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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 1, 95–102

Persistent URL: `http://dml.cz/dmlcz/102204`

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

DUAL SPACES OF TOTALLY ORDERED RINGS

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(Received December 2, 1985)

1. INTRODUCTION

For any abelian o-group, one may define a dual space which reflects the structure of the group to the extent that the evaluation map into the second dual is a one-to-one, order-preserving group-homomorphism. Such a dual space may be defined for any o-ring and it is natural to ask whether convolution may be used to define multiplication on the second dual and, if so, whether the evaluation map then preserves multiplication. In this paper, we will give conditions on an o-ring which will allow us to give an affirmative answer to both these questions. In particular, a power series ring always satisfies these conditions, and because its evaluation map is always onto, multiplication of power series is always abstract convolution of functions.

In what follows, we assume that all rings are associative. We also adopt the notation and definitions of [6] which we apply to rings in the following way. An *o-ring* $(T, +, \cdot, \leqq)$ is an abelian o-group $(T, +, \leqq)$ and a ring $(T, +, \cdot)$ such that if $a < b$ and $0 < x$ in T, then $ax < bx$ and $xa < xb$. (This definition is somewhat stronger than the definitions in [1] and [2].) The construction of the dual space in [6] depends on a Banaschewski function τ. One would expect that in the case of rings the Banaschewski function and the ring multiplication would have to interact in some way; the appropriate way turns out to be the following. For $t \in T$ and $P \in \boldsymbol{P}$, we write $t > P$ to mean that $t > p$ for all $p \in P$, and for $P, Q \in \boldsymbol{P}$, we let

$$P \diamondsuit Q = \{z \in T \mid z \leqslant xy \text{ for all } x > P \text{ and } y > Q\} .$$

A *β-ring* $(T, +, ., \leqq, \tau)$ is then an o-ring $(T, +, ., \leqq)$ together with a function $\tau: \boldsymbol{P} \to \boldsymbol{D}$ such that $(T, +, \leqq, \tau)$ is a β-group (see [6]) and τ satisfies the additional condition

(iii) for all $P, Q \in \boldsymbol{P}$, $\tau(P)\, \tau(Q) \subseteq \tau(P \diamondsuit Q)$.

The foremost example of a β-ring is the *power series ring* $(p\Pi_\Delta \mathbb{R}, +, ., \leqq, \chi)$ formed as follows (cf. [5]): Δ is a totally ordered semigroup which satisfies the con-

*) This work was done while the author visiting Simon Fraser University where he was supported in part by NSERC grant number A4044. The author wishes to thank the Department of Mathematics at Simon Fraser and, in particular, N. R. Reilly for their hospitality.

dition: $\alpha\delta < \beta\delta$ and $\delta\alpha < \delta\beta$ whenever $\alpha, \beta, \delta \in \Delta$ satisfy $\alpha < \beta$. $({}_X\Pi_\Delta \mathbb{R}, +, \leqq)$ is the lexicographically ordered group of functions from Δ to the real numbers $\mathbb{R}$ with inversely well-ordered support (cf. [6]). Multiplication on ${}_X\Pi_\Delta \mathbb{R}$ is defined by

$$(fg)_\delta = \sum_{\alpha\beta=\delta}(r_{\alpha,\beta} f_\alpha g_\beta),$$

where the $r_{\alpha,\beta}$ form a *factor system*: (i) $r_{\alpha,\beta} > 0$ for all $a, \beta \in \Delta$ and (ii) $r_{\alpha\beta,\delta} r_{\alpha,\beta} = r_{\alpha,\beta\delta} r_{\beta,\delta}$ for all $\alpha, \beta, \delta \in \Delta$. It is then easy to see that not only is $({}_X\Pi_\Delta \mathbb{R}, +, \leqq, \chi)$ a strong β-group (cf. [6]) but also $({}_X\Pi_\Delta \mathbb{R}, +, ., \leqq, \chi)$ is a β-ring.

Suppose finally that T is a β-ring and let $x, y \in T$ and $A, B, C \in \boldsymbol{A}$. We use $M[x, y, C]$ to denote the product of all $m(x, U)\, m(y, V)$, where $U \in S(x)$, $V \in S(y)$, and $UV \subseteq C$; we use $M[x, y, A, B, C]$ to denote the product of all $m(x, U)\, m(y, V)$, where $U \in S(x)$ $V \in S(y)$, $UV \subseteq C$, but $U \neq A$ and $V \neq B$. (Note that $M[x, y, C]$ and $M[x, y, A, B, C]$ are well-defined because $S(x)$ and $S(y)$ are inversely well-ordered by Proposition 2.2 of [6].) It was shown in [7] that if $\tau\colon \boldsymbol{P} \to \boldsymbol{D}$ for an o-ring T, then (T, τ) is a β-ring if and only if (i) (T, τ) is a β-group, (ii) for all $A, B \in \boldsymbol{A}$ there exists $C \in \boldsymbol{A}$ such that $AB \subseteq C$ and (iii) for all $x, y \in T$ and $C \in \boldsymbol{A}$

$$M[x, y, C]\,(xy)_C = m(xy, A) \sum_{AB \subseteq C} M[x, y, A, B, C]\,(x_A y_B).$$

We will be using this latter characterization of β-rings in the proofs of § 2 and hence, for these purposes, it may be taken as the definition.

If (T, τ) is a β-ring and $A, B \in \boldsymbol{A}$, then we let $[AB]$ denote the (unique) element of $\boldsymbol{A}$ containing AB.

2. CONVOLUTION

Let T be a β-ring and let $T^{\wedge\wedge}$ be its second dual. One may define convolution on $T^{\wedge\wedge}$ as follows ([3], page 262). For $t \in T$ and $f \in T^\wedge$, define ${}_t f \in T^\wedge$ by ${}_t f(x) = f(tx)$. For $\boldsymbol{G} \in T^{\wedge\wedge}$, define $\boldsymbol{G}^-\colon T^\wedge \to T^\wedge$ by $\boldsymbol{G}^-(f)(t) = \boldsymbol{G}({}_t f)$ for $t \in T$ and $f \in T^\wedge$. For $\boldsymbol{F}, \boldsymbol{G} \in T^{\wedge\wedge}$, define $\boldsymbol{F} * \boldsymbol{G} \in T^{\wedge\wedge}$, the *convolution* of $\boldsymbol{F}$ and $\boldsymbol{G}$, by $\boldsymbol{F}*\boldsymbol{G}(f) = \boldsymbol{F}(\boldsymbol{G}^-(f))$ for $f \in T^\wedge$. We show below (Propositions 2.3, 2.6, and 2.9) that this series of definitions makes sense for any β-ring T, and hence that convolution is a well-defined operation on $T^{\wedge\wedge}$. Our proof shows immediately that $(T^{\wedge\wedge}, +, *, \leqq, \tau^{\wedge\wedge})$ is a β-ring (Theorem 2.10). It is then easy to see that the evaluation map $\Xi\colon T \to T^{\wedge\wedge}$ always preserves multiplication (Theorem 2.11). In particular, it follows that the map $\Xi\colon {}_x\Pi_\Delta \mathbb{R} \to ({}_x\Pi_\Delta \mathbb{R})^{\wedge\wedge}$ is an o-β-isomorphism of rings (Corollary 2.12).

Note: To avoid double subscripts in the following proofs, we will sometimes use $x(A)$ in place of x_A.

Lemma 2.1. *Let $t \in T$, $f \in T^\wedge$, and $B \in \boldsymbol{A}$. Let*

$$(tBf) = \{A \in S(t) \mid AB \in \operatorname{Supp}(f)\}.$$

(a) *(tBf) is finite;*

(b) *for all* $x \in T$.

$$_t f(x) = \lim_{\Phi \in F} \sum_{D \in \Phi^*} m(x, D)^{-1} \sum_{A \in (tDf)} m(t, A)^{-1} f(t_A x_D),$$

where $\Phi^* = \{D \in S(x) \mid AD \in \Phi$ *for some* $A \in S(t)\}$;

(c) *for all* $a \in A$,

$$_t f(a) = \sum_{A \in (tBf)} m(t, A)^{-1} f(t_A a),$$

where the sum is zero if (tBf) *is empty.*

Proof. (a) if (tBf) is not finite, it has an infinite descending chain $A_1 > A_2 > > A_3 > \ldots$. Then $A_1 B > A_2 B > A_3 B > \ldots$ is an infinite descending chain in Supp (f), a contradiction.

(b) Since $f \in T^\wedge$, we have by (a)

$$_t f(x) = f(tx) = \lim_{\Phi \in F} \sum_{C \in \Phi} m(tx, C)^{-1} f((tx)_C) =$$

$$= \lim_{\Phi \in F} \sum_{C \in \Phi} m(tx, C)^{-1} M[t, x, C]^{-1} f[m(tx, C) \sum_{[AD]=C} M[t, x. A, D, C]\, t_A x_D] =$$

$$= \lim_{\Phi \in F} \sum_{[AD] \in \Phi} m(t, A)^{-1} m(x, D)^{-1} f(t_A x_D) =$$

$$= \lim_{\Phi \in F} \sum_{D \in \Phi^*} m(x, D)^{-1} \sum_{A \in (tDf)} m(t, A)^{-1} f(t_A x_D).$$

(Here the limit is taken over the directed set F — see [4], pages 77–78, "Integration Theory, Junior Grade".)

(c) Since (tBf) is finite, the set E, of all $C \in$ Supp (f) such that $[AB] = C$ for some $A \in S(t)$, is finite. Then for all $\Phi \in F$ containing E,

$$\sum_{D \in \Phi^*} m(a, D)^{-1} \sum_{A \in (tDf)} m(t, A)^{-1} f(t_A a_D) = \sum_{A \in (tBf)} m(t, A)^{-1} f(t_A a).$$

Part (c) then follows from part (b).

Lemma 2.2. *If* $f \in T^\wedge$ *and* $0 < a \in A \in \boldsymbol{A}$, *then* $_a f \in T^\wedge$.

Proof. Clearly the function $x \to ax$ is a β-homomorphism of T into T. Its dual map takes $f \in T^\wedge$ to $_a f$ and hence Lemma 2.2 follows from Theorem 4.2 of [6].

Proposition 2.3. *Let* $f \in T^\wedge$ *and* $t \in T$. *Then* $_t f \in T^\wedge$ *and for all* $B \in \boldsymbol{A}$,

$$(_t f)_{B^\wedge} = \sum_{A \in (tBf)} [m(t, A)^{-1}\ {}_{t(A)}(f_{[AB]^\wedge})].$$

Proof. (I) Clearly $_t f$ is a well-defined group-homomorphism from T to $\mathbb{R}$. (II) By Lemma 2.2, for any $B \in \boldsymbol{A}$,

$$\sum_{A \in (tBf)} m(t, A)^{-1}\ {}_{t(A)} f \in T^\wedge,$$

and by Lemma 2.1 (c),

$$_t f|_B = [\sum_{A \in (tBf)} m(t, A)^{-1}\ {}_{t(A)} f]|_B.$$

Hence $_t f|_B$ is comparable to 0 with respect to $\lesssim$. (III) If $B \in$ Supp $(_t f)$ and $[AB] \in \boldsymbol{A} \setminus$ Supp (f) for all $A \in S(t)$, then $(tBf) = \emptyset$ and by Lemma 2.1 (c) $_t f(b) = 0$ for all $b \in B$, a contradiction. Thus for all $B \in$ Supp $(_t f)$, there exists $A \in S(t)$ such that $[AB] \in$ Supp (f). Suppose that $B_1 > B_2 > B_3 > \ldots$ is an infinite descending chain in Supp $(_t f)$, and let $\{A_i\} \subseteq S(t)$ be such that $[A_i B_i] \in$ Supp (f) for all i. Since $S(t)$ is inversely well-ordered, we may assume $A_1 \geqq A_2 \geqq A_3 \geqq \ldots$ by picking a cofinal subset if necessary. Then $[A_1B_2] > [A_2B_2] > [A_3B_3] > \dots$ is an infinite descending chain in Supp (f), a contradiction. Thus $S({}_tf)$ is well-ordered. (IV) By Lemma 2.1 (b) and (c), we have for all $x \in T$,

$$\begin{aligned} {}_tf(x) &= \lim_{\Phi\in F} \textstyle\sum_{B\in\Phi} m(x, B)^{-1} \sum_{A\in(tBf)} m(t, A)^{-1} f(t_A x_B) = \\ &= \lim_{\Phi\in F} \textstyle\sum_{B\in\Phi^*} m(x, B)^{-1}\, {}_tf(x_B) \end{aligned}$$

and therefore ${}_tf \in T^\wedge$. The characterization of $({}_tf)_{B^\wedge}$ then follows from Lemma 2.1 (c).

Lemma 2.4. *Let* $G \in T^{\wedge\wedge}$, $f \in T^\wedge$, *and* $A \in \mathbf{A}$. *Let*

$$(AGf) = \{B \in \mathbf{A} \mid B^\wedge \in \operatorname{Supp}(G) \text{ and } [AB] \in S(f)\}\,.$$

(a) (AGf) *is finite;*

(b) *for all* $x \in T$,

$$G^-(f)(x) = \lim_{\Phi\in F} \textstyle\sum_{D\in\Phi^{**}} m(x, D)^{-1} \sum_{B\in(DGf)} G({}_{x(D)}[f_{[DB]^\wedge}])\,,$$

where $\Phi^{**} = \{A \in S(x) \mid [AB] \in \operatorname{Supp}(f) \text{ for some } B \in \Phi\}$;

(c) *for all* $a \in A$,

$$G^-(f)(a) = \textstyle\sum_{B\in(AGf)} G({}_a[f_{[AB]^\wedge}])\,,$$

where the sum is zero if (AGf) *is empty.*

Proof. (a) Suppose that (AGf) is not finite. Since Supp (G) is well-ordered and the map $C \to C^\wedge$ reverses order (Proposition 3.1 of [6]), AGf must contain an infinite descending chain $B_1 > B_2 > B_3 > \dots$. Then $[AB_1] > [AB_2] > [AB_3] > \dots$ is an infinite descending chain in Supp (f), a contradiction.

(b) Clearly $\Phi^{**} = \bigcup_{B\in\Phi} (xBf)$ and hence by Lemma 2.1 (a), Φ^{**} is finite. It is easy to see that Supp $({}_{x(D)}[f_{[DB]^\wedge}]) = \{B\}$ and hence by Proposition 2.3 and part (a) of this lemma,

$$\begin{aligned} G^-(f)(x) &= G({}_xf) = \lim_{\Phi\in F} \textstyle\sum_{B\in\Phi} G([{}_xf]_{B^\wedge}) = \\ &= \lim_{\Phi\in F} \textstyle\sum_{B\in\Phi} \sum_{D\in(xBf)} m(x, D)^{-1} G({}_{x(D)}[f_{[DB]^\wedge}) = \\ &= \lim_{\Phi\in F} \textstyle\sum_{D\in\Phi^{**}} m(x, D)^{-1} \sum_{B\in(HGf)} G({}_{x(D)}[f_{[DB]^\wedge}) \,. \end{aligned}$$

(c) If $[AD] \in \mathbf{A} \setminus \operatorname{Supp}(f)$ for all $D \in \mathbf{A}$, then $(AGf) = \emptyset$ and hence by (b) $G^-(f)(a) = 0$. Otherwise, for any $\Phi \in F$ for which $A \in \Phi^{**}$,

$$\textstyle\sum_{D\in\Phi^{**}} m(a, D)^{-1} \sum_{B\in(DGf)} G({}_{a(D)}[f_{[DB]^\wedge}]) = \sum_{B\in(AGf)} G({}_a[f_{[AB]^\wedge}])\,.$$

Part (c) then follows from part (b).

Lemma 2.5. *If* $G \in T^{\wedge\wedge}$ *and* $0 < z \in C^\wedge \in \mathbf{A}^\wedge$, *then* $G^-(z) \in T^\wedge$.

Proof. If $\mathbf{A}$ forms a multiplicative group (instead of merely a semigroup), then the function $x \to {}_xz$ is a β-homomorphism whose dual maps $G \to G^-(z)$ and hence (by Theorem 4.2 of [6] $G^-(z) \in T^\wedge$. However, without this assumption, the function need not be dense, and hence we must check the definition directly. (I) Clearly $G^-(z)$ is a group-homomorphism of T into $\mathbb{R}$. (II) Let $0 < d \in D \in \mathbf{A}$ and suppose that

$G^{-}(z)(d) > 0$. Then $G({}_{d}z) > 0$ and hence ${}_{d}z \neq 0$. Thus there exists $B \in \boldsymbol{A}$ such that $[DB] = C$. We then have $0 < {}_{d}z \in B^{\wedge}$ and hence $0 \prec G|_{B^{\wedge}}$. Now let $0 < w \in D$. We have $0 < {}_{w}z \in B^{\wedge}$ and thus $G^{-}(z)(w) = G({}_{w}z) > 0$. Therefore $0 \prec G^{-}(z)|_{D}$ We conclude that for all $D \in \boldsymbol{A}$, $G^{-}(z)|_{D}$ is comparable to 0 with respect to $\lesssim$. (III) Suppose that $D_1 > D_2 > D_3 > \ldots$ is an infinite descending chain in $\mathrm{Supp}\,(G^{-}(z))$. We noted above that for each $D \in \mathrm{Supp}\,(G^{-}(z))$, there exists $B \in \boldsymbol{A}$ such that $[DB] = C$ and $B^{\wedge} \in \mathrm{Supp}\,(G)$. For each D_i, let $B_i \in \boldsymbol{A}$ satisfy these conditions. Then $B_1 < B_2 < B_3 < \ldots$ and hence (Proposition 3.1 of [6]) $B_1^{\wedge} > B_2^{\wedge} > B_3^{\wedge} > \ldots$, a contradiction. Thus $\mathrm{Supp}\,(G^{-}(z))$ is well-ordered. (IV) Clearly $z_{[DB]^{\wedge}} \neq 0$ exactly when $[DB] = C$. Thus by Lemma 2.4,

$$G^{-}(z)(x) = \lim_{\Phi \in F} \sum_{D \in \Phi^{**}} m(x, D)^{-1} \sum_{B \in (DGf)} G({}_{x(D)}[z_{[DB]^{\wedge}}]) =$$
$$= \lim_{\Phi \in F} \sum_{D \in \Phi^{**}} m(x, D)^{-1}\, G^{-}(z)(x_D)\,.$$

Therefore $G^{-}(z) \in T_{\wedge}$.

For $G \in T^{\wedge\wedge}$ and $B \in \boldsymbol{A}$, note that $G^{-}_{B^{\wedge\wedge}} : T^{\wedge} \to T^{\wedge}$ is defined by $G^{-}_{B^{\wedge\wedge}}(f)(x) = G_{B^{\wedge\wedge}}({}_{x}f)$.

Proposition 2.6. *Let $G \in T^{\wedge\wedge}$ and $f \in T^{\wedge}$. Then $G^{-}(f) \in T^{\wedge}$ and for all $A \in \boldsymbol{A}$,*

$$G^{-}(f)_{A^{\wedge}} = \sum_{B \in (AGf)} G^{-}_{B^{\wedge\wedge}}(f_{[AB]^{\wedge}})\,.$$

Proof. (I) Clearly $G^{-}(f)$ is a well-defined group-homomorphism from T to $\mathbb{R}$. (II) By Lemmas 2.4 (a) and 2.5,

$$\sum_{B \in AGf} G^{-}(f_{[AB]^{\wedge}}) \in T^{\wedge} \quad \text{for any} \quad A \in \boldsymbol{A}\,.$$

By Lemma 2.4 (c),

$$G^{-}(f)|_{A} = [\sum_{B \in (AGf)} G^{-}(f_{[AB]^{\wedge}})]|_{A}$$

and hence $G^{-}(f)|_{A}$ is comparable to 0 with respect to $\lesssim$. (III) Let $A \in \boldsymbol{A}$ and suppose that whenever $[AB] \in \mathrm{Supp}\,(f)$, $B^{\wedge} \in \boldsymbol{A}^{\wedge} \setminus \mathrm{Supp}\,(G)$. Then $G^{-}({}_{a}[f_{[AB]^{\wedge}}]) = 0$ for all $B \in \boldsymbol{A}$ and $a \in A$, and hence by Lemma 2.4 (b), $G^{-}(f) = 0$. Thus, we may assume that if $A \in \mathrm{Supp}\,(G^{-}(f))$, there exists $B \in \boldsymbol{A}$ such that $B^{\wedge} \in \mathrm{Supp}\,(G)$ and $[AB] \in \mathrm{Supp}\,(f)$. Now suppose that $A_1 > A_2 > A_3 > \ldots$ is an infinite descending chain in $\mathrm{Supp}\,(G^{-}(f))$ and for each i let $B_i \in \boldsymbol{A}$ be such that $B_i^{\wedge} \in \mathrm{Supp}\,(G)$ and $[A_i B_i] \in \mathrm{Supp}\,(f)$. But $\mathrm{Supp}\,(G)$ is well-ordered and hence, picking a co-final subset if necessary, we may assume that $B_1^{\wedge} \leqq B_2^{\wedge} \leqq B_3^{\wedge} \leqq \ldots$. By Proposition 3.1 of [6], we have $B_1 \geqq B_2 \geqq B_3 \geqq \ldots$ and hence $[A_1B_1] > [A_2B_2] > [A_3B_3] > \ldots$ is an infinite descending chain in $\mathrm{Supp}\,(f)$, a contradiction. Thus, $\mathrm{Supp}\,(G^{-}(f))$ is well-ordered. (IV) By Lemma 2.4 (b) and 2.4 (c), we have for all $x \in T$,

$$G^{-}(f)(x) = \lim_{\Phi \in F} \sum_{D \in \Phi^{**}} m(x, D)^{-1} \sum_{B \in DGf} G({}_{x(D)}[f_{[DB]^{\wedge}}]) =$$
$$= \lim_{\Phi \in F} \sum_{D \in \Phi^{**}} m(x, D)^{-1}\, G^{-}(f)(x_D)\,.$$

Therefore $G^{-}(f) \in T^{\wedge}$. To see that the equation holds, note that if $a \in A$, then ${}_{a}[f_{[AB]^{\wedge}}] \in A^{\wedge}$ and hence

$$G^{-}(f_{[AB]^{\wedge}})(a) = G({}_{a}[f_{[AB]^{\wedge}}]) = G_{B^{\wedge\wedge}}({}_{a}[f_{[AB]^{\wedge}}]) = G^{-}_{B^{\wedge\wedge}}(f_{[AB]^{\wedge}})(a)\,.$$

On the other hand, if $A \neq D \in A$ and $d \in D$, then ${}_d[f_{[AB]^\wedge}] = 0$ or ${}_d[f_{[AB]^\wedge}] \in E^\wedge$, where $[DE] = [AB]$, and since $D \neq A$, $E \neq B$. In either case,

$$G^-_{B^{\wedge\wedge}}(f_{[AB]^\wedge})(d) = G_{B^{\wedge\wedge}}({}_d[f_{[AB]^\wedge}]) = 0 .$$

The equation then follows from Lemma 2.4 (c).

Lemma 2.7. *Let* $F, G \in T^{\wedge\wedge}$.

(a) *For* $C \in A$, *there exist only finitely many pairs* $(A, B) \in A \times A$ *such that* $[AB] = C$ *and* $F_{A^{\wedge\wedge}} \circ G^-_{B^{\wedge\wedge}} = 0$.

(b) *For all* $f \in T^\wedge$,

$$F(G^-(f)) = \lim_{\Phi \in F} \sum_{C \in \Phi^{***}} \sum_{[AB]=C} F_{A^{\wedge\wedge}}(G^-_{B^{\wedge\wedge}}(f_{C^\wedge})) ,$$

where $\Phi^{***} = \{C \in \mathrm{Supp}(f) \mid [AB] = C$ *for some* $A \in \Phi$ *and* $B^\wedge \in \mathrm{Supp}(G)\}$.

(c) *For all* $z \in C^\wedge \in A^\wedge$,

$$F(G^-(z)) = \sum_{[AB]=C} F_{A^{\wedge\wedge}}(G^-_{B^{\wedge\wedge}}(z)) .$$

Proof. (a) For $F_{A^{\wedge\wedge}} \circ G^-_{B^{\wedge\wedge}} \neq 0$, we must have $A^\wedge \in \mathrm{Supp}(F)$ and $B^\wedge \in \mathrm{Supp}(G)$. These sets are both well-ordered and hence there can be only finitely many such (A, B) with $[AB] = C$.

(b) Since $\mathrm{Supp}(F)$ and $\mathrm{Supp}(G)$ are well-ordered and Φ is finite, Φ^{***} is also finite. Since $F \in T^{\wedge\wedge}$, we have by Proposition 2.6,

$$\begin{aligned} F(G^-(f)) &= \lim_{\Phi \in F} \sum_{A \in \Phi} F_{A^{\wedge\wedge}}([G^-(f)]_{A^\wedge}) = \\ &= \lim_{\Phi \in F} \sum_{A \in \Phi} \sum_{B \in AFf} F_{A^{\wedge\wedge}}(G^-_{B^{\wedge\wedge}}(f_{[AB]^\wedge})) = \\ &= \lim_{\Phi \in F} \sum_{C \in \Phi^{***}} \sum_{[AB]=C} F_{A^{\wedge\wedge}}(G_{B^{\wedge\wedge}}(f_{C^\wedge})) . \end{aligned}$$

(c) If $[AB] \neq C$ for all $A \in A$ and $B^\wedge \in \mathrm{Supp}(G)$, then $F(G^-(z)) = 0$ by part (b), and the sum in part (c) also clearly equals 0. Otherwise, for any $\Phi \in F$ such that $C \in \Phi^{***}$,

$$\sum_{D \in \Phi^{***}} \sum_{[AB]=D} F_{A^{\wedge\wedge}}(G^-_{B^{\wedge\wedge}}(z_D)) = \sum_{[AB]=C} F_{A^{\wedge\wedge}}(G^-_{B^{\wedge\wedge}}(z)) .$$

Part (c) then follows from part (b).

Lemma 2.8. *Let* $F \in T^{\wedge\wedge}$ *and* $0 < H \in B^{\wedge\wedge} \in A^{\wedge\wedge}$. *Then* $F^*H \in T^{\wedge\wedge}$.

Proof. If A forms a multiplicative group, then the function $f \to H^-(f)$ is a β-homomorphism whose dual maps $F \to F^*H$ and hence $F^*H \in T^{\wedge\wedge}$. However, without this or some similar assumption, we must check the definition directly. (I) Clarly F^*H is a group-homomorphism. (II) Let $0 < z \in B^\wedge \in A^\wedge$ and suppose that $F^*H(z) > 0$. We wish to show that $F^*H(w) > 0$ whenever $0 < w \in B^\wedge$. Since $F(H^-(z)) > 0$, $H^-(z) \neq 0$ and hence there exists $A \in A$ such that $[AB] = C$ and $H^-(z)(a) = H({}_az) > 0$ for all $0 < a \in A$. Thus we have $0 < H^-(z) \in A^\wedge$ and hence $0 \prec F|_{A^\wedge}$ because $F \in T^{\wedge\wedge}$. But also $H^-(w)(a) = H({}_aw) > 0$ for all $a \in A$ and hence $0 < H^-(w) \in A^\wedge$. Then $F^*H(w) = F(H^-(w)) > 0$ and we conclude that $0 \prec F^*H|_{C^\wedge}$. Therefore $F^*H|_{C^\wedge}$ is comparable to 0 with respect to $\lesssim$. (III) Suppose next that $C_1^\wedge > C_2^\wedge > C_3^\wedge > \dots$ is an infinite descending chain in $\mathrm{Supp}(F^*H)$. We noted above that for each i, there exists $A_i \in \boldsymbol{A}$ such that $A_i^\wedge \in \operatorname{Supp}(\boldsymbol{F})$ and $[A_iC] = C_i$. We have $C_1 < C_2 < C_3 < \dots$ in $\boldsymbol{A}$ and hence $A_1 < A_2 < A_3 < \dots$. Then $A_1^\wedge > A_2^\wedge > A_3^\wedge > \dots$ in $\operatorname{Supp}(\boldsymbol{F})$, a contradiction. Thus, $\operatorname{Supp}(\boldsymbol{F^*H})$ is well-ordered. (IV) Finally, by Lemma 2.7 (b), for all $f \in T^\wedge$, we have

$$\begin{aligned} \boldsymbol{F^*H}(f) &= \lim_{\Phi \in \boldsymbol{F}} \textstyle\sum_{C \in \Phi^{***}} \sum_{[AD]=C} \boldsymbol{F}_{A^{\wedge\wedge}}(\boldsymbol{H}^-_{D^{\wedge\wedge}}(f_{C^\wedge})) = \\ &= \lim_{\Phi \in \boldsymbol{F}} \textstyle\sum_{C \in \Phi^{***}} \sum_{[AB]=C} \boldsymbol{F}_{A^{\wedge\wedge}}(\boldsymbol{H}^-(f_{C^\wedge})) = \\ &= \lim_{\Phi \in \boldsymbol{F}} \textstyle\sum_{C \in \Phi^{***}} (\boldsymbol{F^*H})(f_{C^\wedge}) . \end{aligned}$$

We conclude that $\boldsymbol{F^*H} \in T^{\wedge\wedge}$.

Proposition 2.9. *Let $\boldsymbol{F}, \boldsymbol{G} \in T^{\wedge\wedge}$. Then $\boldsymbol{F^*G} \in T^{\wedge\wedge}$ and for all $C \in \boldsymbol{A}$,*

$$(\boldsymbol{F^*G})_{C^{\wedge\wedge}} = \textstyle\sum_{[AB]=C} \boldsymbol{F}_{A^{\wedge\wedge}} {}^* \boldsymbol{G}_{B^{\wedge\wedge}} .$$

Proof. (I) Clearly $\boldsymbol{F^*G}$ is a well-defined group-homomorphism from $T^\wedge$ to $\mathbb{R}$. (II) By Lemmas 2.7 and 2.8, $\sum_{[AB]=C} \boldsymbol{F}_{A^{\wedge\wedge}} {}^*\boldsymbol{G}_{B^{\wedge\wedge}} \in T^{\wedge\wedge}$ for any $C \in \boldsymbol{A}$. By Lemma 2.7 (c),

$$\boldsymbol{F^*G}|_{C^\wedge} = [\textstyle\sum_{[AB]=C} \boldsymbol{F}_{A^{\wedge\wedge}} {}^*\boldsymbol{G}_{B^{\wedge\wedge}}]|_{C^\wedge}$$

and hence $\boldsymbol{F^*G}|_{C^\wedge}$ is comparable to 0 with respect to $\leqq$. (III) Suppose that $C_1^\wedge > C_2^\wedge > C_3^\wedge > \dots$ is an infinite descending chain in $\operatorname{Supp}(\boldsymbol{F^*G})$. By Lemma 2.7 (c), there exist $A_i^\wedge \in \operatorname{Supp}(\boldsymbol{F})$ and $B_i^\wedge \in \operatorname{Supp}(\boldsymbol{G})$ such that $[A_iB_i] = C_i$ for all i. Picking cofinal subsets if necessary, we must have $A_1^\wedge \leqq A_2^\wedge \leqq A_3^\wedge \leqq \dots$ and $B_1^\wedge \leqq B_2^\wedge \leqq B_3^\wedge \leqq \dots$. Then $[A_1B_1] \geqq [A_2B_2] \geqq [A_3B_3] \geqq \dots$ in $\boldsymbol{A}$. But in $\boldsymbol{A}$, $C_1 < C_2 < C_3 < \dots$, a contradiction. Thus $\operatorname{Supp}(\boldsymbol{F^*G})$ is well-ordered. (IV) By Lemmas 2.7 (b) and 2.7 (c), we have for all $f \in T^\wedge$

$$\begin{aligned} \boldsymbol{F^*G}(f) &= \lim_{\Phi \in \boldsymbol{F}} \textstyle\sum_{C \in \Phi^{***}} \sum_{[AB]=C} \boldsymbol{F}_{A^{\wedge\wedge}}(\boldsymbol{G}^-_{B^{\wedge\wedge}}(f_{C^\wedge})) = \\ &= \lim_{\Phi \in \boldsymbol{F}} \textstyle\sum_{C \in \Phi^{***}} \boldsymbol{F}(\boldsymbol{G}^-(f_{C^\wedge})) . \end{aligned}$$

Therefore $\boldsymbol{F^*G} \in T^{\wedge\wedge}$. The characterization of $(\boldsymbol{F^*G})_{C^{\wedge\wedge}}$ then follows from Lemma 2.7 (c).

Theorem 2.10. *$(T^{\wedge\wedge}, +, {}^*, \leqq, \tau^{\wedge\wedge})$ is a β-ring.*

Proof. By Propositions 2.3, 2.6, and 2.9, convolution is a well-defined operation on $T^{\wedge\wedge}$. Clearly, for $A, B, C \in \boldsymbol{A}$, $AB \subseteq C$ if and only if $A^{\wedge\wedge} {}^* B^{\wedge\wedge} \subseteq C_{\wedge\wedge}$. Thus, by Proposition 2.9 and the discussion in § 1, $(T^{\wedge\wedge}, +, {}^*, \leqq, \tau^{\wedge\wedge})$ is a β-ring.

Theorem 2.11. *The evaluation map $\Xi: T \to T^{\wedge\wedge}$ is an o-β-monomorphism of β-rings.*

Proof. By Theorem 5.1 of [6], Ξ is an o-β-monomorphism of β-groups. It is easy to check that $\Xi(xy) = \Xi(x)^* \Xi(y)$.

Corollary 2.12. *For any totally ordered power series ring ${}_X\Pi_\Delta\mathbb{R}$, the evaluation map $\Xi: {}_X\Pi_\Delta\mathbb{R} \to ({}_X\Pi_\Delta\mathbb{R})^{\wedge\wedge}$ is a o-β-isomorphism of β-rings.*

Proof. The corollary follows from Theorem 2.11 and Proposition 6.8 of [6].

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