_0$ *there is* $\sigma_0 > 0$ *such that* $\deg(\Phi(\lambda_1), 0, B_\sigma(0)) - \deg(\Phi(\lambda_2), 0, B_\sigma(0))$ *is even*)* *for* $0 < \sigma < \sigma_0$,

$\lambda_0 \in J$ *is such that there exist open sets* K^+, K^- *in* $J \times \mathbb{V} \times \mathbb{R}$ *satisfying*

$$\overline{K^+} \cap \overline{K^-} = \{[\lambda_0, 0, 0]\}\,, \tag{3.2}$$

$$C \cap B_S(\lambda_0, 0, 0) \subset K^+ \cup K^- \cup \{[\lambda_0, 0, 0]\} \quad \textit{for some} \quad S > 0\,, \tag{3.3}$$

(3.4) *for any* $\gamma > 0$ *there exist* $\gamma_+ \in (0, \gamma)$, $\gamma_- \in (0, \gamma_+\rangle$ *such that* $\lambda_0 \pm \gamma_\pm \notin \Lambda_0$, $\deg(\Phi(\lambda_0 + \gamma_+), 0, W^+_{\sigma,\delta_1}) - \deg(\Phi(\lambda_0 - \gamma_-), 0, W^+_{\sigma,\delta_1})$ *is odd for any* $\sigma > 0$ *small enough,*

where $W^+_{\sigma,\delta_1} = \{x \in X;\ [\lambda_0, x] \in K^+,\ \sigma < |||x||| < \delta_1\}$, $\Lambda_0 = \{\lambda \in J;\ [\lambda, 0, 0] \in C\} \subset \Lambda$ (see Remark 3.2 below) and $\delta_1 > 0$ *is such that*

$$\Phi(\lambda_0)(x) \neq 0 \quad \textit{if} \quad 0 < |||x||| \leqq \delta_1\,. \tag{3.5}$$

Throughout Sections 1, 2, the set Λ was considered instead of Λ_0, but in fact only points from Λ_0 played a role in the conditions just mentioned (precisely, see Remark 3.2). Hence, we can replace the assumption (1.1) by

$$\Lambda_0 \quad \textit{is nowhere dense}\,. \tag{3.6}$$

Under the assumptions of Theorem 1.1, (3.1) was automatically fulfilled with Λ instead of Λ_0 (precisely, see Remark 3.3 below) and (3.2), (3.3) were satisfied with $K^+ = K^+_\eta$, $K^- = K^-_\eta$ (see Remark 2.3). The condition (3.4) was proved only under the assumption (3.5) (i.e. (2.8), see Lemma 2.2). The assumption (3.5) (i.e. also (3.4)) was removed in Lemma 2.5, but the linearity of $T(\lambda_0)$ (the Fredholm alternative) was used in its proof. If we do not explicitly suppose the linearity of $T(\lambda_0)$ then the assumptions (3.4), (3.5) cannot be omitted. Further, suppose that

$$C \cap B_\varepsilon(\lambda_0, 0, 0) \cap K^+ = \{[\lambda, u, \tau] \in C \setminus M;\ \lambda \in J_0\} \cap B_\varepsilon(\lambda_0, 0, 0) \quad \textit{for} \quad \varepsilon > 0 \quad \textit{small enough}\,, \tag{3.7}$$

which is equivalent to (1.3), (1.4) for $K^\pm = K^\pm_\eta$ under the assumptions of Theorem 1.1 (see Remarks 2.2, 2.3).

Now, we can formulate an abstract version of Theorem 1.1 for a mapping $T: J \times \mathbb{V} \to \mathbb{V}$ ($T(\lambda)(u) = T(\lambda, u)$, $T(\lambda)$ nonlinear in general):

Theorem 3.1. *Let* (0.4), (0.5), (3.6), (3.1) *be fulfilled, let* $\lambda_0 \in J$ *be such that there exist open sets* K^+, K^- *and* $\delta_1 > 0$ *satisfying* (3.2)–(3.5). *Consider a set M closed in* $J \times \mathbb{V} \times \mathbb{R}^+$ *and an open bounded interval* J_0 *such that* $\lambda_0 \in \bar{J}$, $J_0 \cup \{\lambda_0\} \subset J$,

*) Particularly, zero is admissible.

and (3.7) *holds. Then there exists a closed connected set* $C_0^+ \subset C$ *satisfying at least one of the conditions* (a)–(d) *from Theorem* 1.1 *with* Λ *replaced by* Λ_0 *in* (b).

Remark 3.2. Recall that $\Lambda_0 \subset \Lambda$ under the assumptions of Theorem 1.1 (see Remark 2.2) and this was the basic property of the set of all critical points. This property is not preserved if $T(\lambda)$ are nonlinear because $E_T(\lambda) = \{0\}$ can happen even if $[\lambda, 0, 0] \in C$. On the other hand, all points $\lambda \in \Lambda$ playing a role in Sections 1, 2 lie in fact in Λ_0 with the exception of λ_0 which is not in Λ_0 by the assumption (but $\lambda_0 \in \Lambda_0$ follows from Theorem 1.1). The assumption $\lambda_0 \in \Lambda$, $E_T(\lambda_0) = \text{Lin}\{w_0\}$ is replaced by the existence of K^+, K^- and δ_1 satisfying (3.2)–(3.5). Hence, to replace Λ by Λ_0 is natural in our present setting.

Remark 3.3. Recall that $\deg(I - T(\lambda), 0, B_\sigma(0))$ was defined for $\lambda \notin \Lambda$ and $\sigma > 0$ small enough in the situation of Sections 1, 2 and that the precise formula for this degree was given in Remark 2.5. But this was used only for the proof of (3.4) in Lemma 2.2, and in the form of the condition (3.1) in the proof of Lemmas 2.2, 2.4 (with Λ instead of Λ_0). In our present situation, $\deg(\Phi(\lambda), 0, B_\sigma(0))$ is defined for any $\lambda \notin \Lambda_0$ and $\sigma > 0$ small enough (such that $\Phi(\lambda)(x) \neq 0$ for $0 < |||x||| \leqq \sigma$), and (3.1), (3.4) are explicitly supposed. Hence, we need no further information about $\deg(I - T(\lambda), 0, B_\sigma(0))$. The set Λ_0 takes the role of Λ and therefore (1.1) must be replaced by (3.6).

The idea of the proof of Theorem 3.1 is the same as that of Theorem 1.1 in Section 2. The whole proof is formally simpler because all the basic conditions are now summarized in the assumptions (3.1)–(3.7) while it was necessary to prove some of them on the basis of the assumptions (0.3), (1.2) in Section 2. First, Lemma 2.1 remains valid in the present situation. In its proof, it is sufficient to replace (1.3) by (3.7). (Remark 2.2 is useless because Λ is replaced by Λ_0.) Lemma 2.2 is useless because of the assumption (3.4). In the formulation of Lemma 2.4 we replace K_η^- by K^- (and realize that (2.8) = (3.5)). In its proof, it is sufficient to replace the oddness condition from (1.2) by that from (3.4), Lemma 2.2 by (3.4), Remark 2.5 by (3.1) and Remark 2.3 by (3.3). We need no analogue of Lemma 2.5 under the assumption (3.5) (=(2.8)).

Proof of Theorem 3.1 is now the same as that of Theorem 1.1 in Section 2. The only difference is that (2.21) follows now directly from (3.7) and Lemma 2.1. We need no Lemma 2.5 because (3.5) (=(2.8)) is supposed.

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