Czechoslovak Mathematical Journal

Eduard Feireisl

On the existence of periodic solutions of a semilinear wave equation with a superlinear forcing term

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 1, 78–87

Persistent URL: http://dml.cz/dmlcz/102202

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

ON THE EXISTENCE OF PERIODIC SOLUTIONS OF A SEMILINEAR WAVE EQUATION WITH A SUPERLINEAR FORCING TERM

EDUARD FEIREISL, Praha

(Received October 25, 1985)

1. INTRODUCTION

The problem of the existence of periodic solutions of a wave equation has been studied very extensively at the present time. There exists a vaste literature concerning both the homogeneous case (free vibrations. see [4], [6]) and the nonhomogeneous one (forced vibrations, e.g. [2]). In the latter situation all up to now known results are dealing with a sublinear forcing term, which is supposed to satisfy some growth conditions connected with the spectrum of the corresponding linear operator. No satisfactory results seem to be known in the superlinear case (except the paper [6] of P. H. Rabinowitz dealing with an autonomous equation).

A. Bahri and H. Berestycki obtained in [1] positive results for Hamiltonian systems. Unfortunately, the technique they used does not seem to be applicable in the case of partial differential equations like a wave equation.

The paper presents some results in this direction. It is shown that for every forcing term (right-hand side of the equation) satisfying some growth conditions there exists a positive integer T in such a way that the equation possesses a solution which is $2\pi/T$-periodic if a force is $2\pi/T$-periodic with respect to the t-variable.

Remark. After having completed the paper, the author was informed of the works of K. Tanaka (see e.g. [7]). In case the function representing the "force" is a perturbation of an odd function or of a time-independent one, Tanaka's results are better and deeper than ours. All the same, our approach makes it possible to treat more general situations concerning the forcing term.

2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

We are going to investigate the problem $\{P_T\}$:

$$u_{tt}(x, t) - u_{xx}(x, t) + f(x, t, u(x, t)) = 0 , \tag{1}$$

where the unknown function $u = u(x, t)$ is defined for all $x \in [0, \pi]$, $t \in R^1$ and u

satisfies the Dirichlet boundary conditions

(2) $$u(0, t) = u(\pi, t) = 0 \quad \text{for all} \quad t \in R^1 .$$

Moreover, u is to be periodic in t with the period $2\pi/T$, i.e.

(3) $$u(x, t + 2\pi/T) = u(x, t) \quad \text{for all} \quad x \in [0, \pi] , \quad t \in R^1 ,$$

where T is a positive integer.

The function f is supposed to satisfy the following conditions:

(F 1) The continuity condition:
$f = f(x, t, u)$ is continuous on the set $[0, \pi] \times R^1 \times R^1$.

(F 2) The periodicity condition:
$f(x, t + 2\pi/T, u) = f(x, t, u)$ for all $x \in [0, \pi]$, $t, u \in R^1$.

(F 3) The monotonicity condition:
If $u_2 \geqq u_1$, then $f(x, t, u_2) \geqq f(x, t, u_1)$ for all $x \in [0, \pi]$, $t \in R^1$.

(F 4) The growth condition:
There exist positive constants c_1, c_2, c_3, c_4 and a number $p, p \in (2, +\infty)$ satisfying

(i) $|f(x, t, u)| \leqq c_1 |u|^{p-1} + c_2$,

(ii) $|f(x, t, u)| \geqq c_3 |u|^{p-1} - c_4$
for all $x \in [0, \pi]$, $t, u \in R^1$ and there exists $\delta > 0$ such that

(iii) $c_3/2 \geqq c_1/p + \delta$ (c_3, c_1 may depend on x, t as well).

Let us denote $\alpha = (c_1, c_2, c_3, c_4, p)$. The vector α will be considered as a parameter of our problem.

Before presenting the main theorem, let us define the solution of the problem $\{P_T\}$ in a weak sense. Let us denote

$$Q_T = \{(x, t) | \, x \in [0, \pi], \; t \in [0, 2\pi/T]\} .$$

Definition. The function u is a solution of the problem $\{P_T\}$ if $u \in L_1(Q_T)$, $f(\cdot, u) \in$ $\in L_1(Q_T)$ and

(4) $$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) + f(\cdot, u) \, \varphi = 0$$

holds for all functions φ which are both sufficiently smooth and satisfying the conditions (2), (3).

Our main goal is to prove the following existence theorem:

Theorem 1. *Let a parameter α and a nonnegative number K be given. Then there is an integer $T_0 = T_0(\alpha, K)$ such that for every $T \geqq T_0$, $T \in N$ and for every function f, satisfying (F1)–(F4) with α and T, the problem $\{P_T\}$ has a solution u. Moreover, u belongs to the class $L_\infty(Q_T)$ and $\|u\|_{L_\infty(Q_T)} \geqq K$.*

3. THE PROBLEM $\{P_T\}$ AS AN ABSTRACT OPERATOR EQUATION

First we are going to reformulate our problem $\{P_T\}$. We shall write $Q, L_p, \| \|_p$ instead of $Q_1, L_p(Q_1), \| \|_{L_p(Q_1)}$, respectively. Let us define a function g by

$$g(x, t, u) =^{\text{def}} f(x, t/T, u)\,.$$

Observe that if f satisfies (F1)–(F4) with a parameter α then g satisfies (F1)–(F4) with the same parameter and for $T = 1$. Moreover, it is clear that u satisfying

$$u(x, t) =^{\text{def}} v(x, Tt)$$

solves the problem $\{P_T\}$ only if v is a solution of the problem $\{P'_T\}$ given by

$$T^2\, v_{tt}(x, t) - v_{xx}(x, t) + g(x, t, v(x, t)) = 0\,, \tag{5}$$

$$v(0, t) = v(\pi, t) = 0\,, \tag{6}$$

$$v(x, t + 2\pi) = v(x, t) \quad \text{for all} \quad x \in [0, \pi]\,, \quad t \in R^1\,, \tag{7}$$

Obviously we have $\|v\|_\infty = \|u\|_{L_\infty(Q_T)}$. Consequently it suffices to find solutions of the problem $\{P'_T\}$.

Let us consider the linear operator

$$D_T = T^2 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

defined for smooth functions satisfying (6), (7). D_T has a selfadjoint extension on L_2 (denoted for simplicity D_T again). The system of eigenvectors of D_T

$$e_{kj}(x, t) = \begin{cases} \dfrac{\sqrt{2}}{\pi} \sin(kx) \sin(jt) & \text{for} \quad k \in N\,, \quad j \in N \\ 1/\pi \sin(kx) & \text{for} \quad k \in N\,, \quad j = 0 \\ \dfrac{\sqrt{2}}{\pi} \sin(kx) \cos(jt) & \text{for} \quad k \in N\,, \quad -j \in N \end{cases}$$

(N denotes the set of all positive integers) forms an orthonormal basis in L_2. The corresponding eigenvalues represent the spectrum Λ_T of D_T

$$\Lambda_T = \{k^2 - j^2 T^2 \mid k \in N,\ j \in Z\}$$

(Z denotes the set of all integers). Let us define Fourier coefficients for $u \in L_1$ by

$$a_{kj}(u) = \int_Q u e_{kj} \quad \text{for all} \quad k \in N\,, \quad j \in Z\,.$$

Now the operator D_T has a spectral resolution

$$D_T v = \sum_{\substack{k \in N \\ j \in Z}} (k^2 - j^2 T^2)\, a_{kj}(v)\, e_{kj}\,.$$

It seems to be convenient to introduce the notation

$$|\Lambda_T \leqq z| = \operatorname{span}\{e_{kj} \mid k \in N,\ j \in Z,\ k^2 - j^2 T^2 \leqq z\} \quad \text{for} \quad z \in R^1\,.$$

Further we shall write $\sum$ instead of $\sum_{\substack{k \in N \\ j \in Z}}$.

Finally let us denote λ_T the greatest negative eigenvalue belonging to Λ_T.

We are going to prove the following lemma:

Lemma 1.

(i) *For arbitrary* $a > 1$ *there exists the constant* $c_5(a)$, $c_5(a)$ *does not depend on* T *and*

(8) $$\sum_{k^2 - j^2T^2 \neq 0} |k^2 - j^2T^2|^{-a} < c_5(a) < +\infty .$$

(ii) *The following estimate holds*

(9) $$|\lambda_T| \geqq T .$$

(iii) *The nullspace of the operator* D_T, *i.e. the* L_2*-closure of* $|\Lambda_T = 0|$ *is characterised by*

(10) $$\mathcal{N}(D_T) = \{u \mid u(x, t) = q(t + Tx) - q(t - Tx) ,$$
$$q \in L_2[0, 2\pi],\ q(s + 2\pi) = q(s) \quad \textit{for all} \quad s \in R^1,\quad \textstyle\int_0^{2\pi} q(s)\, ds = 0\} .$$

Proof.

(i) $$\sum_{k^2 - j^2T^2 \neq 0} |k^2 - j^2T^2|^{-a} \leqq \sum_{k^2 - j^2 \neq 0} |k^2 - j^2|^{-a} \leqq$$
$$\leqq 2 \sum_{m \in N} m^{-a} \sum_{n \in N} n^{-a} \leqq c_5(a) .$$

(ii) $$|\lambda_T| = |k_0 - j_0T|\, |k_0 + j_0T| \geqq l\, .\, T .$$

(iii) See for example [2]. ■

Now we need some estimates concerning the function g. Let us set

$$G(x, t, v) =^{\text{def}} \textstyle\int_0^v g(x, t, s)\, ds \quad \text{for all} \quad x \in [0, \pi] , \quad t \in R^1 .$$

Using (F4) (i), (ii) we get immediately

(11) $$G(x, t, v) \leqq c_1/p\, |v|^p + c_2|v| ,$$

(12) $$G(x, t, v) \geqq c_3/p\, |v|^p - c_4|v| \quad \text{for all} \quad x \in [0, \pi] , \quad t, v \in R^1 .$$

Combining it with (F4) (iii) we have

(13) $$\tfrac{1}{2}v\, g(x, t, v) - G(x, t, v) \geqq \delta|v|^p - (c_2 + c_4)\, |v|$$
$$\text{for all} \quad x \in [0, \pi] , \quad t, v \in R^1 .$$

Finally let us define the function H

(14) $$H(v) =^{\text{def}} \sup \{g(x, t, v)|\ x \in [0, \pi],\ t \in [0, 2\pi]\} -$$
$$- \inf \{g(x, t - v)|\ x \in [0, \pi],\ t \in [0, 2\pi]\} .$$

Observe that according to (F3) H is nondecreasing in v. Moreover the assumptions (F4) (i), (ii) imply

(15) $$\lim_{v \to -\infty} H(v) = -\infty .$$

Let us consider the scale of Hilbert spaces H_s^T, defined for $s \in [0, 1]$, where H_s^T is a completion of $|\Lambda_T \neq 0|$ according to the norm

$$|||v|||_{s,T} = \{\sum_{k^2 - j^2T^2 \neq 0} |k^2 - j^2T^2|^s a_{kj}^2(v)\}^{1/2}.$$

We have for every $v \in H_s^T$

$$\|v\|_\infty \leqq \sqrt{2}/\pi \sum_{k^2 - j^2T^2 \neq 0} |a_{kj}(v)| \leqq$$

$$\leqq \sqrt{2}/\pi \sqrt{c_5(a)} \{\sum_{k^2 - j^2T^2 \neq 0} |k^2 - j^2T^2|^a a_{kj}^2(v)\}^{1/2},$$

where $a > 1$ arbitrary (lemma 1). Interpolation theory gives

$$\|v\|_p \leqq \left(\frac{\sqrt{2}\, c_5(a)}{\pi}\right)^{(p-2)/p} \{\sum_{k^2 - j^2T^2 \neq 0} |k^2 - j^2T^2|^{a(p-2)/2} a_{kj}^2(v)\}^{1/2}$$

since $p \in (2, +\infty)$. We can choose $a > 1$ such that

$$r = \frac{a(p-2)}{p} < 1.$$

Thus we have obtained an important estimate

$$\|v\|_p \leqq c_6 |||v|||_{r,T} \quad \text{for all} \quad v \in H_r^T. \tag{16}$$

The constant $c_6 > 0$ does not depend on T.

One easily verifies that v is a solution of the problem $\{P'_T\}$ (see definition in § 2) only if

$$v \in L_1,$$

$$g(\cdot, v) \in L_1,$$

$$(k^2 - j^2T^2)\, a_{kj}(v) + a_{kj}(g(\cdot, v)) = 0 \tag{17}$$

holds for all $k \in N$, $j \in Z$.

4. THE FINITE DIMENSIONAL APPROXIMATION

We shall approximate our problem given by (17). Let us consider the sequence of finite dimensional Hilbert spaces

$$E_n = \operatorname{span}\{e_{kj} \mid k \leqq n,\ |j| \leqq n\} \quad \text{for} \quad n \in N$$

with a norm induced by $\| \ \|_2$. We define the functional I_n^T on the space E_n by

$$I_n^T(v) = \tfrac{1}{2} \sum (k^2 - j^2T^2)\, a_{kj}^2(v) + \int_Q G(\cdot, v).$$

Clearly I_n^T is of the class $C^1(E_n, R^1)$ with the gradient

$$\langle \operatorname{grad} I_n^T(v), w \rangle = \sum (k^2 - j^2T^2)\, a_{kj}(v)\, a_{kj}(w) + \int_Q g(\cdot, v)\, w.$$

We get according to (12) that I_n^T is coercive on E_n, i.e.

$$\lim_{\|v\|_2 \to \infty} I_n^T(v) = +\infty. \tag{18}$$

Our aim is to find some appropriate critical points of the functional I_n^T on E_n. We shall use the following assertion.

Lemma 2. *Let us choose $z \in R^1$ arbitrary. Then there exists a constant $c_7(z) \in R^1$, $c_7(z)$ depends neither on T nor on n, such that*

$$(19) \qquad I_n^T(v) \geqq c_7(z) \quad \textit{for all} \quad v \in |\Lambda_T \geqq z| \cap E_n\,.$$

Proof. Let us choose $v \in |\Lambda_T \geqq z| \cap E_n$. We have

$$I_n^T(v) = \tfrac{1}{2} \sum_{k^2 - j^2T^2 \geqq z} (k^2 - j^2T^2)\, a_{kj}^2(v) + \int_Q G(\cdot, v) \geqq z/2\, \|v\|_2^2 + c^3/p\, \|v\|_p^p - c_4\|v\|_1\,.$$

We have used the estimate (12). Further we get

$$I_n^T(v) \geqq z/2\, \|v\|_2^2 + c_8\|v\|_2^p - c_9\|v\|_2$$

where $c_8, c_9 > 0$ depend on α only. Thus we obtain

$$I_n^T(v) \geqq \inf_{x \geqq 0}(z/2\; x^2 + c_8 x^p - c_9 x) \geqq c_7(z)\,. \quad \blacksquare$$

Let us denote the unit sphere in H_r^T (r from (16)) by

$$SP_r^T = \{v \mid v \in H_r^T,\ |||v|||_{r,T} = 1\}\,.$$

We are going to prove the following lemma.

Lemma 3. *Let $z \in R^1$ be a given number. Then there is $T_0 = T_0(z)$, $T_0 \in N$ (T_0 does not depend on n) such that for all $T \geqq T_0$*

$$(20) \qquad I_n^T(v) \leqq z$$

whenever v belongs to $SP_r^T \cap |\Lambda_T < 0| \cap E_n$.

Proof. For $v \in SP_r^T \cap |\Lambda_T < 0| \cap E_n$ we have

$$I_n^T(v) = \tfrac{1}{2} \sum_{k^2 - j^2T^2 < 0} (k^2 - j^2T^2)\, a_{kj}^2(v) + \int_Q G(\cdot, v)$$

$$\leqq -\tfrac{1}{2}|\lambda_T|^{1-r}\, |||v|||_{r,T}^2 + c_1/p\, \|v\|_p^p + c_2\|v\|_1\,.$$

Now according to (9), (16) we can conclude

$$\leqq -\tfrac{1}{2}T^{1-r} + c_{10}\,,$$

where c_{10} does not depend on T, n. If T is sufficiently large, then (20) holds. $\blacksquare$

Now we are ready to show the existence of critical points of the functional I_n^T belonging to a critical level which is bounded independently on n. This fact will enable us to carry out a limit process.

Let us choose a number $d < 0$ arbitrary, $d < c_7(0)$. According to lemma 3 we can find $T = T(d)$ satisfying

$$(21) \qquad I_n^T(v) \leqq d \quad \text{for all} \quad v \in SP_r^T \cap |\Lambda_T < 0| \cap E_n\,.$$

In what follows, $T = T(d)$ will remain fixed. Thus we can drop the subscript T

for the sake of convenience. Set

$$c_{11} = \min (c_7(\lambda), d) - 1 .$$

Denote by P_n the orthogonal projection

$$P_n\colon E_n \to |A < 0| \cap E_n .$$

Suppose that there is not a critical value of I_n in the interval $[c_{11}, d]$ i.e.

(22) If $v \in \{v \mid \operatorname{grad} I_n(v) \equiv 0\}$, then $I_n(v) \in (-\infty, c_{11}) \cup (d, +\infty)$.

Since (18) holds, it can be shown (see [5] for example) that there is a homotopy h satisfying

$$h\colon \{v \mid I_n(v) \leqq d\} \times [0, 1] \to E_n ,$$

$$h(v, 0) = v \quad \text{for all} \quad v ,$$

(23) $$I_n(h(v, t)) \leqq d + \varepsilon < c_7(0), \quad \varepsilon > 0, \quad \text{for all} \quad v, t$$

(according to (21)),

(24) $$h(\{v \mid I_n(v) \leqq d\}, 1) \subseteq \{v \mid I_n(v) \leqq c_{11}\} .$$

Let us denote the unit sphere in $E_n \cap |A < 0|$ by

$$S_n^- = \{v \mid v \in E_n \cap |A < 0|, \ \|v\|_2 = 1\} .$$

Clearly there is the homeomorphism ϱ from S_n^- onto $SP_r \cap |A < 0| \cap E_n$. Now according to (23), (21)

$$P_n(h(v, t)) \neq 0 \quad \text{for all} \quad v \in SP_r \cap |A < 0| \cap E_n .$$

Thus it is correct to define a new homotopy

$$\hat{h}\colon S_n^- \times [0, 1] \to S_n^- ,$$

$$\hat{h}(v, t) = \frac{P_n\, h(\varrho(v), t)}{\|P_n\, h(\varrho(v), t)\|_2} .$$

Now the mapping $\hat{h}(\cdot, 0)$ is essential because it maps S_n^- onto S_n^-.

On the other hand if n is sufficiently large (in order to $|A = \lambda| \subseteq E_n$), there exists $e \in S_n^- \cap |A = \lambda|$. According to (24)

$$e \notin \hat{h}(S_n^-, 1) .$$

Consequently $\hat{h}(\cdot, 1)$ is homotopically trivial. But this is impossible and thus (22) must be false.

We have just obtained the following result: There exists the sequence $\{v_n\}_{n=n_0}^{\infty}$ of approximate solutions of the problem $\{P'_T\}$ satisfying

(25) $$\tfrac{1}{2} \sum (k^2 - j^2 T^2)\, a_{kj}^2(v_n) + \int_Q G(\cdot, v_n) \in [c_{11}, d] ,$$

(26) $$\sum (k^2 - j^2 T^2)\, a_{kj}(v_n)\, a_{kj}(w) + \int_Q g(\cdot, v_n)\, w = 0 \quad \text{for all} \quad w \in E_n .$$

We are going to carry out the limit process in the sequence $\{v_n\}_{n=n_0}^{\infty}$. First let us set $w = v_n$ in (26) and combining it with (25), we get

(27) $$\tfrac{1}{2}\int_Q g(\cdot, v_n)\, v_n - \int_Q G(\cdot, v_n) \in [-d, -c_{11}]\,.$$

Now using (13) we obtain the existence of the constants $c_{12} > 0$ and $c_{13} > 0$ (by (F3))

(28) $$\|v_n\|_p < c_{12}\,,$$

(29) $$\|g(\cdot, v_n)\|_{p'} < c_{13} \quad \text{for all} \quad n \geqq n_0$$

where $1/p + 1/p' = 1$. Further we need the following lemma.

Lemma 4. *For arbitrary* $\varepsilon > 0$ *there exists* $l(\varepsilon) > 0$ *satisfying*

(30) $$\sum_{|k^2 - j^2T^2| \geqq l(\varepsilon)} |k^2 - j^2T^2|\, a_{kj}^2(v_n) < \varepsilon \quad \textit{for all} \quad n \geqq n_0\,.$$

Proof. Let us set

$$w_n = \sum_{|k^2 - j^2T^2| \geqq l} \operatorname{sgn}(k^2 - j^2T^2)\, a_{kj}(v_n)\, e_{kj}$$

in (26). Thus we get

$$\sum_{|k^2 - j^2T^2| \geqq l} |k^2 - j^2T^2|\, a_{kj}^2(v_n) \leqq c_{13}\|w_n\|_p \leqq c_{13}\, c_6 |||w_n|||_r \leqq$$
$$\leqq c_{13}c_6 l^{(r-1)/2}\{\sum_{|k^2 - j^2T^2| \geqq l} |k^2 - j^2T^2|\, a_{kj}^2(v_n)\}^{1/2}.$$

Since $r < 1$, we can choose $l > 0$ such that

$$c_{13}c_6 l^{(r-1)/2} < \varepsilon^2\,. \quad \blacksquare$$

Consider now the orthogonal projection

$$P: L_2 \to H_0\,.$$

According to (30) we have $\{Pv_n\}_{n=n_0}^{\infty}$ is totally bounded and consequently precompact in H_1. Combining it with (28), (29) we get the existence of a subsequence (denoted $\{v_n\}_{n=1}^{\infty}$ for simplicity) satisfying

(31) $$v_n \to v \quad \text{weakly in} \quad L_p\,,$$
$$g(\cdot, v_n) \to \varphi \quad \text{weakly in} \quad L_{p'}\,,$$
$$Pv_n \to v \quad \text{strongly in} \quad H_1\,.$$

For fixed $w \in E_n$ we can pass to the limit in (26) now. We get

(32) $$\sum (k^2 - j^2T^2)\, a_{kj}(v)\, a_{kj}(w) + \int_Q \varphi w = 0\,.$$

Setting $w = v_n$ in (26) we get

(33) $$\lim_{n\to\infty} \int_Q g(\cdot, v_n)\, v_n = -|||v|||_1\,.$$

Now we can insert $w = v_n$ in (32) and pass to the limit

(34) $$-|||v|||_1 = \int_Q \varphi v\,.$$

Combining (33), (34) and (31) with (F3) we get

(35) $$\varphi = g(\cdot, v)$$

using standard arguments of monotone operator theory (see [3]). Thus (32) is equivalent to (17) and we conclude that the function v is a solution of the problem $\{P'_T\}$ belonging to the space L_p.

Moreover from (27) using (33), (34), (35) and the convexity of G, we have

$$\tfrac{1}{2} \int_\Omega g(\cdot, v)\, v - \int_\Omega G(\cdot, v) \geqq -d\,.$$

Applying (F4) (i), (ii) we get an estimate

(36) $$c_1/2\, \|v\|_p^p + (c_2 + c_4)\, \|v\|_1 \geqq -d\,.$$

6. REGULARITY OF THE SOLUTION v

It remains only to show that v is of the class L_∞. The estimate (36) then gives $\|v\|_\infty \geqq K$ if we choose $d < 0$ sufficiently small. In order to prove this, we use an analogous technique as in [2].

Consider the following decomposition

$$v = v_1 + v_2$$

where $v_1 = Pv$ and $v_2 = (\mathrm{Id} - P)\, v$. It is known that $\|v_1\|_\infty \leqq M$ (see [2]) for some constant M. Now v_2 represents the nullspace component of v according to D_T. Now we have

(37) $$\int_0^\pi g(x, t + Tx, v(x, t + Tx)) - g(x, t - Tx, v(x, t - Tx))\, \mathrm{d}x = 0 \quad \text{for a.e.} \quad t \in [0, 2\pi]$$

since $g(\cdot, v)$ is orthogonal to $\mathcal{N}(D_T)$ given by (10) (see [2] for details). Now v_2 can be written as

$$v_2(x, t) = q(t + Tx) - q(t - Tx)\,, \quad q \text{ as in (10)}\,, \quad q \in L_p[0, 2\pi]\,.$$

Thanks to the assumption (F3) we get from (37)

(38) $$\int_0^\pi g(x, t + Tx, M + q(t + 2Tx) - q(t)) - g(x, t - Tx, -M - q(t - 2Tx) + q(t))\, \mathrm{d}x \geqq 0\,.$$

Consequently after an easy computation

(39) $$\int_0^{2\pi} H(M + q(s) - q(t))\, \mathrm{d}s \geqq 0 \quad \text{for a.e.} \quad t \in [0, 2\pi]\,.$$

Suppose that there is a sequence $\{t_n\}_{n=1}^\infty \subseteq [0, 2\pi]$, $q(t_n) \geqq n$ and

$$\operatorname{meas}\{t \mid t \in [0, 2\pi],\ q(t_n) \geqq n\} > 0\,.$$

We can insert $t = t_n$ in (39) now. According to monotonicity of H we can pass to the limit on both sides of (39). But the limit on the left-hand side equals $-\infty$ according to (15). Thus

$$\operatorname*{ess\,sup}_{s\in[0,2\pi]} q(s) < +\infty .$$

Similarly we prove

$$\operatorname*{ess\,sup}_{s\in[0,2\pi]} -q(s) < +\infty .$$

and consequently $v_2 \in L_\infty$.

Theorem 1 has been proved.

References

[1] *Bahri, A., Berestycki, H.:* Forced vibrations of superquadratic Hamiltonian systems. Preprint, Université Paris VI, 1981.

[2] *Bahri, A., Brezis, H.:* Periodic solutions of a nonlinear wave equation. Proc. Roy. Soc. Edinburgh Sect. A., *85* (1980), 313—320.

[3] *Lions, J. L:* Quelques méthodes de resolution des problèmes aux limites non linéaires. Dunod,Gauthier-Willars, Paris (1969).

[4] *Lovicar, V.:* Free vibrations for the equation $u_{tt} - u_{xx} + f(u) = 0$ with f sublinear. Proc. of Equadiff 5, Teubner-Texte zur Mathematik, Band 47, 228—230.

[5] *Palais, R. S.:* Critical point theory and the minimax principle. Proc. Sympos. Pure Math. *15* (1970), 185—212.

[6] *Rabinowitz, P. H.:* Large amplitude time periodic solutions of a semilinear wave equation. Comm. Pure. Appl. Math. *37* (1984), 189—206.

[7] *Tanaka, K.:* Infinitely many periodic solutions for the equation $u_{tt} - u_{xx} \pm |u|^{s-1} u = f(x, t)$. Proc. Japan. Acad. *61* (1985), Ser. A, 70—73.

Author's address: 120 00 Praha 2, Karlovo nám. 13, Czechoslovakia (FSI ČVUT).