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ON MEANS OF POSITIVE DEFINITE MATRICES

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1. INTRODUCTION

Let A denote an $n \times n$ positive definite symmetric matrix and p a nonzero real number. Lutwak [5] defined the (absolute) p-mean of A, $\Phi_p[A]$, by

$$\Phi_p[A] = \left(\frac{1}{nw_n} \int_{S^{n-1}} (u, Au)^{p/2} \, \mathrm{d}S(u)\right)^{2/p},$$

where w_n denotes the volume of the unit ball in the Euclidean space $\mathbb{R}^n$, $\mathrm{d}S(u)$ denotes the area element of S^{n-1} at u and $(,)$ denotes the usual inner product in $\mathbb{R}^n$. For $p = -\infty$, 0, or ∞ the (absolute) p-mean of A is defined by

$$\Phi_p[A] = \lim_{q \to p} \Phi_q[A] .$$

The (absolute) p-means of A are related to some of the fundamental scalar functions (such as the determinant and trace) of A. It is therefore worthwhile to use the properties of p-means and prove results for the matrices themselves. In particular, we consider finite sequences of matrices and introduce the concept of relative p-means.

For a finite sequence $\mathscr{A} = (A_1, \ldots, A_k)$, $k \geqq 2$, of $n \times n$ positive definite symmetric matrices we define the relative p-mean of A_i, $\mathrm{rel}(\Phi_p[A_i]) \equiv \Phi_{p,i,\mathscr{A}}$ **), by

$$\Phi_{p,i,\mathscr{A}} = \Phi_p[A_i * \bar{A}]^{1/2} \quad (i = 1, \ldots, k),$$

where

$$\bar{A} = \frac{1}{k}(A_1 + \ldots + A_k)$$

and $*$ denotes the Hadamard (elementwise) product. For $k = 1$ we define the relative p-mean of A to be equal to its (absolute) p-mean.

If A and B are positive definite symmetric matrices, then we shall write $A \leqq B$ if $B - A$ is positive semidefinite. All matrices are assumed to be of order n.

The importance of relative p-means is given by the following Proposition, the proof of which appears at the end of section 2.

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**) If $\mathscr{A}$ is specified then we shall write $\Phi_{p,i}$ instead of $\Phi_{p,i,\mathscr{A}}$.

Proposition 1. *Let* $\mathscr{A} = (A_1, \ldots, A_k)$ *and* $\mathscr{B} = (B_1, \ldots, B_k)$, $k \geqq 2$, k: *even, be two finite sequences of positive definite symmetric matrices. Assume that* $A_1 * \bar{A} \leqq \ldots \leqq A_k * \bar{A}$, $B_1 * \bar{B} \leqq \ldots \leqq B_k * \bar{B}$ *and* $\bar{A} \leqq \bar{B}$. *Suppose that there exists a subset* K *of* $\{1, \ldots, k\}$ *of cardinality* $k/2$ *and a permutation* σ *of* K *such that*:

(a) $A_i * \bar{A} \leqq B_{\sigma(i)} * \bar{B}$ *for all* $i \in K$

(b) *either* 1 *or* k *is a member of* K.

(c) $A_i * \bar{A} = A_j * \bar{A}$ *for all* $i, j \in K$.

If for some p, $-\infty < p < \infty$

$$(*) \qquad \sum_{i \in K} \Phi_{p,i,\mathscr{A}} = \sum_{i \in K} \Phi_{p,i,\mathscr{B}} = \sum_{i \notin K} \Phi_{p,i,\mathscr{B}},$$

then

$$A_1 * I = \ldots = A_k * I = B_1 * I = \ldots = B_k * I,$$

where I is the unit matrix.

If in addition all the entries of $\bar{A}$ and $\bar{B}$ are nonzero, then

$$A_1 = \ldots = A_k = B_1 = \ldots = B_k.$$

Remark 1. If $A_1 \leqq \ldots \leqq A_k$ and $B_1 \leqq \ldots \leqq B_k$ then the hypothesis of Proposition 1 is satisfied; since $\bar{A} > 0$, $\bar{B} > 0$, $A_i \leqq A_j$ and $B_i \leqq B_j$ imply $A_i * \bar{A} \leqq A_j * \bar{A}$ and $B_i * \bar{B} \leqq B_j * \bar{B}$.

Remark 2. No comparison is assumed between all the members of $\mathscr{A}$ and $\mathscr{B}$, though the assumption that $\bar{A} \leqq \bar{B}$ is significant.

Problems which involve a comparison of matrices in the sense $A \leqq B$ are useful in statistics and especially in linear models. The reader is referred to [7] and the references therein.

The aim of this paper is to investigate the relative p-means of a finite sequence of positive definite symmetric matrices $\mathscr{A} = (A_1, \ldots, A_k)$, $k \geqq 2$. In particular, we find lower bounds of the product and the sum of the relative p-means of $\mathscr{A}$. Since some of the p-means of A are related to some of the fundamental scalar functions of A, then the inequalities proved in sections 3 and 4 automatically hold for these functions. This gives us a general method to prove inequalities for these functions. Moreover, more applications will be given in section 5.

2. PRELIMINARY RESULTS

The following is the first major result on Hadamard products.

Theorem 1 (Schur [6]). *If* A, B *are positive definite hermitian matrices, then* $A * B$ *is positive definite hermitian.*

Also, we shall need the following result.

Theorem 2. *If* A_i, B_i $(i = 1, \ldots, k)$ *are positive definite hermitian matrices and*

$A_i \leqq B_i$ $(i = 1, \ldots, k)$, *then*

$$A_1 * \ldots * A_k \leqq B_1 * \ldots * B_k .$$

Proof. [1, p. 224]

The basic properties of the (absolute) p-mean were proved by Lutwak in [5] We shall continuously make use of the following properties. (All matrices A, B_i are assumed to be positive definite.)

1. For all p and all positive scalars λ,

$$\Phi_p[\lambda A] = \lambda\, \Phi_p[A] .$$

2. For a fixed matrix A, $\Phi_p[A]$ is continuous in p.

3. If $A \leqq B$, then

$$\Phi_p[A] \leqq \Phi_p[B]$$

for all real p, with equality if and only if $A = B$.

4. If $p > 2$, then

$$\Phi_p[A + B] \leqq \Phi_p[A] + \Phi_p[B] ,$$

with equality if and only if $A = \lambda B$. If $p = 2$, then

$$\Phi_2[A + B] = \Phi_2[A] + \Phi_2[B] .$$

If $p < 2$, then

$$\Phi_p[A + B] \geqq \Phi_p[A] + \Phi_p[B] ,$$

with equality if and only if $A = \lambda B$, where λ is a positive scalar.

5. $$\Phi_2[A] = (1/n) \operatorname{tr} A ,$$

where tr A denotes the trace of A.

6. $$\Phi_\infty[A] = \lambda_1(A) \quad \text{and} \quad \Phi_{-\infty}[A] = \lambda_n(A) ,$$

where $\lambda_1(A)$ and $\lambda_n(A)$ denote the largest and smallest of the eigenvalues of A, respectively.

7. $$\Phi_{-n}[A] = (\det A)^{1/n} ,$$

where det A denotes the determinant of A.

The following results will be used in sections 3 and 4.

Lemma 1. *If A is a positive definite symmetric matrix, then for all $p > 0$*

$$\Phi_p[A^{-1}] \geqq \Phi_p[A]^{-1} ,$$

with equality if and only if $A = \lambda I$, λ a positive number.

Proof. If we integrate both sides of the following inequality [2, p. 69]

$$(u, Au)\,(u, A^{-1}u) \geqq (u, u)^2 , \tag{1}$$

and then apply Schwarz's inequality [4, p. 132] to the functions on the left we obtain the result. The conditions of equality follow from the observation that equality in (1) holds if and only if $A = \lambda I$.

Lemma 2. *Let $A_1, \ldots, A_k$ be positive definite symmetric matrices. If $p < 2$, then*

$$\Phi_p[\sum_{i=1}^{k} A_i] \geqq \sum_{i=1}^{k} \Phi_p[A_i],$$

with equality if and only if $A_i = \lambda_i A_k$, $i = 1, \ldots, k-1$ and λ_i are positive numbers. If $p > 2$, then

$$\Phi_p[\sum_{i=1}^{k} A_i] \leqq \sum_{i=1}^{k} \Phi_p[A_i],$$

with equality if and only if $A_i = \lambda_i A_k$, $i = 1, \ldots, k-1$.

Proof. Let $p < 2$, then the case $k = 2$ is property 4. If $k > 2$ then using property 6 repeatedly we obtain

$$\Phi_p[A_1 + A_2 + \ldots + A_k] \geqq$$
$$\geqq \Phi_p[A_1] + \Phi_p[A_2 + \ldots + A_k] \geqq \ldots \Phi_p[A_1] + \ldots + \Phi_p[A_k].$$

Equality holds if and only if equality holds in all the inequalities above and so by property 4 we obtain

$$\begin{aligned} A_1 &= \lambda'_1(A_2 + \ldots + A_k) \\ A_2 &= \lambda'_2(A_3 + \ldots + A_k) \\ &\vdots \\ A_{k-1} &= \lambda'_{k-1} A_k . \end{aligned}$$

Hence the result.

A similar argument is used for the case $p > 2$.

Lemma 3. *For a positive definite hermitian matrix A the following holds*

$$A * A \geqq \lambda_n(A)\, A * I,$$

where $\lambda_n(A)$ is the smallest eigenvalue of A.

Proof. [1, p. 238].

Lemma 4. *If $D_1, \ldots, D_k$ are positive definite diagonal matrices and if $\alpha_1, \ldots, \alpha_k$ are positive numbers such that $\sum_{i=1}^{k} \alpha_i = 1$, then the following holds*

$$\sum_{i=1}^{k} \alpha_i D_i \geqq \left(\sum_{i=1}^{k} \alpha_i D_i^{-1}\right)^{-1}.$$

Equality holds if and only if $D_1 = \ldots = D_k$.

Proof. The result follows from the arithmetic-harmonic mean inequality of numbers.

Kantorovich's inequality will be used in sections 3 and 4. We shall use a formulation of this important inequality, due to Clausing [3].

Let $0 < m \leqq m_i \leqq M\,(i = 1, \ldots, k)$. Suppose $0 < \alpha_i\,(i = 1, \ldots, k)$, and $\sum_{i=1}^{k} \alpha_i = 1$. Set $\gamma_I = \sum_{i \in I} \alpha_i$ for $I \subset \{1, \ldots, k\}$. Let $I_0 \subset \{1, \ldots, k\}$ be such that

$$|\gamma_{I_0} - \tfrac{1}{2}| \leqq |\gamma_I - \tfrac{1}{2}| \quad \text{for all} \quad I \subset \{1, \ldots, k\}.$$

Define $$C_3(m, M) \equiv 1 + \gamma_{I_0}(1 - \gamma_{I_0}) \frac{(M - m)^2}{mM}.$$

Lemma 5. (Kantorovich's inequality). *The following inequality holds*

$$\left(\sum_{i=1}^{k} \alpha_i m_i\right)\left(\sum_{i=1}^{k} \frac{\alpha_i}{m_i}\right) \leqq C_3(m, M),$$

with equality if and only if there is a subset I of $\{1, \ldots, k\}$ such that $\sum_{i\in I} \alpha_i = \frac{1}{2}$.

Proof. [3].

If $n = 2$ then $$C_3(m, M) = 1 + \alpha_1\alpha_2 \frac{(M - m)^2}{mM}.$$

Also, in the special case $\alpha_i = 1/k$ $(i = 1, \ldots, k)$

$$C_3(m, M) = 1 + \frac{(M - m)^2}{4mM}, \quad \text{if } k \text{ is even}$$

$$C_3(m, M) = 1 + \left(1 - \frac{1}{k^2}\right) \frac{(M - m)^2}{4mM}, \quad \text{if } k \text{ is odd}.$$

Lemma 6. If A is a matrix with the main diagonal consisting of real numbers, then

$$\Phi_2[A * A] \geqq \Phi_2[A]^2,$$

with equality if and only if $A * I = \lambda I$.

Proof. By property 5 $\Phi_2[A] = (1/n) \operatorname{tr} A$. Then the result follows from Cauchy's inequality.

We give now the proof of Proposition 1.

Using property 3 and the fact that $A_i * \bar{A} \leqq B_{\sigma(i)} * \bar{B}$ for every $i \in K$, we obtain that $\Phi_p[A_i * \bar{A}] \leqq \Phi_p[B_{\sigma(i)} * \bar{B}]$ for every $i \in K$, and hence

$$\sum_{i\in K} \Phi_{p,i,\mathscr{A}} \leqq \sum_{i\in K} \Phi_{p,\sigma(i),\mathscr{B}} = \sum_{i\in K} \Phi_{p,i,\mathscr{B}}.$$

By $(*)$, the above inequality and property 3 we deduce that

$$A_i * \bar{A} = B_{\sigma(i)} * \bar{B} \quad \text{for all} \quad i \in K.$$

Hence $B_i * \bar{B} = B_j * \bar{B}$ for all $i, j \in K$. Set $A_i * \bar{A} = A$ and $B_i * \bar{B} = B$ for $i \in K$. Clearly $A = B$. We may assume, without loss of generality, that for all indices j not belonging to K, the sequences $\{A_j * \bar{A}\}$ and $\{B_j * \bar{B}\}$ contain only distinct elements. From (b) of Proposition 1 we have the following three possible cases:

(I) If 1 is in K, then

$$\underbrace{A = \ldots = A}_{k/2\text{-times}} \leqq A_i * \bar{A} \leqq \ldots \leqq A_j * \bar{A}$$

$$\underbrace{B = \ldots = B}_{k/2\text{-times}} \leqq B_i * \bar{B} \leqq \ldots \leqq B_j * \bar{B}$$

(II) If k is in K, then

$$A_i * \bar{A} \leqq \ldots \leqq A_j * \bar{A} \leqq \underbrace{A = \ldots = A}_{k/2\text{-times}}$$

$$B_i * \bar{B} \leqq \ldots \leqq B_j * \bar{B} \leqq \underbrace{B = \ldots = B}_{k/2\text{-times}}$$

(III) If 1 and k are in K, then

$$A = \ldots = A \leqq A_i * \bar{A} \leqq \ldots \leqq A_j * \bar{A} \leqq A = \ldots = A$$

$$B = \ldots = B \leqq B_i * \bar{B} \leqq \ldots \leqq B_j * \bar{B} \leqq B = \ldots = B,$$

where all the indices are not members of K.

If (I) occurs, then $\sum_{i \notin K} \Phi_{p,i,\mathscr{B}} \geqq k\, \Phi_p[B]^{1/2}$. Therefore by (*) $\Phi_{p,i,\mathscr{B}} = \Phi_p[B]$ for all $i \notin K$. This implies that $B_i * \bar{B} = B$ for all $i \notin K$. Since $\bar{A} \leqq \bar{B}$, then by Theorem 2 and (I) we obtain

$$\sum_{i \notin K} A_i * \bar{A} \leqq \sum_{i \notin K} B_i * \bar{B} = \frac{k}{2} B.$$

Now (I) implies that $\frac{1}{2}kA \leqq \sum_{i \notin K} A_i * \bar{A}$. The last two inequalities imply that

$$\sum_{i \notin K} A_i * \bar{A} = (\tfrac{1}{2}k)\, A$$

and so using (I), we deduce that

$$A_i * \bar{A} = A \quad \text{for all} \quad i \notin K.$$

So, we have proved that

$$A_i * \bar{A} = B_j * \bar{B} \quad \text{for all} \quad 1 \leqq i, \quad j \leqq k,$$

and hence

$$A_i * I = B_j * I \quad \text{for all} \quad 1 \leqq i, \quad j \leqq k,$$

since $\bar{A}$ and $\bar{B}$ are positive definite.

The proof for the case (II) is similar to the previous one and therefore is ommitted. The proof for the case (III) is trivial. Hence we have proved that

$$A_i * \bar{A} = B_j * \bar{B} \quad \text{and} \quad A_i * I = B_j * I \quad \text{for all} \quad 1 \leqq i, \quad j \leqq k.$$

Assume now that all the entries of $\bar{A}$ and $\bar{B}$ are nonzero. Then,

$$A_i * \bar{A} = A_j * \bar{A} \quad \text{implies} \quad (A_i - A_j) * \bar{A} = 0,$$

and hence $A_i = A_j$ for all $1 \leqq i, j \leqq k$. Similarly, $B_i = B_j$ for all i, j. Using the previous equalities and the fact that $\bar{A} \leqq \bar{B}$, we obtain

$$A_i \leqq B_j \quad \text{for all} \quad 1 \leqq i, \quad j \leqq k.$$

But $A_i * \bar{A} = B_j * \bar{B}$ for all i, j and hence $\bar{A} = \bar{B}$ and $A_i = B_j$ for all i, j. This completes the proof.

3. INEQUALITIES FOR THE CASE $p < 2$

In this section we assume that $-\infty \leqq p < 2$. The conditions of equality hold for $-\infty < p < 2$.

Proposition 2. *For any sequence $\mathscr{A} = (A_1, \ldots, A_k)$ of positive definite symmetric matrices and for any sequence $\alpha_1, \ldots, \alpha_k$ of positive numbers such that $\sum_{i=1}^{k} \alpha_i = 1$, the inequality*

$$\prod_{i=1}^{k} \Phi_{p,i}^{\alpha_i} \geqq \frac{\min\limits_{1 \leqq i \leqq k} \Phi_{p,i} k^{-1/2}}{\Phi_p\left[\sum_{i=1}^{k} \alpha_i^2 A_i * \bar{A}\right]^{1/2}} \prod_{i=1}^{k} \left(\Phi_{-\infty}[A_i]\, \Phi_p[A_i * I]\right)^{\alpha_i/2}$$

holds, where $\bar{A} = (1/k)(A_1 + \ldots + A_k)$. If $A_{\sigma(1)} \leqq \ldots \leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, k\}$, then equality holds if and only if $A_i = \lambda I$ and $\alpha_i = (1/k)$ $(i = 1, \ldots, k)$.

Proof. Using the properties of the (absolute) p-mean we obtain the following inequalities

$$(2) \qquad \frac{\Phi_p\left[\sum_{i=1}^{k} \alpha_i^2 A_i * \bar{A}\right]}{\min\limits_{1 \leqq i \leqq k} \Phi_{p,i}^2} \geqq \Phi_p\left[\sum_{i=1}^{k} \alpha_i^2 \frac{A_i * \bar{A}}{\Phi_{p,i}^2}\right] \quad \text{(by property 3)}$$

$$= \frac{1}{k} \Phi_p\left[\sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_i^2 \frac{A_i * A_j}{\Phi_{p,i}^2}\right] \quad \text{(by property 1)}$$

$$= \frac{1}{k} \Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right)^{*2} + \sum_{1 \leqq i < j \leqq k} A_i * A_j \left(\frac{\alpha_i}{\Phi_{p,i}} - \frac{\alpha_j}{\Phi_{p,j}}\right)^2\right]$$

$$(3) \qquad \geqq \frac{1}{k} \Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right)^{*2}\right] \quad \text{(by property 4)}.$$

$$(4) \qquad \geqq \frac{1}{k} \Phi_{-\infty}\left[\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right] \Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right) * I\right]$$

(by property 3, Lemma 2 and Property 1)

$$= \frac{1}{k} \Phi_{-\infty}\left[\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right] \Phi_p\left[\sum_{i=1}^{k} \alpha_i \frac{A_i * I}{\Phi_{p,i}}\right]$$

$$(5) \qquad \geqq \frac{1}{k}\left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_{-\infty}[A_i]}{\Phi_{p,i}}\right)\left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_p[A_i * I]}{\Phi_{p,i}}\right) \quad \text{(by property 4)}$$

$$(6) \qquad \geqq \frac{1}{k} \prod_{i=1}^{k} \left(\frac{\Phi_{-\infty}[A_i]\, \Phi_p[A_i * I]}{\Phi_{p,i}^2}\right)^{\alpha_i}$$

(by the arithmetic-geometric mean inequality).

Hence the result.

For $p = -\infty$ or 0, a limit argument yields the result.

Equality holds in (2) if and only if

$$\Phi_{p,i} = \Phi_{p,j} \quad \text{for all} \quad 1 \leqq i, \quad j \leqq k,$$

by property 3. By property 4 and the fact that $\Phi_p[A_i * A_j] > 0$ for all $1 \leqq i, j \leqq k$ (Theorem 1), equality holds in (3) if and only if $a_i = a_j$ for all $1 \leqq i, j \leqq k$. So, by the previous remarks and property 3 equality holds in (4) if and only if

$$(7) \qquad (\sum_{i=1}^{k} A_i) * (\sum_{i=1}^{k} A_i) = \lambda_n(\sum_{i=1}^{k} A_i)(\sum_{i=1}^{k} A_i) * I.$$

Now, by Lemma 2 equality holds in (5) if and only if $A_i * I = \lambda_i A_k * I$ where λ_i ($i = 1, \ldots, k - 1$) are positive numbers. Also, equality holds in (6) if and only if $\Phi_{-\infty}[A_i] = \Phi_{-\infty}[A_j]$ and $\Phi_p[A_i * I] = \Phi_p[A_j * I]$ for all $1 \leqq i, j \leqq k$. So we get $\lambda_i = 1$ ($i = 1, \ldots, k - 1$) since $\Phi_p[A_i * I] \neq 0$ for all $1 \leqq i \leqq k$. This means that $\operatorname{tr} A_i = \operatorname{tr} A_j$, and so using property 3 we get $A_1 = \ldots = A_k = A$.

Now, (7) reduces to

$$(8) \qquad A * A = \lambda A * I$$

for some positive number λ. By (8), A is diagonal. Let $A = (a_{ii})$. Then by (8), $a_{ii}^2 = \lambda a_{ii}$ so that $a_{ii} = \lambda$, since $a_{ii} \neq 0$ for all $i = 1, \ldots, n$. This completes the proof.

Proposition 3. *Let $\mathscr{A} = (A_1, \ldots, A_k)$ be a sequence of positive definite symmetric matrices such that $A_{\sigma(1)} \leqq \ldots \leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, k\}$. Let $\alpha_1, \ldots, \alpha_k$ be positive numbers such that $\sum_{i=1}^{k} \alpha_i = 1$; then the inequality*

$$(\sum_{i=1}^{k} \alpha_i \Phi_{p,i})^2 \geqq \frac{\min_{1 \leqq i \leqq k} \Phi_{p,i} C_3(m_2, M_2)^{-1}}{C_3(m_1, M_1) \,.\, k \,.\, \Phi_p[\sum_{i=1}^{k} \alpha_i^2 A_i * \bar{A}]} (\sum_{i=1}^{k} \alpha_i \Phi_{-\infty}[A_i]) (\sum_{i=1}^{k} \alpha_i \Phi_p[A_i * I])$$

holds, where $\bar{A} = (1/k)(A_1 + \ldots + A_k)$, $m_1 = \Phi_p[A_{\sigma(1)} * I]$, $M_1 = \Phi_p[A_{\sigma(k)} * I]$, $m_2 = \Phi_{-\infty}[A_{\sigma(1)}]$ and $M_2 = \Phi_{-\infty}[A_{\sigma(k)}]$. Equality holds if and only if k is even,

$$A_1 = \ldots = A_k = \lambda I \quad \text{and} \quad \alpha_1 = \ldots = a_k = 1/k.$$

Proof. Using the arithmetic-harmonic mean inequality in the right-hand side of (5) we obtain

$$(9) \qquad \left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_{p,i}}{\Phi_p[A_i * I]}\right)\left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_{p,i}}{\Phi_{-\infty}[A_i]}\right) \geqq \frac{\min_{1 \leqq i \leqq k} \Phi_{p,i}}{k \,.\, \Phi_p[\sum_{i=1}^{k} \alpha_i^2 A_i * \bar{A}]}.$$

If $A_{\sigma(i)} \leqq A_{\sigma(j)}$ then $A_{\sigma(i)} * I \leqq A_{\sigma(j)} * I$ and $A_{\sigma(i)} * \bar{A} \leqq A_{\sigma(j)} * \bar{A}$ by Theorem 2. So, $\Phi_p[A_{\sigma(i)} * I] \leqq \Phi_p[A_{\sigma(j)} * I]$, $\Phi_{p,\sigma(i)} \leqq \Phi_{p,\sigma(j)}$ and $\Phi_{-\infty}[A_{\sigma(i)}] \leqq \Phi_{-\infty}[A_{\sigma(j)}]$.

This means that the sequences

$$(\Phi_{p,i}),\ \left(\frac{1}{\Phi_p[A_i * I]}\right) \quad \text{and} \quad (\Phi_{p,i}),\ \left(\frac{1}{\Phi_{-\infty}[A_i]}\right) \quad (i = 1, \ldots, k)$$

are oppositely ordered and so Tchebychef's inequality [4, p. 43] applies,

$$\left(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}\right)\left(\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_p[A_i * I]}\right) \geqq \sum_{i=1}^{k} \alpha_i \frac{\Phi_{p,i}}{\Phi_p[A_i * I]}$$

(10) and

$$\left(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}\right)\left(\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_{-\infty}[A_i]}\right) \geqq \sum_{i=1}^{k} \alpha_i \frac{\Phi_{p,i}}{\Phi_{-\infty}[A_i]}.$$

By Lemma 5 we have

$$\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_p[A_i * I]} \leqq \frac{C_3(m_1, M_1)}{\sum_{i=1}^{k} \alpha_i \Phi_p[A_i * I]}$$

(11) and

$$\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_{-\infty}[A_i]} \leqq \frac{C_3(m_2, M_2)}{\sum_{i=1}^{k} \alpha_i \Phi_{-\infty}[A_i]}.$$

Combining inequalities (9), (10) and (11) we obtain the result .

As in the proof of Proposition 2 equality holds in (9) if and only if $A_1 = \ldots = A_k = \lambda I$ and $\alpha_1 = \ldots = \alpha_k = 1/k$. Also, by Lemma 5 equality holds in (11) if and only if there is a subset I of $\{1, \ldots, k\}$ such that $\sum_{i \in I} \alpha_i = \frac{1}{2}$. The last condition on α_i's forces k to be even.

4. INEQUALITIES FOR THE CASE $p > 2$

In this section we assume that $2 \leqq p \leqq \infty$. The conditions of equality hold for $2 < p < \infty$.

Proposition 4. *Let $\mathscr{A} = (A_1, \ldots, A_k)$ be a sequence of positive definite symmetric matrices such that $A_{\sigma(1)} \leqq \ldots \leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, k\}$. Let $\alpha_1, \ldots, \alpha_k$ be positive numbers such that $\sum_{i=1}^{k} \alpha_i = 1$, the the following inequality holds*

$$\left(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}\right) \prod_{i=1}^{k} \Phi_{p,i}^{\alpha_i} \geqq \frac{\left(\sum_{i=1}^{k} \alpha_i^2\right)^{-1}}{k} \frac{\prod_{i=1}^{k} \Phi_{-\infty}[A_i]^{\alpha_i}}{\sum_{i=1}^{k} \alpha_i \Phi_p[(A_i * I)^{-1}]}.$$

Equality holds if and only if $A_i = \ldots = A_k = \lambda I$ and $\alpha_1 = \ldots = \alpha_k = 1/k$.

Proof. Using the properties of the (absolute) p-mean, we obtain the following inequalities

$$\sum_{i=1}^{k} \alpha_i^2 \geqq \Phi_p\left[\sum_{i=1}^{k} \alpha_i^2 \frac{A_i * \bar{A}}{\Phi_{p,i}^2}\right] \quad \text{(by property 4)}$$

$$= \frac{1}{k}\Phi_p\left[\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i^2 \frac{A_i * A_j}{\Phi_{p,i}^2}\right] \quad \text{(by property 1)}$$

$$= \frac{1}{k}\Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right)^{*2} + \sum_{1 \leqq i < j \leqq k} A_i * A_j\left(\frac{\alpha_i}{\Phi_{p,i}} - \frac{\alpha_j}{\Phi_{p,j}}\right)^2\right]$$

$$\geqq \frac{1}{k}\Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right)^{*2}\right] \quad \text{(by property 3)}$$

$$\geqq \frac{1}{k}\Phi_{-\infty}\left[\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right]\Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{p,i}}\right) * I\right] \quad \text{(by property 3, Lemma 2 and property 1)}$$

$$\geqq \frac{1}{k}\left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_{-\infty}[A_i]}{\Phi_{p,i}}\right)\Phi_p\left[\sum_{i=1}^{k} \alpha_i \frac{A_i * I}{\Phi_{p,i}}\right] \quad \text{(by property 4 and property 1)}$$

(12) $$\geqq \frac{1}{k}\left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_{-\infty}[A_i]}{\Phi_{p,i}}\right)\Phi_p\left[\left(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}(A_i * I)^{-1}\right)^{-1}\right]$$

(by Lemma 3 and property 3)

$$\geqq \frac{1}{k}\prod_{i=1}^{k}\left(\frac{\Phi_{-\infty}[A_i]}{\Phi_{p,i}}\right)^{\alpha_i}\Phi_p\left[\sum_{i=1}^{k} \alpha_i \Phi_{p,i}(A_i * I)^{-1}\right]^{-1}$$

(by the arithmetic-geometric mean inequality and Lemma 1).

So we obtain

(13) $$\sum_{i=1}^{k} \alpha_i \Phi_{p,i}\Phi_p[(A_i * I)^{-1}] \geqq \Phi_p\left[\sum_{i=1}^{k} \alpha_i \Phi_{p,i}(A_i * I)^{-1}\right] \geqq$$

$$\geqq \frac{\left(\sum_{i=1}^{k} \alpha_i^2\right)^{-1}}{k}\prod_{i=1}^{k}\left(\frac{\Phi_{-\infty}[A_i]}{\Phi_{p,i}}\right)^{\alpha_i},$$

where the left-hand inequality is obtained by property 4.

If $A_{\sigma(i)} \leqq A_{\sigma(j)}$ then by Theorem 2, $A_{\sigma(i)} * I \leqq A_{\sigma(j)} * I$ and $A_{\sigma(i)} * \bar{A} \leqq A_{\sigma(j)} * \bar{A}$. So, $(A_{\sigma(i)} * I)^{-1} \geqq (A_{\sigma(j)} * I)^{-1}$ and by property 3 we get $\Phi_p[(A_{\sigma(i)} * I)^{-1}] \geqq$ $\geqq \Phi_p[(A_{\sigma(j)} * I)^{-1}]$ and $\Phi_{p,\sigma(i)} \leqq \Phi_{p,\sigma(j)}$. This mans that the sequences

$$(\Phi_p[(A_i * I)^{-1}]), \quad (\Phi_{p,i}) \quad (i = 1, \ldots, k)$$

are oppositely ordered and so Tchebychef's inequality [4, p. 43] applies,

(14) $$\left(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}\right)\left(\sum_{i=1}^{k} \alpha_i \Phi_p[A_i * I)^{-1}]\right) \geqq \sum_{i=1}^{k} \alpha_i \Phi_{p,i}\Phi_p[(A_i * I)^{-1}].$$

If we combine (13) and (14), we obtain the result. For $p = \infty$, a limit argument yields the result.

A similar argument used in the proof of Proposition 1 yields the case of equality.

Corollary 1. *Under the assumptions of Proposition* 4 *the following inequality holds*

$$(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}) \geqq \frac{(\sum_{i=1}^{k} \alpha_i^2)^{-1}}{C_3(m_2, M_2)\, k} (\sum_{i=1}^{k} \alpha_i \Phi_{-\infty}[A_i]) (\sum_{i=1}^{k} \alpha_i \Phi_p[(A_i * I)^{-1}])^{-1},$$

where $m_2 = \Phi_{-\infty}[A_{\sigma(1)}]$ *and* $M_2 = \Phi_{-\infty}[A_{\sigma(k)}]$. *Equality holds if and only if* k *is even,* $A_1 = \ldots = A_k = \lambda I$ *and* $\alpha_1 = \ldots = \alpha_k = 1/k$.

Proof. If we combine (12) and Lemma 3 we obtain

$$\sum_{i=1}^{k} \alpha_i \Phi_{p,i} \Phi_p[(A_i * I)^{-1}] \geqq \frac{(\sum_{i=1}^{k} \alpha_i^2)^{-1}}{k} \sum_{i=1}^{k} \alpha_i \frac{\Phi_{-\infty}[A_i]}{\Phi_{p,i}}$$

and if we use the arithmetic-harmonic mean inequality in the right-hand inequality above we obtain

$$(15) \qquad (\sum_{i=1}^{k} \alpha_i \Phi_{p,i} \Phi_p[(A_i * I)^{-1}]) \left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_{p,i}}{\Phi_{-\infty}[A_i]} \right) \geqq \frac{(\sum_{i=1}^{k} \alpha_i^2)^{-1}}{k}.$$

As in the proof of Proposition 4 the sequences

$$(\Phi_{p,i}), \left(\frac{1}{\Phi_{-\infty}[A_i]} \right) \quad (i = 1, \ldots, k)$$

are oppositely ordered and so Tchebychef's inequality [4, p. 43] applies,

$$(16) \qquad (\sum_{i=1}^{k} \alpha_i \Phi_{p,i}) \left(\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_{-\infty}[A_i]} \right) \geq \sum_{i=1}^{k} \alpha_i \frac{\Phi_{p,i}}{\Phi_{-\infty}[A_i]}.$$

If we combine (11), (14), and (15) we obtain the result. For $p = \infty$, a limit argument yields the result.

The equality conditions follow from Proposition 4 and the facts that the conditions of equality of Lemma 5 and $\alpha_1 = \ldots = \alpha_k = 1/k$ imply that k is even.

Corollary 2. *Under the assumptions of Proposition* 4 *the following inequality holds*

$$(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}) \geqq \frac{(\sum_{i=1}^{k} \alpha_i^2)^{-1}}{C_3(m_2, M_2)} \frac{k^{-1}}{C_3(m_3, M_3)} (\sum_{i=1}^{k} \alpha_i \Phi_{-\infty}[A_i]) (\sum_{i=1}^{k} \alpha_i \Phi_p[A_i * I]),$$

where $m_3 = \Phi_p[(A_{\sigma(k)} * I)^{-1}]$ *and* $M_3 = \Phi_p[(A_{\sigma(1)} * I)^{-1}]$. *Equality holds if and only if* k *is even,* $A_1 = \ldots = A_k = \lambda I$ *and* $\alpha_1 = \ldots = \alpha_k = 1/k$.

Proof. By Kantorovich's inequality we have

$$(17)\qquad \sum_{i=1}^{k} \alpha_i \Phi_p[(A_i * I)^{-1}] \leqq \frac{C_3(m_3, M_3)}{\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_p[(A_i * I)^{-1}]}} .$$

If we combine (11), (15) and (17) we obtain

$$\Big(\sum_{i=1}^{k} \alpha_i \Phi_{p,i}\Big)^2 \geqq \frac{\big(\sum_{i=1}^{k} \alpha_i^2\big)^{-1}}{C_3(m_2, M_2)} \frac{k^{-1}}{C_3(m_3, M_3)} \Big(\sum_{i=1}^{k} \alpha_i \Phi_{-\infty}[A_i]\Big) \left(\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_p[(A_i * I)^{-1}]}\right)^{-1} .$$

By Lemma 1, $\Phi_p[A_i * I]^{-1} \leqq \Phi_p[(A_i * I)^{-1}]$ for $i = 1, \ldots, k$, and so

$$\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_p[(A_i * I)^{-1}]} \leqq \sum_{i=1}^{k} \alpha_i \Phi_p[A_i * I] .$$

Finally, If we combine the last two inequalities we obtain the result.

For $p = \infty$, a limit argument yields the result.

The conditions of equality follow from Proposition 4 and Corollary 1.

5. APPLICATIONS

For a positive definite symmetric matrix A from properties 6 and 7 we have

$$\Phi_{-\infty}[A] = \lambda_n(A) , \quad \Phi_{-n}[A] = (\det A)^{1/n} , \quad \Phi_{\infty}[A] = \lambda_1(A) ,$$

where $\lambda_n(A)$ denotes the smallest eigenvalue of A, $\det A$ denotes the determinant of A and $\lambda_1(A)$ denotes the largest eigenvalue of A. So, the inequalities in sections 3 and 4 hold for these functions. In particular the results in sections 3 and 4 can be extended in the case where the matrices $A_1, \ldots, A_k$ are positive definite hermitian. This is true because Lemma 2 and property 3 hold for positive definite hermitian matrices when $p = -\infty$, $-n$, or ∞ (see e.g. [2]). Hence, we have the following

"The inequalities in sections 3 and 4 hold for a finite sequence $\mathscr{A} = (A_1, \ldots, A_k)$ of positive definite hermitian matrices if $p = -\infty$, $-n$ and ∞."

Let A be a positive definite hermitian matrix. Then, using the well-known inequality (see e.g. [8])

$$(18)\qquad \Phi_p[A^*A] \geqq \Phi_p[A]^2 ,$$

for $p = -\infty$ or $-n$, we can reformulate the inequalities in section 3. This can be done by using (18) in (3) in the proof of Proposition 2. For the case $p = -n$ see [9].

We are dealing now with the case $p = 2$.

Proposition 5. *Let $\mathscr{A} = (A_1, \ldots, A_k)$ be a sequence of positive definite symmetric matrices and let $\alpha_1, \ldots, \alpha_k$ be positive numbers such that $\sum_{i=1}^{k} \alpha_i = 1$. Then the fol-*

lowing inequality holds

$$\prod_{i=1}^{k} \Phi_{2,i}^{\alpha_i} \geqq (k \sum_{i=1}^{k} \alpha_i^2)^{-1/2} \prod_{i=1}^{k} \Phi_2[A_i]^{\alpha_i} .$$

If $A_{\sigma(1)} \leqq \ldots \leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, k\}$, then equality holds if and only if $A_1 = \ldots = A_k$, $A_1 * I = \ldots = A_k * I = \lambda I$ and $\alpha_1 = \ldots = \alpha_k = 1/k$.

Proof. Using the linearity of $\Phi_2[\cdot]$, we obtain

$$\sum_{i=1}^{k} \alpha_i^2 = \Phi_2 \left[\sum_{i=1}^{k} \alpha_i^2 \frac{A_i * \bar{A}}{\Phi_{2,i}^2} \right] = \frac{1}{k} \Phi_2 \left[\sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_i^2 \frac{A_i * A_j}{\Phi_{2,i}^2} \right] =$$

$$= \frac{1}{k} \Phi_2 \left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{2,i}} \right)^{*2} + \sum_{1 \leqq i < j \leqq k} A_i * A_j \left(\frac{\alpha_i}{\Phi_{2,i}} - \frac{\alpha_j}{\Phi_{2,j}} \right)^2 \right] \geqq$$

$$\geqq \frac{1}{k} \Phi_2 \left[\left(\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{2,i}} \right)^{*2} \right] \geqq \frac{1}{k} \Phi_2 \left[\sum_{i=1}^{k} \alpha_i \frac{A_i}{\Phi_{2,i}} \right]^2 \quad \text{(by Lemma 6)}$$

$$(19) \qquad = \frac{1}{k} \left(\sum_{i=1}^{k} \alpha_i \frac{\Phi_2[A_i]}{\Phi_{2,i}} \right)^2 \geqq \frac{1}{k} \prod_{i=1}^{k} \left(\frac{\Phi_2[A_i]}{\Phi_{2,i}} \right)^{2\alpha_i}$$

(by the arithmetic-geometric mean inequality)

Hence the result.

By the proof above, equality holds if and only if $a_i/\Phi_{2,i} = a_j/\Phi_{2,j}$ for all $1 \leqq i, j \leqq k$ and so by Lemma 6 equality holds if and only if

$$(23) \qquad \sum_{i=1}^{k} A_i * I = \lambda I .$$

Now equality holds in (19) if and only if

$$\Phi_2[A_1] = \ldots = \Phi_2[A_k] .$$

Using property 3, we obtain

$$A_1 = \ldots = A_k .$$

So by (23) $A_i * I = \mu I$ $(i = 1, \ldots, k)$ where $\mu = \lambda/k$. Therefore, we obtain that $\Phi_{2,i} = \Phi_{2,j}$ for all i, j, and so $\alpha_1 = \ldots = \alpha_\mu = 1/k$. This completes the proof.

Proposition 6. *Let $\mathscr{A} = (A_1, \ldots, A_k)$ be a sequence of positive definite symmetric matrices such that $A_{\sigma(1)} \leqq \ldots \leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, k\}$. Let $\alpha_1, \ldots, \alpha_k$ be positive numbers such that $\sum_{i=1}^{k} \alpha_i = 1$. Then, the inequality*

$$\sum_{i=1}^{k} \alpha_i \Phi_{2,i} \geqq \frac{(k \sum_{i=1}^{k} \alpha_i^2)^{-1/2}}{C_3(m_4, M_4)} \sum_{i=1}^{k} \alpha_i \Phi_2[A_i]$$

holds, where $\bar{A} = (1/k)(A_1 + \ldots + A_k)$, $m_4 = \Phi_2[A_{\sigma(1)}]$ and $M_4 = \Phi_2[A_{\sigma(1)}]$.

Equality holds if and only if k is even, $A_1 = \dots = A_k$, $A_1 * I = \dots = A_k * I = \mu I$ *and* $\alpha_1 = \dots = \alpha_k = 1/k$.

Proof. Using the arithmetic-harmonic mean inequality in the right-hand side of (19) we obtain

$$\sum_{i=1}^{k} \alpha_i \frac{\Phi_{2,i}}{\Phi_2[A_i]} \geqq \left(k \sum_{i=1}^{k} \alpha_i^2\right)^{-1/2} . \tag{20}$$

If $A_{\sigma(i)} \leqq A_{\sigma(j)}$ then by Theorem 2, $A_{\sigma(i)} * \bar{A} \leqq A_{\sigma(j)} * \bar{A}$. So, by property 3 $\Phi_2[A_{\sigma(i)}] \leqq \Phi_2[A_{\sigma(j)}]$ and $\Phi_{2,\sigma(i)} \leqq \Phi_{2,\sigma(j)}$. This means that the sequences

$$(\Phi_{2,i}), \left(\frac{1}{\Phi_2[A_i]}\right) \quad (i = 1, \dots, k)$$

are oppositely ordered and so Tchebychef's inequality [4, p. 43] applies,

$$\left(\sum_{i=1}^{k} \alpha_i \Phi_{2,i}\right)\left(\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_2[A_i]}\right) \geqq \sum_{i=1}^{k} \alpha_i \frac{\Phi_{2,i}}{\Phi_2[A_i]} . \tag{21}$$

Also by Kantorovich's inequality (Lemma 5) we have

$$\sum_{i=1}^{k} \frac{\alpha_i}{\Phi_2[A_i]} \leqq \frac{C_3(m_4, M_4)}{\sum_{i=1}^{k} \alpha_i \Phi_2[A_i]} . \tag{22}$$

If we combine (20), (21) and (22) we obtain the result.

The conditions of equality are obtained by Proposition 5 and Lemma 5.

References

[1] *T. Ando:* Concavity of Certain Maps on Positive Definite Matrices and Applications to Hadamard products, Linear Algebra Appl. *26* (1979) 203—241.

[2] *E. F. Beckenbach* and *R. Bellman:* Inequalities, Springer-Verlag 1965.

[3] *Achim Clausing:* Kantorovich-type inequalities, American Mathematical Monthly, *89* (1982) 314—330.

[4] *K. H. Hardy, J. E. Littlewood,* and *K. Pólya:* Inequalities, Cambridge University Press 1952.

[5] *Erwin Lutwak:* On Power Means of Positive Quadratic Forms, Linear Algebra Appl. *57* (1984) 13—19.

[6] *I. Schur:* Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. Reine Angew. Math. *140* (1911) 1—28.

[7] *Czesław Stepniak,* Ordering of Nonnegative Definite Matrices with Applications to Comparison of Linear Models, Linear Algebra and Appl. *70* (1985) 67—71.

[8] *K. P. H. Styan:* Hadamard products and multivariate statistical analysis, Linear Algebra and Appl. *6* (1973) 217—240.

[9] *Theodore Tollis:* An inequality involving determinants of positive definite hermitian matrices, Abstracts of AMS, January 1985.

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