Czechoslovak Mathematical Journal

Ivan Dobrakov

On extension of vector polymeasures

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 1, 88–94

Persistent URL: `http://dml.cz/dmlcz/102203`

ON EXTENSION OF VECTOR POLYMEASURES

IVAN DOBRAKOV, Bratislava

(Received November 28, 1985)

Vector and operator valued polymeasures were introduced in [2], where some of their basic properties needed for the subsequent integration theory were deduced. In this paper we treat the problem of extension of a vector polymeasure from a Cartesian product of rings to the corresponding Cartesian product of generated σ-rings. Since, by definition, a non trivial vector polymeasure has no further extension to the generated product σ-ring, our setting of the extension problem for vector polymeasures is the right one. On the other hand, it is also important to know whether such further extension to the product σ-ring exists. For this latter problem we refer to References [1], [20], [21], [32] and [40] of the paper [2].

For uniform vector polymeasures, see Definition 1 in [2]; in Theorem 2 we give a necessary and sufficient condition for their extension. The proof of this result is based on Kluvánek's theorem on extension of vector measures, see [8] or Section I.5 in [1]. Under the assumption of existence of control polymeasures, see Section 3 in [2], in Theorem 5 we give a necessary and sufficient condition for extension of a not necessarily uniform vector polymeasure. Hence solutions of Problem 1 from [2] is of great importance for the extension problem for vector polymeasures.

We adopt the notations from [2] without repeating their meanings.

Our first theorem reduces the extension problem for operator valued polymeasures which are separately countably additive in the strong operator topology to the extension problem for vector polymeasures.

Theorem 1. *Let $\Gamma_0\colon \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology and for each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$ let there be a separately countably additive vector d-polymeasure $\gamma_{(i)}\colon \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ such that $\gamma_{(x_i)}(R_i) = \Gamma_0(R_i)(x_i)$ for each $(R_i) \in \mathsf{X}\mathscr{R}_i$. Put $\Gamma(A_i)(x_i) = \gamma_{(x_i)}(A_i)$ for $(A_i) \in \mathsf{X}\sigma(\mathscr{R}_i)$ and $(x_i) \in \mathsf{X}X_i$. Then $\Gamma\colon \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to L^{(d)}(X_1, \ldots, X_d; Y)$, and Γ is the operator valued d-polymeasure separately countably additive in the strong operator topology which extends Γ_0.*

Proof. Let $(A_i) \in \mathsf{X}\sigma(\mathscr{R}_i)$. Then the d-linearity of $\Gamma(A_i)$ follows immediately from the d-linearity of $\Gamma_0(R_i)$ for each $(R_i) \in \mathsf{X}\mathscr{R}_i$ by the application of Lemma 4 from [2]. The separate continuity of $\Gamma(A_i)$ is a consequence of the following facts: For a fixed $(x_i) \in \mathsf{X}X_i$ the set $\{\gamma_{(x_i)}(A'_i), (A'_i) \in \mathsf{X}\sigma(\mathscr{R}_i)\}$ is bounded in Y by the Nikodým uniform boundedness theorem for polymeasures, see (N) in [2]. Hence the set $\{\Gamma(R_i), (R_i) \in \mathsf{X}\mathscr{R}_i\} \subset L^{(d)}(X_1, \ldots, X_d; Y)$ is pointwise bounded. Thus by the uniform boundedness principle for operators, see [5], this set is norm bounded in $L^{(d)}(X_1, \ldots, X_d; Y)$. From here even the uniform boundedness of $\{\Gamma(A_i), (A_i) \in \mathsf{X}\sigma(\mathscr{R}_i)\}$ follows by Lemma 4 in [2]. The theorem is proved.

Let us recall that a set function $v: \mathscr{R} \to Y$ is called *exhaustive* (also *strongly bounded*, or *strongly additive*) if $v(A_n) \to 0$ for any sequence of pairwise disjoint sets $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$. The next extension theorem for uniform vector polymeasures is based on Kluvánek's extension theorem for vector measures, see [8], or Section I.5 in [1].

Theorem 2. *Let $\gamma_0: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ be a uniform vector d-polymeasure. Then there is a unique uniform vector d-polymeasure $\gamma: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ which extends γ_0 if and only if γ_0 is separately uniformly exhaustive.*

Proof. By the extension theorem of Kluvánek, see [8], or Section I.5 in [1], for each $(A_2, \ldots, A_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$ there is a unique countably additive extension $\gamma_1(\cdot, A_2, \ldots, A_d): \sigma(\mathscr{R}_1) \to Y$ of γ_0. According to Theorem 11 in [3], or to [9], these extensions $\gamma_1(\cdot, A_2, \ldots, A_d): \sigma(\mathscr{R}_1) \to Y$, $(A_2, \ldots, A_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$ are uniformly countably additive. Hence by the well known result of Theorem IV.9.2 in [5], see also Theorem 3.10 in [6] and Theorem 7 in [3], there is a countably additive measure $\lambda_1: \sigma(\mathscr{R}_1) \to [0, +\infty)$ such that the vector measures $\gamma_1(\cdot, A_2, \ldots, A_d)$, $(A_2, \ldots, A_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$, are uniformly absolutely continuous with respect to λ_1 on $\sigma(R_1)$. Now, by Theorem D in § 13 in [7], for each $A_1 \in \sigma(\mathscr{R}_1)$ there is a sequence $A_{1,n} \in \mathscr{R}_1$, $n = 1, 2, \ldots$ such that $\lambda_1(A_1 \Delta A_{1,n}) \to 0$. But then $\gamma_1(A_{1,n}, A_2, \ldots, A_d) \to \gamma_1(A_1, A_2, \ldots, A_d)$ uniformly with respect to $(A_2, \ldots, A_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$ by the above mentioned uniform absolute continuity. From here and from the assumption that γ_0 is a uniform vector d-polymeasure on $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$ we easily see that $\gamma_1: \sigma(\mathscr{R}_1) \times \mathscr{R}_2 \times \ldots \times \mathscr{R}_d \to Y$ is a uniform vector d-polymeasure.

Repeating the argument, we successively obtain extensions $\gamma_{1,2}: \sigma(\mathscr{R}_1) \times \sigma(\mathscr{R}_2) \times \mathscr{R}_3 \times \ldots \times \mathscr{R}_d \to Y, \ldots, \gamma_{1,\ldots,d}: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ which are uniform vector d-polymeasures. Hence $\gamma = \gamma_{1,\ldots,d}$ is the required unique extension of γ_0. The theorem is proved.

According to the remarkable result of K. Ylinen, see Theorem 4.4 in [10] and (Y) in [2], each scalar bimeasure $\beta: \mathscr{S}_1 \times \mathscr{S}_2 \to K$ is a uniform bimeasure. Hence we have

Corollary 1. A scalar bimeasure $\beta_0: \mathscr{R}_1 \times \mathscr{R}_2 \to K$ can be uniquely extended to a necessarily uniform scalar bimeasure $\beta: \sigma(\mathscr{R}_1) \times \sigma(\mathscr{R}_2) \to K$ if and only if β_0 is separately uniformly exhaustive on $\mathscr{R}_1 \times \mathscr{R}_2$.

From Corollary 2 of Theorem 2 in [2] we know that each vector d-polymeasure $\gamma: 2^N \times \ldots \times 2^N \to Y$ $(N = \{1, 2, \ldots\})$ is uniform. Hence we have also

Corollary 2. *Let Φ be the ring of all fintte subsets of $N = \{1, 2, \ldots\}$. Then a vector d-polymeasure $\gamma_0: \Phi \times \ldots \times \Phi \to Y$ can be uniquely extended to a necessarily uniform vector d-polymeasure $\gamma: 2^N \times \ldots \times 2^N \to Y$ if and only if γ_0 is separately uniformly exhaustive on $\Phi \times \ldots \times \Phi$.*

Unfortunately the author knows no further types of uniform polymeasures on Cartesian products of σ-rings, except the trivial case of polymeasures with bounded variations (clearly uniform) which have extensions with bounded variations.

Theorem 3. *Let Y be a reflexive Banach space, and let $\gamma_0: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ be a vector d-polymeasure such that the scalar d-polymeasure $y^*\gamma_0$ can be extended to a d-polymeasure on $\sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d)$ for each $y^* \in Y^*$. Then there is a unique vector d-polymeasure $\gamma: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ which extends γ_0.*

Proof. Let $(A_i) \in \mathsf{X}\sigma(\mathscr{R}_i)$ be fixed, and for $y^* \in Y^*$ put $\gamma(A_i)(y^*) = \gamma_{y^*}(A_i)$, where γ_{y^*} is the unique extension of $y^*\gamma_0$. By Lemma 4 in [2], $\gamma(A_i): Y^* \to K$ is clearly linear. Since for each $y^* \in Y^*$ the range of the scalar d-polymeasure γ_{y^*} is bounded on $\mathsf{X}\sigma(\mathscr{R}_i)$, see the Nikodým theorem (N) in [2], the uniform boundedness principle for operators implies that the set $\{\gamma_0(R_i), (R_i) \in \mathsf{X}\mathscr{R}_i\}$ embedded in Y^{**} is bounded, see Theorem II.3.20 in [5]. Hence there is a constant $C > 0$ such that $|\gamma_0(R_i)(y^*)| \leqq C \,.\, |y^*|$ for each $(R_i) \in \mathsf{X}\mathscr{R}_i$ and each $y^* \in Y^*$. But then by Lemma 4 in [2] we have the inequality $|\gamma(A_i)(y^*)| = |\gamma_{y^*}(A_i)| \leqq C \,.\, |y^*|$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{R}_i)$ and each $y^* \in Y^*$. Thus $\gamma(A_i) \in Y^{**} = Y$. Now the separate countable additivity of γ follows from the Orlicz-Pettis theorem, see the beginning of Section 1 in [2]. The theorem is proved.

Since polymeasures with bounded variations (the variation of a polymeasure was introduced in Definition 3 in [2]) are evidently uniform, using Theorem 2 we obtain

Corollary 1. *Let Y be a reflexive Banach space, and let $\gamma_0: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ be a vector d-polymeasure such that the variatton $v(y^*\gamma_0, (\ldots))$ is a bounded d-polymeasure on $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$ for each $y^* \in Y^*$. Then there is a unique vector d-polymeasure $\gamma: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ which extends γ_0.*

Using Corollary 1 of Theorem 2 we also obtain

Corollary 2. *Let Y be a reflexive Banach space. Then a vector bimeasure $\beta_0: \mathscr{R}_1 \times \times \mathscr{R}_2 \to Y$ can be uniquely extended to a vector bimeasure $\beta: \sigma(\mathscr{R}_1) \times \sigma(\mathscr{R}_2) \to Y$ if and only if for each $y^* \in Y^*$ the scalar bimeasure $y^*\beta_0: \mathscr{R}_1 \times \mathscr{R}_2 \to K$ is separately uniformly exhaustive.*

It is an interesting open problem whether Theorem 3 remains valid if Y is a Banach space not containing an isomorphic copy of the space c_0.

We shall say that a set function $v_1: \mathscr{R} \to Y_1$ is $(\delta \to \varepsilon)$ – absolutely continuous with respect to a set function $v: \mathscr{R} \to Y$ if $|v_1(A)| < \varepsilon$ whenever $A \in \mathscr{R}$ and $\bar{v}(A) < \delta$, where $\bar{v}(A) = \sup\{|v(B)|, B \in A \cap \mathscr{R}\}$. For countably additive vector measures on a σ-ring the $(\delta \to \varepsilon)$-absolute continuity coincides with the (we may say $(0 \to 0)$ –)

absolute continuity of Definition 2 in [2], see [3]. The next theorem is a generalization of Theorem 12 from [3].

Theorem 4. *Let $\gamma: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ be a vector d-polymeasure and let $\lambda_i: \sigma(\mathscr{R}_i) \to [0, +\infty)$, $i = 1, \ldots, d$, be countably additive measures. Then γ is separately $(\delta \to \varepsilon)$-absolutely continuous with respect to the polymeasure $\lambda_1 \times \ldots \times \lambda_d$ on $\sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d)$ if and only if its restriction to $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$ is separately $(\delta \to \varepsilon)$-absolutely continuous with respect to the restriction $\lambda_1 \times \ldots \times \lambda_d: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to [0, +\infty)$.*

Proof. The necessity implication is trivial. Conversely, let $\gamma: R_1 \times \ldots \times \mathscr{R}_d \to Y$ be separately $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_1 \times \ldots \times \lambda_d: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to [0, +\infty)$. According to Theorem 12 in [3], $\gamma(\cdot, R_2, \ldots, R_d): \sigma(\mathscr{R}_1) \to Y$ is $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_1: \sigma(\mathscr{R}_1) \to [0, +\infty)$ on $\sigma(\mathscr{R}_1)$ for each $(R_2, \ldots, R_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$. Hence if $N_1 \in \sigma(\mathscr{R}_1)$ and $\lambda_1(N_1) = 0$, then $\gamma(N_1, R_2, \ldots, R_d) = 0$ for each $(R_2, \ldots, R_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$. Now let $(A_2, \ldots, A_d) \in \sigma(\mathscr{R}_2) \times \ldots \times \sigma(\mathscr{R}_d)$. Since $\gamma(N_1, \ldots): \sigma(\mathscr{R}_2) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ is a vector $(d-1)$-polymeasure, by Lemma 4 in [2] there are $A_{i,n} \in \mathscr{R}_i$, $i = 2, \ldots, d$, $n = 1, 2, \ldots$ such that

$$\gamma(N_1, A_2, \ldots, A_d) = \lim_{n \to \infty} \gamma(N_1, A_{2,n}, \ldots, A_{d,n}) = 0 .$$

Hence $\gamma(\cdot, A_2, \ldots, A_d): \sigma(\mathscr{R}_1) \to Y$ is $(0 \to 0)$; hence equivalently $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_1: \sigma(\mathscr{R}_1) \to [0, +\infty)$ on $\sigma(\mathscr{R}_1)$ for each $(A_2, \ldots, A_d) \in \sigma(\mathscr{R}_2) \times \ldots \times \sigma(\mathscr{R}_d)$. By symmetry in coordinates, analogous assertions hold for $i = 2, \ldots, d$. The theorem is proved.

Theorem 5. *Let $\gamma_0: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ be a vector d-polymeasure and suppose that there are countably additive measures $\lambda_i: \sigma(\mathscr{R}_i) \to [0, +\infty)$, $i = 1, \ldots, d$, such that γ_0 is separately $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_1 \times \ldots \times \lambda_d$ on $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$. Then there is a unique separately countably additive extension $\gamma: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ of γ_0 if and only if the following condition holds:*

(1) *if $A_{i,n} \in \mathscr{R}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$, and $\chi_{A_{i,n}}(\cdot)$ converges a.e. $\lambda_i - (\lambda_i: \sigma(\mathscr{R}_i) \to [0, +\infty))$ on T_i for each $i = 1, \ldots, d$ then*
$$\lim_{n_1, \ldots, n_d \to \infty} \gamma(A_{1,n_1}, \ldots, A_{d,n_d}) \in Y \text{ exists.}$$

In that case $\lambda_1 \times \ldots \times \lambda_d$ is a control d-polymeasure for γ in the sense of Definition 4 in [2].

Proof. Let $(R_1, \ldots, R_{d-1}) \in \mathscr{R}_1 \times \ldots \times \mathscr{R}_{d-1}$. Since by assumption $\gamma_0(R_1, \ldots, R_{d-1}, .): \mathscr{R}_d \to Y$ is $(\delta \to \varepsilon)$-absolutely continuous with respect to the bounded measure $\lambda_d: \mathscr{R}_d \to [0, +\infty)$. by Kluvánek's theorem on extension, see [8] or Section I.5 in [1], there is a unique countably additive extension $\gamma_d(R_1, \ldots, R_{d-1})(.): \sigma(\mathscr{R}_d) \to Y$ of $\gamma_0(R_1, \ldots, R_{d-1}, .): \mathscr{R}_d \to Y$. According to Theorem 12 in [3], see Theorem 4 and its proof, this extension is $(\delta \to \varepsilon)$-absolutely continuous with respect to λ_d: $\sigma(\mathscr{R}_d) \to [0, +\infty)$. By symmetry in coordinates, analogous assertions hold if d is replaced by any $i = 1, \ldots, (d-1)$.

Suppose (1) and let $(A_1, \ldots, A_d) \in \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d)$. Due to Theorem D in § 13 and Theorem D in § 22 in [7] there are $A_{i,n} \in \mathscr{R}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ such that $\chi_{A_{i,n}}(.) \to \chi_{A_i}(.)\ \lambda_{A_i}$ a.e. on T_i for each $i = 1, \ldots, d$. Hence by (1) we may unambigously define $\gamma(A_1, \ldots, A_d) = \lim\limits_{n_1, \ldots, n_d \to \infty} \gamma_0(A_{1,n_1}, \ldots, A_{d,n_d})$. (If $A'_{i,n} \in \mathscr{R}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$, and $\chi_{A'_{i,n}}(.) \to \chi_{A_i}(.)\ \lambda_i^-$ a.e. on T_i, then $\chi_{A''_{i,n}}(.) \to \chi_{A_i}(.)$ λ_i^- a.e. on T_i, where $A_{A''_{i,n}} = A_{i,n}$ for n odd and $= A'_{i,n}$ for n even.)

Clearly γ extends γ_0, so it remains to show that γ is separately countably additive. Since our assumptions are symmetric with respect to the coordinates, it is enough to prove that $\gamma(A_1, \ldots, A_{d-1}, .)$: $\sigma(\mathscr{R}_d) \to Y$ is countably additive for each $(A_1, \ldots, A_{d-1}) \in \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_{d-1})$.

Let $(A_1, \ldots, A_{d-1}) \in \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_{d-1})$ be fixed. Then for each $A_d \in \sigma(\mathscr{R}_d)$,

$$\gamma(A_1, \ldots, A_d) = \lim_{n_1, \ldots, n_d \to \infty} \gamma_0(A_{1,n_1}, \ldots, A_{d,n_d}) =$$

$$= \lim_{n_1, \ldots, n_{d-1} \to \infty} \lim_{n_d \to \infty} \gamma_0(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}}, A_{d,n_d}) =$$

$$= \lim_{n_1, \ldots, n_{d-1} \to \infty} \gamma_d(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}})(A_d),$$

since $\chi_{A_{d,n_d}}(.) \to \chi_{A_d}(.)\ \lambda_d^-$ a.e. on T_d, and since $\gamma_d(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}})(.)$: $\sigma(\mathscr{R}_d) \to Y$ is $(\delta \to \varepsilon)$-absolutely continuous with respect to λ_d. Consequently, since $\gamma_d(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}})(.)$: $\sigma(\mathscr{R}_d) \to Y$, $n_1, \ldots, n_{d-1} = 1, 2, \ldots$ are countably additive vector measures, the countable additivity of $\gamma(A_1, \ldots, A_{d-1}, .)$: $\sigma(\mathscr{R}_d) \to Y$ follows from the (VHSN) – theorem, see Theorem I.4.8 in [1]. Hence the sufficiency is proved.

Conversely, let γ: $\sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ be the unique separately countably additive extension of γ_0. First we show that $\lambda_1 \times \ldots \times \lambda_d$ is a control d-polymeasure for γ on $\sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d)$. By symmetry in the coordinates it is enough to deduce that $A_d \in \sigma(\mathscr{R}_d)$ and $\lambda_d(A_d) = 0$ implies $\gamma(A_1, \ldots, A_d) = 0$ for each $(A_1, \ldots, A_{d-1}) \in \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_{d-1})$. Hence let A_d be such and let $(A_1, \ldots, A_{d-1})$ be fixed. According to Lemma 4 in [2] there are $A_{i,n} \in \mathscr{R}_i$, $i = 1, \ldots, d-1$, $n = 1, 2, \ldots$ such that $\gamma(A_1, \ldots, A_d) = \lim\limits_{n \to \infty} \gamma(A_{1,n}, \ldots, A_{d-1,n}, A_d)$. But $\gamma(A_{1,n}, \ldots, A_{d-1,n}, A_d) = \gamma_d(A_{1,n}, \ldots, A_{d-1,n})(A_d) = 0$ for each $n = 1, 2, \ldots$ by the $(0 \to 0)$-absolute continuity of $\gamma_d(A_{1,n}, \ldots, A_{d-1,n})(.)$: $\sigma(\mathscr{R}_d) \to Y$ with respect to λ_d, which was proved at the beginning of the proof. Now using this control d-polymeasure for γ we immediately obtain condition (1) as a direct consequence of Theorem 1 in [2]. The theorem is proved.

Let us note that for $d = 1$ condition (1) in the just proved theorem is a consequence of the assumed $(\delta \to \varepsilon)$-absolute continuity of γ_0 with respect to λ_1. Whether the analogue holds for $d > 1$ remains an open problem.

Using Theorem 11 in [2] we immediately obtain the following

Corollary. *Let each $\mathscr{R}_i$, $i = 1, 2, \ldots, d$, be a countable ring. Then a vector d-polymeasure $\gamma_0\colon \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ can be uniquely extended to a vector d-polymeasure $\gamma\colon \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ if and only if there are countably additive measures $\lambda_i\colon \sigma(\mathscr{R}_i) \to [0, +\infty)$, $i = 1, \ldots, d$, such that γ_0 is separately $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_1 \times \ldots \times \lambda_d$ on $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$, and (1) of Theorem 5 holds.*

Let us note that if it turns out that each vector d-polymeasure $\gamma\colon \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ has a control d-polymeasure, see Section 3 in [2], then the above corollary will be true for any rings $\mathscr{R}_i$, $i = 1, \ldots, d$.

In some situations the following result may be useful.

Theorem 6. *Let $1 \leqq d_1 < d$ be a positive integer, let $\gamma_0\colon \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1} \times \mathscr{R}_{d_1+1} \times \ldots \times \mathscr{R}_d \to Y$ be a vector d-polymeasure, and let there be countably additive measures $\lambda_i\colon \sigma(\mathscr{R}_i) \to [0, +\infty)$, $i = d_1 + 1, \ldots, d$, such that $\gamma_0(A_1, \ldots, A_{d_1}, \ldots)\colon \mathscr{R}_{d_1+1} \times \ldots \times \mathscr{R}_d \to Y$ is separately $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_{d_1+1} \times \ldots \times \lambda_d$ for each $(A_1, \ldots, A_{d_1}) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1}$. Then γ_0 can be uniquely extended to a vector d-holymeasure $\gamma\colon \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1} \times \sigma(\mathscr{R}_{d_1+1}) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ if and only if $\gamma_0(A_1, \ldots, A_{d_1}, \ldots)\colon \mathscr{R}_{d_1+1} \times \ldots \times \mathscr{R}_d \to Y$ can be extended to a $(d - d_1)$-polymeasure $\gamma_{d-d_1}(A_1, \ldots, A_{d_1})\colon \sigma(\mathscr{R}_{d_1+1}) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ for each $(A_1, \ldots, A_{d_1}) \in \mathscr{S}_1 \times \ldots \mathscr{S}_{d_1}$. (In that case $\gamma(A_1, \ldots, A_d) = \gamma_{d-d_1}(A_1, \ldots, A_{d_1})(A_{d_1+1}, \ldots, A_d)$ for each $(A_1, \ldots, A_d) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1} \times \sigma(\mathscr{R}_{d_1+1}) \times \ldots \times \sigma(\mathscr{R}_d)$ by the uniqueness of extensions.)*

Proof. The necessity is obvious. Conversely, suppose that the extensions $\gamma_{d-d_1}(A_1, \ldots, A_{d_1})\colon \sigma(\mathscr{R}_{d_1+1}) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ exist for each $(A_1, \ldots, A_{d_1}) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1}$. Define γ by the equality in the brackets above. We have to show that $\gamma(\ldots, A_{d_1+1}, \ldots, A_d)\colon \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1} \to Y$ is separately countably additive for each $(A_{d_1+1}, \ldots, A_d) \in \sigma(\mathscr{R}_{d_1+1}) \times \ldots \times \sigma(\mathscr{R}_d)$. Let $A_i \in \sigma(\mathscr{R}_i)$, $i = d_1 + 1, \ldots, d$. According to Theorem D in § 13 in [7] there are $A_{i,n} \in \mathscr{R}_i$, $i = d_1 + 1, \ldots, d$, $n = 1, 2, \ldots$ such that $\lambda_i(A_i \,\Delta\, A_{i,n}) \to 0$. Hence $\gamma(A_1, \ldots, A_{d_1}, A_{d_1+1,n}, \ldots, A_{d,n}) \to \gamma(A_1, \ldots, A_d)$ for each $(A_1, \ldots, A_{d_1}) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1}$ by Theorem 12 in [2]. Since by assumption $\gamma(\ldots, A_{d_1+1,n}, \ldots, A_{d,n})\colon \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1} \to Y$ are vector d_1-polymeasures for each $n = 1, 2, \ldots$, the (VHSN)-theorem for polymeasures, see the beginning in [2], implies the required separate countable additivity of $\gamma(\ldots, A_{d_1+1}, \ldots, A_d)\colon \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1} \to Y$. The theorem is proved.

Using Theorem 2 in [4], Theorem 10 and Corollary of Theorem 14 in [2] we easily obtain our last

Theorem 7. *Let T_i, $i = 1, \ldots, d$, be locally compact Hausdorff topological spaces, and let $\mathscr{C}_i$ $(\mathscr{C}_{0,i})$ denote the class of compact (compact G_δ) subsets of T_i. Then*

1) *for each separately regular vector Borel d-polymeasure*

$\gamma\colon \sigma(\mathscr{C}_1) \times \ldots \times \sigma(\mathscr{C}_d) \to Y$ *and for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{C}_i)$ *there is a d-tuple* $(B_i) \in \in \mathsf{X}\sigma(\mathscr{C}_{0,i})$ *such that* $\gamma(A_i) = \gamma(B_i)$;

2) *each vector Baire d-holymeasure* $\gamma_0\colon \mathsf{X}\sigma(\mathscr{C}_{0,i}) \to Y$, *which has a control d-polymeasure, can be uniquely extended to a separately regular vector Borel d-polymeasure* $\gamma\colon \mathsf{X}\sigma(\mathscr{C}_i) \to Y$, *and*

3) *each uniform vector Baire d-polymeasure* $\gamma_0\colon \mathsf{X}\sigma(\mathscr{C}_{0,i}) \to Y$ *can be uniquely extended to a separately uniformly regular vector Borel d-polymeasure* $\gamma\colon \mathsf{X}\sigma(\mathscr{C}_i) \to \to Y$.

The analogues hold if $\sigma(\mathscr{C}_i)$ *is replaced by* $\sigma(\mathscr{U}_i)$, $i = 1, \ldots, d$, *where* $\mathscr{U}_i$ *denotes the class of all open subsets of* T_i.

References

[1] *Diestel, J., Uhl, J. J.:* Vector measures, Amer. Math. Soc. Surveys, No. 15, Providence, 1977.

[2] *Dobrakov, I.:* On integration in Banach spaces, VIII (Polymeasures), Czech. Math. J.

[3] *Dobrakov, I., Farková, J.:* On submeasures II, Math. Slovaca *30*, 1980, 65—81.

[4] *Dobrakov, I.:* On extension of Baire submeasures, Math. Slovaca *35*, 1985, 37—42.

[5] *Dunford, N., Schwartz, J. T.:* Linear operators, Part I, Interscience, New York, 1958.

[6] *Gould, G. G.:* Integration over vector-valued measures, Proc. London Math. Soc. (3) *15*, 1965, 193—225.

[7] *Halmos, P. R.:* Measure theory, D. Van Nostrand, 1950.

[8] *Kluvánek, I.:* The extension and closure of vector measures, in: Vector and operator valued measures and applications, Edited by D. H. Tucker, H. B. Maynard, Academic Press, Inc., New York—London, 1973, 175—190.

[9] *Walker, H. D.:* Uniformly additive families of measures, Bull. Math. Soc. Sci. Math. Roum., *18*, 1974, 217—224.

[10] *Ylinen, K.:* On vector bimeasures, Annali Mat. Pura Appl. (4) *117*, 1978, 115—138.

Author's address: 814 73 Bratislava, Obrancov mieru 49, Czechoslovakia (Matematický ústav SAV).