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NATURAL TRANSFORMATIONS IN DIFFERENTIAL GEOMETRY

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The aim of this paper is to develop an efficient tool for handling natural transformations in differential geometry, like the foot point projection of the tangent bundle, the canonical flip mapping on the second tangent bundle, vertical lift and vertical projection, fibre addition and fibre multiplication. The natural setting for this is the study of product preserving (multiplicative) functors on the category $\mathcal{M}f$ of smooth finite dimensional manifolds. We prove that (under some mild conditions) any such functor F is given by the action of a Weil-algebra A (see 1.6) on manifolds, in particular that $F(M) = \operatorname{Hom}(C^\infty(M), A)$ for connected M (see theorem 3.5, 3.6). The main ingredient for this is the study of ideals of finite codimension in $C^\infty(M)$ and with this we begin our paper.

Let us also indicate that the theory of Weil algebras and multiplicative functors gives also a rigorous setting (and vast generalisation) for the theory of infinitesimals in the sense of Newton, Leibnitz, Euler etc. as follows: $\operatorname{Hom}(C^\infty(M), \mathbf{R}) = M$ for any manifold and consequently $\operatorname{Hom}(C^\infty(M), C^\infty(N)) = C^\infty(N, M)$. So the category of algebras $C^\infty(M)$ and their algebra homomorphisms is dual to the the category $\mathcal{M}f$ of manifolds and smooth mappings. But let now D be the algebra generated by 1 and ε with $\varepsilon^2 = 0$ (sometimes called the algebra of dual numbers or Study numbers). Then $\operatorname{Hom}(C^\infty(M), D) = TM$, the tangent bundle of M. If D were the algebra of smooth functions on a manifold, this manifold would consist of one point only but would have non trivial tangent vectors (infinitesimals). Moreover $\operatorname{Hom}(C^\infty(M), D \otimes D) = TTM$.

We may summarize: With the eyes of today's (standard) mathematics the classical infinitesimals can be seen in the (dual) mirror image of function algebras. This point of view is very similar to synthetic differential geometry, see Kock [3] and especially Moerdijk and Reyes [7].

When this paper was finished we received preprints of D. Eck [15] and O. O. Luciano [16] containing some of our results. We want to thank I. Kolář, J. Slovák, D. Eck and the referee for helpful comments and pointing out mistakes.

Contents. 1. Ideals of smooth functions of finite codimension. 2. Functors on manifolds defined by Weil algebras. 3. Multiplicative functors on manifolds. 4. Applications in differential geometry.

1.1. Theorem. *Let M be a smooth finite dimensional manifold. Let I be an ideal of finite codimension in $C^\infty(M)$. Then there are finitely many points $x_1, \ldots, x_k \in M$ and $n_i \in N_0$ such that I contains the ideal of all functions f which vanish at x_i up to order n_i for $i = 1, \ldots, k$.*

1.2. For the proof we need some preparation. If A is a closed subset of a smooth manifold M, let $C^\infty(A)$ be the algebra of all real valued functions on A which are restrictions of smooth functions on M. For $f \in C^\infty(A)$ put $Z_f = \{x \in A: f(x) = 0\}$ and for a set $A \subseteq C^\infty(A)$ put $Z_A = \bigcap\{Z_f: f \in A\}$.

Lemma. *Let I be an ideal of finite codimension in $C^\infty(A)$. Then Z_I is a finite subset of A, and $Z_I = \emptyset$ if and only if $I = C^\infty(A)$.*

Proof. Z_I is finite since $C^\infty(A)/I$ is finite dimensional. In order to show that $I \neq C^\infty(A)$ implies $Z_I \neq \emptyset$ we claim that then $\{Z_f, f \in I\}$ consists of closed nonempty subsets of A, is downwards directed and contains a compact set. If $Z_f = \emptyset$, then f is invertible and $I = C^\infty(A)$. Clearly $Z_f \cap Z_g = Z_{f^2+g^2}$.

Choose $\varphi \in C^\infty(M)$, $\varphi > 0$, with $\sup\{\varphi(x): x \in C\} = \infty$ for each closed non compact subset $C \subseteq M$.

Put $\psi = \varphi \mid A \in C^\infty(A)$. Then $\{\psi, \psi^2, \psi^3, \ldots\}$ is linearly dependent mod I, so $f = \sum_{i=1}^{n} \lambda_i \psi^i \in I$ for some $(\lambda_i) \neq 0$ in R^n. Then clearly Z_f is compact. qed.

1.3. Lemma. *If I is an ideal of finite codimension in $C^\infty(A)$ and if $f \in C^\infty(A)$ vanishes near Z_I then $f \in I$.*

Proof. Let $Z_I \subset U_1 \subset \bar{U}_1 \subset U_2$, where U_1, U_2 are open in A and $f \mid U_2 = 0$. The restriction $C^\infty(A) \to C^\infty(A \setminus U_1)$ is surjective, so the image of I is an ideal I' in $C^\infty(A \setminus U_1)$ of finite codimension, but clearly $Z_{I'} = \emptyset$, so $I' = C^\infty(A \setminus U_1)$ by 1.2. Thus there is some $g \in I$ such that $g \mid A \setminus U_1 = f \mid A \setminus U_1$.

Choose $\varphi \in C^\infty(A)$, $\varphi \equiv 0$ on U_1 and $\varphi \equiv 1$ on $A \setminus U_2$. Then $f = f \cdot \varphi = g \cdot \varphi \in I$.

1.4. Let $\mathscr{E}_n$ be the ring of germs at 0 of smooth real valued functions on R^n, and let $\mathscr{M} = \mathscr{M}_n$ be the maximal ideal of germs vanishing at 0.

Lemma. *Let I be an ideal of finite codimension in $\mathscr{E}_p$. Then I contains $\mathscr{M}_p^{k+1}$, the ideal of germs vanishing at 0 up to order k, for some $k \in N$.*

Proof. If $(\mathscr{M}_p^{k+1} \setminus \mathscr{M}_p^{k+2}) \cap I = \emptyset$ for all k then I cannot have finite codimension. Thus $\mathscr{M}_p^{k+1} \subset I + \mathscr{M}_p^{k+2}$ for some k and by Nakayama's lemma we have $\mathscr{M}_p^{k+1} \subset I$. qed.

1.5. Proof of theorem 1.1. Let I be a nontrivial ideal of finite codimension in $C^\infty(M)$. Then by 1.2. Z_I is a nonempty finite subset, $Z_I = \{x_1, \ldots, x_k\}$. Let I_j be the ideal generated by $I \cup \{f \in C^\infty(M)$: the germ of f at x_j is $0\}$. Then $I = \bigcap_{j=1}^{k} I_j$ by lemma 1.3. So we may treat each I_j separately, factor I_j to an ideal of finite codimension in the ring of germs at x_j and use lemma 1.4. qed.

1.6. Definition. A *Weil algebra* is a real finite dimensional commutative algebra with unit (r f c a) which is generated by its idempotent and nilpotent elements. It is called *irreducible* if the unit is its only idempotent element.

If A is any r f c a then the idempotents and nilpotents in A generated a subalgebra $W(A)$ of A which is a Weil algebra. $W(.)$ is clearly a functor, right adjoint to the inclusion of Weil algebras.

Lemma. *Let A be a Weil algebra. Then there is a maximal decomposition $1 = e_1 + \ldots + e_k$ where $e_i e_j = \delta_{ij} e_i$, which is unique up to order. Then $A = A_1 \oplus \ldots \oplus A_k$, where $A_i = A \,.\, e_i$ is an ideal, $A_i A_j = 0$, $A_i = \mathbf{R} \,.\, e_i \oplus N_i$ where N_i consists of nilpotent elements only.*

The proof is routine. The irreducible Weil algebras are those considered by A. Weil [12] and in synthetic differential geometry [3], [5], [6].

1.7. Corollary. *If Λ is a closed subset in a smooth manifold M and I is an ideal of finite codimension in $C^\infty(\Lambda)$, then $C^\infty(\Lambda)/I$ is a Weil algebra.*

If A is a r f c a *the* Hom $(C^\infty(\Lambda), A) =$ Hom $(C^\infty(\Lambda), W(A))$, *where* Hom (B, A) *is the set of all algebra homomorphisms $B \to A$.*

Proof. Restriction $r: C^\infty(M) \to C^\infty(\Lambda)$ is surjective, and $r^{-1}(I)$ is an ideal of finite codimension in $C^\infty(M)$, so by 1.1. $C^\infty(\Lambda)/I \cong C^\infty(M)/r^{-1}(I)$ is a Weil algebra. If $\varphi: C^\infty(\Lambda) \to A$ is an algebra homomorphism, then $\ker \varphi$ is an ideal of finite codimension, so φ factors over a Weil algebra and thus takes its image in $W(A)$. qed.

1.8. Corollary. ("Milnor's exercise"): *The evaluation mapping* $\mathrm{ev}: M \to \mathrm{Hom}\,(C^\infty(M), \mathbf{R})$, $\mathrm{ev}(x)(f) = f(x)$, *is bijective.*

1.9. Corollary. *The mapping* $C^\infty(M_1, M_2) \to \mathrm{Hom}\,(C^\infty(M_2), C^\infty(M_1))$, $f \to (g \to g \circ f)$, *is bijective for all manifolds M_1, M_2.*

2. FUNCTORS ON MANIFOLDS DEFINED BY WEIL ALGEBRAS

2.1. Let $\mathcal{M}f$ denote the category of smooth (finite dimensional second countable) manifolds and smooth mappings. A functor $F: \mathcal{M}f \to \mathcal{M}f$ is called *multiplicative* if F preserves products:

$$F(M_1) \xleftarrow{F(\mathrm{pr}_1)} F(M_1 \times M_2) \xrightarrow{F(\mathrm{pr}_2)} F(M_2)$$

is always a product. Then $F(\text{point}) = \text{point}$.

Note that smooth manifolds are exactly the smooth retracts of open subsets U

of $\mathbf{R}^n$'s, i.e. the sets $p(U)$ where $p \in C^\infty(U, U)$ with $p \circ p = p$ (see Federer [1], p. 232). So any functor defined on the category of open subsets of $\mathbb{R}^n$'s and smooth mappings extends uniquely to a functor $\mathcal{M}f \to \mathcal{M}f$. We owe this remark to Bill Lawvere.

In this section we will construct a functor $A = \mathrm{Fu}_A\colon \mathcal{M}f \to \mathcal{M}f$ for each Weil algebra A.

The first construction goes back to A. Weil [13] and is a main ingredient in some models of synthetic differential geometry.

2.2. Let $A = A_1 \oplus \ldots \oplus A_k$ be a Weil algebra with its decomposition into irreducible components (1.6).

We construct a functor $\mathrm{Fu}_A = A\colon \mathcal{M}f \to \mathcal{M}f$.

1.: If $p(t)$ is a real polynomial and $x = x_1 + \ldots + x_k \in A$, $x_i \in A_i$, then $p(x) = p(x_1) + \ldots + p(x_k)$ since $x_i x_j = 0$.

2.: Let $f \in C^\infty(\mathbf{R})$. In view of 1 we define $A_i f\colon A_i \to A_i$ and then put $A f(x) = A_1 f(x_1) + \ldots + A_k f(x_k)$.

To define $A_i f(\lambda e_i + n)$ let $j^\infty f(\lambda)(t) = \sum_{j=0}^{\infty} (f^{(j)}(\lambda)/j!)\, t^j$ be the Taylor expansion of f at $\lambda \in \mathbf{R}$ and put

$$A_i f(\lambda e_i + n) = f(\lambda)\, e_i + \sum_{j=1}^{\infty} \frac{f^{(j)}(\lambda)}{j!}\, n^j\,,$$

which is a finite sum (see 1.6). Clearly then $A(\mathrm{Id}_{\mathbf{R}}) = \mathrm{Id}_A$ and $Af \circ Ag = A(f \circ g)$.

3. Let $f \in C^\infty(\mathbf{R}^n, \mathbf{R})$. We want to define the value of $A_i f$ at the vector $(\lambda_1 e_i + v_1, \ldots, \lambda_n e_i + v_n) \in A_i^n$. So let

$$j^\infty f(\lambda)(t) = \sum_{\alpha \in \mathbf{N}^n} \frac{1}{\alpha!}\, \delta^\alpha f(\lambda)\, t^\alpha$$

be the Taylor series at $\lambda \in \mathbf{R}^n$ for $t \in \mathbf{R}^n$. Put

$$A_i f(\lambda_1 e_i + v_1, \ldots, \lambda_n e_i + v_n) = f(\lambda) \,.\, e_i + \sum_{|\alpha| \geqq 1} \frac{1}{\alpha!}\, \partial^\alpha f(\lambda)\, v_1^{\alpha_1} \ldots v_n^{\alpha_n}\,,$$

a finite sum. Again for $x = x_1 + \ldots + x_k \in A^n$, $x_i \in A_i^n$, we put $A f(x) = A_1 f(x_1) + \ldots + A_k f(x_k)$.

4. For $f \in C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ we apply the construction of step 3 for each component $f_j\colon \mathbf{R}^n \to \mathbf{R}$ and get $Af\colon A^n \to A^m$. Clearly $A(\mathrm{Id}_{\mathbf{R}^n}) = \mathrm{Id}_{A^n}$ and $A(f \circ g) = Af \circ Ag$. If $\varphi\colon A \to B$ is an algebra homomorphism between Weil algebras, then $\varphi^m \circ Af = Af \circ \varphi^n$.

5. Let $p = p_A\colon A = \overset{k}{\oplus} A_i \to \mathbf{R}^k$ the projection onto the subalgebra generated by all idempotent elements of A. This is a surjective algebra homomorphism, so for $f \in C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ the following diagram commutes by step 4:

$$\begin{tikzcd}
A^n \arrow[r, "Af"] \arrow[d, "p_A^n"'] & A^m \arrow[d, "p_A^m"] \\
\mathbf{R}^{k\cdot n} \arrow[r, "f^k"] & \mathbf{R}^{k\cdot m}
\end{tikzcd}$$

If U is open in $\mathbf{R}^n$ we put $AU := (p_A^n)^{-1}(U^k)$, which is an open subset in A^n of the form $U^k \times N^n$, where N is the nilradical of A. If $f\colon U \to V$ is smooth between open subsets in $\mathbf{R}^n$ and $\mathbf{R}^m$, resp., then we get a smooth commutative diagram

$$\begin{tikzcd}
U^k \times N^n \cong \arrow[dr, "\mathrm{pr}_1"'] & AU \arrow[r, "Af"] \arrow[d, "p_A^n"] & AV \arrow[d, "p_A^n"'] & \cong V^k \times N^m \arrow[dl, "\mathrm{pr}_1"] \\
& U^k \arrow[r, "f^k"'] & V^k &
\end{tikzcd}$$

and clearly $A(f \circ g) = Af \circ Ag$ and $A(\mathrm{Id}_U) = \mathrm{Id}_{AU}$.

6. So A is now a functor on the category of open subsets of $\mathbf{R}^m$'s and smooth mappings and by the remark in 2.1. A extends uniquely to a functor $Fu_A = $ A: $\mathcal{M}f \to \mathcal{M}f$.

7. More concretely we use any atlas (U_α, u_α) of a manifold M with cocycle $(u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}\colon u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta))$ of transition functions, such that (U_α^k) is still an open cover of M^k. Then we glue the open sets $A(u_\alpha(U_\alpha)) \cong u_\alpha(U_\alpha)^k \times N^n$ ($n = \dim M$) via the cocycle of transition functions

$$\begin{tikzcd}
u_\beta(U_\alpha \cap U_\beta)^k \times N^n \cong \arrow[dr, "\mathrm{pr}_1"'] & A(u_\beta(U_\alpha \cap U_\beta)) \arrow[r, "Au_{\alpha\beta}"] \arrow[d, "p_A^k"] & A(u_\alpha(U_\alpha \cap U_\beta)) \arrow[d, "p_A^k"'] & \cong u_\alpha(U_\alpha \cap U_\beta)^k \times N^n \arrow[dl, "\mathrm{pr}_1"] \\
& u_\beta(U_\alpha \cap U_\beta)^k \arrow[r, "u_{\alpha\beta}^k"] & u_\alpha(U_\alpha \cap U_\beta)^k &
\end{tikzcd}$$

The result is the manifold $AM = \mathrm{Fu}_A(M)$. It is Hausdorff by 2.3.1 below. For $f \in C^\infty(M_1, M_2)$ we define $Af\colon AM_1 \to AM_2$ by glueing.

8. If $A \neq 0$ then AM does not depend on the choice of the atlas (U_α, u_α). If $A = 0$ we take an atlas as small as possible (so the U_α have many connected components) then $\mathrm{Fu}_0(M) = $ point. If we choose an atlas consisting of connected charts only we get the the functor π_0 computing the connected components of M, which we will distinguish from Fu_0.

2.3. Theorem. *Let A be a Weil algebra with k irreducible components, $A = \overset{k}{\oplus} A_i$, let N be the nilradical of A.*

1. *Then A defines uniquely a functor* $\mathrm{Fu}_A\colon Mf \to Mf$ *by the construction in* 2.2, *such that* $(\mathrm{Fu}_A M, \mathrm{Fu}(p_A), M^k, N^{\dim M})$ *is a smooth fibre bundle over* M^k *with typical fibre* $N^{\dim M}$. *For any* $f \in C^\infty(M_1, M_2)$ *we have a commutative diagram*

$$\begin{tikzcd} \mathrm{Fu}_A M_1 \arrow[r, "\mathrm{Fu}_A(f)"] \arrow[d, "\mathrm{Fu}(p_A)_{M_1}"] & \mathrm{Fu}_A M_2 \arrow[d, "\mathrm{Fu}(p_A)_{M_2}"] \\ M_1^k \arrow[r, "f^k"] & M_2^k \end{tikzcd}$$

This fibre bundle is a natural vector bundle (for all M) if and only if N is nilpotent of order 2.

2. *The functor* $\mathrm{Fu}(A)$ *is multiplicative.*

3. $\mathrm{Fu}_A = \mathrm{Fu}_{A_1} \times \ldots \times \mathrm{Fu}_{A_k}$.

4. $\mathrm{Fu}_A\colon C^\infty(M_1, M_2) \to C^\infty(\mathrm{Fu}_A M_1, \mathrm{Fu}_A M_2)$ *is continuous for the compact* C^∞*-topologies, and maps each of the following classes of mappings into itself: immersions, embeddings, closed embeddings, submersions, surjective submersions, fibre bundle projections.*

5. *If* (U_α^k) *is an open cover of* M^k, *then* $\mathrm{Fu}_A(U_\alpha)$ *is also an open cover of* $\mathrm{Fu}_A(M)$.

6. *Any algebra homomorphism* $\varphi\colon A \to B$ *between Weil algebras induces a natural transformation*

$$\mathrm{Fu}(\varphi)\colon \mathrm{Fu}_A \to \mathrm{Fu}_B \quad \textit{and} \quad \mathrm{Fu}(\varphi \circ \psi) = \mathrm{Fu}(\varphi) \circ \mathrm{Fu}(\psi)\,, \quad \mathrm{Fu}(\mathrm{Id}_A) = \mathrm{Id}_{\mathrm{Fu}(A)}\,.$$

If φ *is injective, then* $\mathrm{Fu}(\varphi)_M\colon \mathrm{Fu}_A(M) \to \mathrm{Fu}_B(M)$ *is a closed embedding for each* $M \in Mf$.

If φ *is surjective, then* $\mathrm{Fu}(\varphi)_M$ *is a fibre bundle projection for each M.*

Proof. 1. The main assertion is clear from 2.2.6 and 7.

$$\mathrm{Fu}_A M \xrightarrow{\mathrm{Fu}(p_A)} M^k$$

is a vector bundle for all M if and only if for each transition function $u_{\alpha\beta}$ of an atlas (U_α, u_α) of M the mapping $\mathrm{Fu}_A(u_{\alpha\beta})$ in the diagram of 2.2.7 is fibre linear, so only the first derivatives of f act on N, so N is nilpotent of order 2 (the multiplication on N is zero) by 2.2.3.

2. is obvious by 2.2.5 and 6.

3. is obvious by 2.2.2 and 3.

4. is obvious by looking at charts of M_1 and M_2.

5. follows from 1.

6. the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "p_A"] & B \arrow[d, "p_B"] \\ R^k \arrow[r, "\bar{\varphi}"] & R^1 \end{tikzcd}$$

commutes for some homomorphism $\bar{\varphi}$, thus $\bar{\varphi}(e_i) = \sum_{e' \in I_i} e'$, where (I_i) is a pairwise disjoint family of subsets of the set of the minimal idempotents of B. So $\bar{\varphi}$ is composed of diagonal mappings $\boldsymbol{R} \to \boldsymbol{R}^j$ and 0's.

The diagram

$$\begin{tikzcd}
\mathrm{Fu}_A(M) \arrow[r, "\mathrm{Fu}(\varphi)_M"] \arrow[d, "\mathrm{Fu}(p_A)_M"'] & \mathrm{Fu}_B(M) \arrow[d, "\mathrm{Fu}(p_B)_M"] \\
M^k \arrow[r, "\mathrm{Fu}(\bar{\varphi})_M"] & M^1
\end{tikzcd}$$

commutes, so $\mathrm{Fu}(\varphi)_M$ is a fibre respecting mapping, and in each fibre bundle chart coming from a chart of M as in 2.2.7 the mapping $\mathrm{Fu}(\varphi)_M$ is a linear mapping $N^n \to N^n$ for each fibre N^n.

So if φ is injective, $\mathrm{Fu}(\bar{\varphi})_M$ consists of diagonal mappings only, so it is a closed embedding, and $\mathrm{Fu}(\varphi)_M$ is fibrewise injective and linear in each canonical fibre bundle chart as in 2.2.7, so $\mathrm{Fu}(\varphi)_M$ is a closed embedding.

If φ is surjective, then for those minimal idempotents e_i in A with $\varphi(e_i) = 0$ the manifold $\mathrm{Fu}_{A_i}(M)$ is a trivial factor in the fibre of $\mathrm{Fu}(\varphi)_M$ by 3. If $\varphi(e_j) \neq 0$, then $\varphi(e_j)$ is a minimal idempotent of B, so $\varphi | A_j$: $A_j \to B_m$ for some M. Thus we may reduce to the case that A and B are irreducible. Then $\bar{\varphi} = \mathrm{Id}_{\boldsymbol{R}}$: $\boldsymbol{R} \to \boldsymbol{R}$, and in each canonical fibre bundle chart of

$$\mathrm{Fu}_A(M) \xrightarrow{\mathrm{Fu}(p_A)} M$$

as in 2.2.7., with $N_1 = \ker \varphi$, $V \subset N_A$ a vector subspace such that $N_1 \oplus V = N_A$, we have

$$\begin{tikzcd}
\mathrm{Fu}_A(M) \supset \arrow[d, "\mathrm{Fu}(\varphi)_M"'] & A(U_\alpha) \arrow[r, "A(u_\alpha)"] \arrow[d, "\mathrm{Fu}(\varphi)U_\alpha"] & u_\alpha(U_\alpha) \times N_A^n \cong \arrow[d, "\mathrm{Id} \times (\varphi \mid N_1)^n"] & u_\alpha(U_\alpha) \times V^n \times N_1^n \arrow[d, "\mathrm{Id} \times \mathrm{Iso} \times 0"] \\
\mathrm{Fu}_B(M) \supset & B(U_\alpha) \arrow[r, "B(u_\alpha)"] & u_\alpha(U_\alpha) \times N_B^n \cong & u_\alpha(U_\alpha) \times N_B^n \times 0
\end{tikzcd}$$

So $\mathrm{Fu}_A(\varphi)$ is a fibre bundle projection with fibre N_1^n. qed.

2.4. Theorem. *There are bijective mappings* $\eta_{M,A}$: $\mathrm{Hom}(C^\infty(M), A) \to AM = \mathrm{Fu}_A M$ *for any smooth manifold* M *and Weil algebra* A, *which are natural in* $M \in \mathcal{Mf}$ *and* $A \in \mathcal{WAl}$. *Via* η *the set* $\mathrm{Hom}(C^\infty(M), A)$ *becomes a smooth manifold and* $\mathrm{Hom}(C^\infty(\cdot), A)$ *is a global expression for the functor* Fu_A: $\mathcal{Mf} \to \mathcal{Mf}$.

Proof. The decomposition of A into irreducible elements gives

$$\mathrm{Hom}(C^\infty(M), A) = \prod_{i=1}^{k} \mathrm{Hom}(C^\infty(M), A_i), \quad \text{by 2.3.3} \quad \mathrm{Fu}_A(M) = \prod_{i=1}^{k} \mathrm{Fu}_{A_i} M,$$

so we may assume that A is irreducible whenever it is convenient.

Step 1. Evaluation at $\mathrm{Id}_{\boldsymbol{R}}$: $\mathrm{Hom}\,(C^\infty(\boldsymbol{R}), A) \to A$ is bijective and natural in A by the Yoneda lemma.

Step 2. Let (x^i) be coordinate functions on $\boldsymbol{R}^n$. Let $\varphi \in \mathrm{Hom}\,(C^\infty(\boldsymbol{R}^n), A)$, assume A irreducible, then by 1.1 there is exactly one point $x(\varphi) = (x^1(\varphi), \ldots, x^n(\varphi)) \in \boldsymbol{R}^n$ such that $\ker \varphi$ contains the ideal of all $f \in C^\infty(\boldsymbol{R}^n)$ vanishing at $x(\varphi)$ up to some order k, so $\varphi(x^i) = x^i(\varphi)\,.\,1 + \varphi(x^i - x^i(\varphi))$, the latter summand being nilpotent in A of order $\leqq k$, so

$$\varphi(f) = \sum_{|\alpha| \leqq k} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x(\varphi))\, \varphi(x^1 - x^1(\varphi))^{\alpha_1} \ldots \varphi(x^n - x^n(\varphi))^{\alpha_n} = A f(\varphi(x^1), \ldots, \varphi(x^n))\,.$$

So φ is uniquely determined by the elements $\varphi(x^i)$ in A, the mapping $\eta_{\boldsymbol{R}^n, A}$: $\mathrm{Hom}\,(C^\infty(\boldsymbol{R}^n), A) \to A^n$, $\eta(\varphi) = (\varphi(x^1), \ldots, \varphi(x^n))$ is injective. Furthermore for $g = (g^1, \ldots, g^m) \in C^\infty(\boldsymbol{R}^n, \boldsymbol{R}^m)$ we have

$$\begin{aligned}(\eta_{\boldsymbol{R}^m, A} \circ (g^*)^*)(\varphi) &= (\varphi(y^1 \circ g), \ldots, \varphi(y^m \circ g)) = (\varphi(g^1), \ldots, \varphi(g^m)) = \\ &= (Ag^1(\varphi(x^1), \ldots, \varphi(n^n)), \ldots, Ag^m(\varphi(x^1), \ldots, \varphi(x^n))) = \mathrm{Fu}_A(g)\, \eta_{\boldsymbol{R}^n, A}(\varphi)\,.\end{aligned}$$

$\eta_{\boldsymbol{R}^n, A}$ is also bijective since any $(a_1, \ldots, a_n) \in A^n$ defines a homomorphism φ: $C^\infty(\boldsymbol{R}^n) \to$ $\to A$ by $\varphi(f) = Af(a_1, \ldots, a_n)$.

Step 3. Let p_A: $A \to \boldsymbol{R}^k$ be the projection onto the subalgebra of idempotents (see 2.2. step 5). Let i: $U \to \boldsymbol{R}^n$ be the embedding of an open subset. Then the image of the mapping

$$\mathrm{Hom}\,(C^\infty(U), A) \xrightarrow{(i^*)^*} \mathrm{Hom}\,(C^\infty(\boldsymbol{R}^n), A) \xrightarrow{\eta_{\boldsymbol{R}^n}} A^n$$

equals the set $(p_A^n)^{-1}(U^k) = AU$, and $(i^*)^*$ is injective.

To see this let A be irreducible and $\varphi \in \mathrm{Hom}\,(C^\infty(U), A)$. By 1.1 $\ker \varphi$ contains the ideal of all f vanishing up to some order m at exactly one point $x(\varphi) \in U \subseteq \boldsymbol{R}^n$, and since

$$\varphi(x^i) = x^i(\varphi)\,.\,1 + \varphi(x^i - x^i(\varphi))\,,$$

$$p_A(\eta_{\boldsymbol{R}^n}(\varphi \circ i^*)) = p_A(\varphi(x^1), \ldots, \varphi(x^n)) = x(\varphi) \in U\,.$$

As in step 2 we see that

$$p_A^{-1}(U) \in (a_1, \ldots, a_n) \to (C^\infty(U) \ni f \to Af(a_1, \ldots, a_n))$$

is the inverse to $\eta_{\boldsymbol{R}^n} \circ (i^*)^*$.

Step 4. The two functors $\mathrm{Hom}\,(C^\infty(\cdot), A)$ and Fu_A: $\mathcal{M}f \to \mathcal{S}et$ coincide at all open subsets of $\boldsymbol{R}^n$'s and smooth mappings, thus by the remark in 2.1 they coincide at all manifolds. Alternatively one may check easily that the glueing process described in 2.2 step 6 workes also for $\mathrm{Hom}\,(C^\infty(\cdot), A)$ and gives $\mathrm{Hom}\,(C^\infty(M), A)$.

Step 5. Naturality in A of $\eta_{M,A}$ is obvious. qed.

2.5. Remark. Let $+ : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$, m: $\boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ be addition and multiplication, λ: point $\to \boldsymbol{R}$ the mapping onto $\lambda \in \boldsymbol{R}$, m_λ: $\boldsymbol{R} \to \boldsymbol{R}$ multiplication with $\lambda \in \boldsymbol{R}$.

Then $\mathrm{Fu}_A(+)$, $\mathrm{Fu}_A(m)$: $A^2 \to A$ are addition and multiplication in A for any Weil algebra;

$\mathrm{Fu}_A(0)$, $\mathrm{Fu}_A(1)$: point $\to A$ are 0 and 1 resp, and $\mathrm{Fu}_A(m_\lambda)$: $A \to A$ is the scalar multiplication.

2.6. Let A be an irreducible Weil algebra, $A = \boldsymbol{R} . 1 \oplus N$. Let $p = \mathrm{Fu}\,(p_A)_M$: $\mathrm{Fu}_A(M) \to M$ be the fibre bundle projection, for some manifold M. For $f \in C^\infty(M)$, $\mathrm{Fu}_A(f)$: $\mathrm{Fu}_A(M) \to \mathrm{Fu}_A(R) = \boldsymbol{R} . 1 \oplus N$ is of the form $\mathrm{Fu}_A(f) = f \circ p \oplus N(f)$ for smooth $N(f)$: $\mathrm{Fu}_A(M) \to N$.

Corollary. 1. *Each* $v \in \mathrm{Fu}_A(M)_x = p^{-1}(x)$, $x \in M$, *defines an* $\boldsymbol{R}$*-linear mapping* X_v: $C^\infty(M) \to N$ *by* $X_v(f) = N(f)(v) = \mathrm{Fu}_A(f)(v) - f(x)$, *which satisfies* $X_v(f . g) = X_v(f) . g(x) + f(x) . X_v(g) + X_v(f) . X_v(g)$.

2. *Any mapping* X: $C^\infty(M) \to N$ *with the property of* 1 *is of the form* X_v *for some unique* $v \in \mathrm{Fu}_A(M)_x$.

3. *The* $\boldsymbol{R}$*-linear mappings* X: $C^\infty(M) \to C^\infty(M, N) = C^\infty(M) \otimes N$ *which satisfy* $X(f . g) = X(f) . g + f . X(g) + X(f) . X(g)$ *are exactly those induced* (via 1 and 2) *by the smooth sections of* p: $\mathrm{Fu}_A(M) \to M$.

Remark. Compare with the properties of tangent vectors and vector fields. We will enlarge this view in § 4.

Proof. 1 and 2. For $\varphi \in \mathrm{Hom}\,(C^\infty(M), A) \cong \mathrm{Fu}_A(M)$ let

$$p(\eta_{M,A}(\varphi)) = \eta_{M,\boldsymbol{R}}(p_A \circ \varphi) = x \in M .$$

Then

$$\varphi(f) = f(x) . 1 + v(f) , \quad v: C^\infty(M) \to N .$$

The homomorphism property of φ is equivalent to the property of 1 for v.

3. For any $x \in M$ the mapping $f \to X(f)(x)$ is of the form $X_{v(x)}$ for $v(x) \in \mathrm{Fu}_A(M)_x$ by 2 and clearly $v(x)$ depends smoothly on x. qed.

3. MULTIPLICATIVE FUNCTORS ON MANIFOLDS

3.1. Recall the mappings $+$, m: $\boldsymbol{R}^2 \to \boldsymbol{R}$, m_λ: $\boldsymbol{R} \to \boldsymbol{R}$, λ: point $\to \boldsymbol{R}$.

Theorem. *Let* F: $Mf \to Mf$ *be a multiplicative functor. Then* $F(\boldsymbol{R})$ *is a Weil algebra with operations* $F(+)$, $F(m)$, $F(m_\lambda)$, $F(0)$, $F(1)$, *which is called* $\mathrm{Al}(F)$. *If* φ: $F_1 \to F_2$ *is a natural transformation between multiplicative functors, then* $\mathrm{Al}(\varphi) = \varphi_{\boldsymbol{R}}$: $\mathrm{Al}(F_1) \to \mathrm{Al}(F_2)$ *is an algebra homomorphism.*

Proof. Remember from 2.1 that $F(\text{point}) = \text{point}$. All the laws for a commutative ring with unit can be formulated by commutative diagrams of mappings between products of the ring and the point. Do this for the ring $\boldsymbol{R}$ and apply the multiplicative functor F, so you get the laws for the commutative ring $F\boldsymbol{R}$ with unit $F(1)$. $F(+)$ and $F(m)$ are morphisms in Mf, thus smooth and continuous.

For $\lambda \in \boldsymbol{R}$ the mapping $F(m_\lambda)$: $F\boldsymbol{R} \to F\boldsymbol{R}$ equals multiplication with the element

$F(\lambda) \in FR$, since the following diagram commutes:

$$\begin{tikzcd}
FR \arrow[d, equal] \arrow[dr, "{(\mathrm{Id},\, F(\lambda))}"] \arrow[drr, "F(m_\lambda)"] & & \\
FR \times \text{point} \arrow[r, "\mathrm{Id} \times F(\lambda)"] \arrow[d, equal] & FR \times FR \arrow[r, "F(m)"] \arrow[d, equal] & FR \\
F(R \times \text{point}) \arrow[r, "F(\mathrm{Id} \times \lambda)"] & F(R \times R) \arrow[ur, "F(m)"'] &
\end{tikzcd}$$

So in order to show that the scalar multiplication $\lambda \to F(m_\lambda)$ induces a continuous mapping $R \times FR \to FR$ it suffices to show that $R \to FR$, $\lambda \to F(\lambda)$ is continuous.

$(FR, F(+), F(m_{-1}), F(0))$ is a commutative real Lie group, and it is second countable as a manifold since $FR \in Mf$. Let $\exp: \mathfrak{L} \to FR$ be the exponential mapping of this group, where $\mathfrak{L}$ is the Lie algebra. Then $\exp(\mathfrak{L})$ is an open subgroup of FR, and if $F(\lambda) \notin \exp(\mathfrak{L})$ for all $\lambda \neq 0$, then $FR/\exp \mathfrak{L}$ is a discrete uncountable subgroup, so FR has uncountably many connected components, in contradiction to $FR \in Mf$. So there is $\lambda_0 \neq 0$ in $\mathbb{R}$ and $v_0 \neq 0$ in $\mathfrak{L}$ such that $F(\lambda_0) = \exp(v_0)$. Clearly for each $v \in \mathfrak{L}$ and $r \in N$, hence $r \in Q$, we have $F(m_r) \exp v = \exp(r \,.\, v)$. Now we claim that for any sequence $\lambda_n \to \lambda$ in R we have $F(\lambda_n) \to F(\lambda)$ in FR. If not then there is $\lambda_n \to \lambda$ in R such that $F(\lambda_n) \in FR \setminus U$ for some neighbourhood U of $F(\lambda)$ in FR, and by considering a suitable subsequence we may also assume that $2^{n^2}(\lambda_n - \lambda)$ is bounded. By lemma 3.2 below there is a C^∞-function $f: R \to R$ with $f(\lambda_0/2^n) = \lambda_n$ and $f(0) = \lambda$. Then we have

$$\begin{aligned} F(\lambda_n) &= F(f)\, F(m_{2^{-n}})\, F(\lambda_0) = F(f)\, F(m_{2^{-n}}) \exp(v_n) = \\ &= F(f) \exp(2^{-n} \,.\, v_0) \to F(f) \exp(0) = F(f(0)) = F(\lambda)\,, \end{aligned}$$

contrary to the assumption that $F(\lambda_n) \notin U$ for all n. So $\lambda \to F(\lambda)$ is continuous: $R \to FR$, so FR, with its manifold topology, is a real finite dimensional commutative algebra, denoted by $\mathrm{Al}(F)$.

The mapping $\varepsilon: \mathrm{Hom}(C^\infty(R), \mathrm{Al}(F)) \to \mathrm{Al}(F)$, $\varepsilon(\varphi) = \varphi(\mathrm{Id}_R)$, is bijective, since it has the right inverse $x \to (f \to F(f)(x))$. But by 1.7 the mapping ε has values in $W(\mathrm{Al}(F))$, so $\mathrm{Al}(F)$ is a Weil algebra.

3.2. Lemma (Kriegl [5]): *Let $\lambda_n \to \lambda$ in R, let $t_n \in R$, $t_n > 0$, $t_n \to 0$ strictly monotone, such that*

$$\left\{ \frac{\lambda_n - \lambda_{n+1}}{(t_n - t_{n+1})^k}, \quad n \in N \right\}$$

is bounded for all k. Then there is a C^∞-function $f: R \to R$ with $f(t_n) = \lambda_n, f(0) = \lambda$, such that f' is flat at each t_n.

Proof. Let $\varphi \in C^\infty(R)$, $\varphi = 0$ near 0, $\varphi = 1$ near 1, and $0 \leqq \varphi \leqq 1$ elsewhere. Then put $f(t) = \lambda$ for $t \leqq 0$,

$$f(t) = \varphi\left(\frac{t - t_{n+1}}{t_n - t_{n+1}}\right)(\lambda_n - \lambda_{n+1}) + \lambda_{n+1} \quad \text{for} \quad t_{n+1} \leqq t \leqq t_n\,,$$

$$f(t) = \lambda_1 \quad \text{for} \quad t_1 \leqq t\,. \qquad \text{qed.}$$

3.3. Definition. A multiplicative functor $F\colon Mf \to Mf$ is called *local*, if it has the following property: If (U_α) is the open cover of a connected manifold M consisting of all its contractible subsets with embeddings $i_\alpha\colon U_\alpha \to M$, then $\bigcup_\alpha F(i_\alpha)\, F(U_\alpha) = F(M)$.

Note that Fu_A is local for any Weil algebra A by 2.3.6.

Example. Let $M = \bigcup_\alpha M_\alpha$ be the disjoint union of its connected components and put $\tilde{H}_1(M) = \bigcup_\alpha H_1(M_\alpha, \mathbb{R})$. If M is compact, then $\tilde{H}_1(M) \in Mf$ and $\tilde{H}_1\colon cp\ Mf \to Mf$ is a multiplicative functor which is not local.

3.4. We define the natural transformation

$$\chi_F\colon F \to \mathrm{Hom}\,(C^\infty(\cdot), \mathrm{Al}(F)) \xrightarrow{\simeq\eta} \mathrm{Fu}(\mathrm{Al}(F))$$

by

$$\chi_{F,M}(x)\,(f) = F(f)\,(x)\,.$$

Lemma. $\chi_{F,M}\colon F(M) \to \mathrm{Fu}(\mathrm{Al}(F))\,(M)$ *is smooth and natural in* F *and* M.

Proof. Naturality in F and M is obvious. To show that χ is smooth we proceed as follows.

Let $h = (h^1, \ldots, h^n)\colon M \to R^n$ be a closed embedding into some high dimensional R^n. Then by 2.3.5 the mapping $\mathrm{Fu}_A(h)\colon \mathrm{Fu}_A M \to \mathrm{Fu}_A(R^n)$ is also a closed embedding.

By 2.4, step 2 of the proof, $\eta_{R^n,A}\colon \mathrm{Hom}\,(C^\infty(R^n), A) \to A^n$ is given by $\eta_{R^n,A}(\varphi) = (\varphi(x^i))_{i=1}^n$, where $(x^i)_{i=1}^n$ are coordinate functions on R^n. Now we consider the mappings

$$\begin{array}{ccc}
F(M) & & \\
\downarrow{\scriptstyle \chi_M} & & \\
\mathrm{Hom}(C^\infty(M), A) & \xrightarrow{\eta_{M,A}} & \mathrm{Fu}_A(M) \\
\downarrow{\scriptstyle (h^*)^*} & & \downarrow{\scriptstyle \mathrm{Fu}_A(h)} \\
\mathrm{Hom}(C^\infty(R^n), A) & \xrightarrow{\eta_{R^n,A}} & \mathrm{Fu}_A(R^n) \cong A^n \cong (FR)^n \cong F(R^n).
\end{array}$$

$$(\eta_{R^n,A} \circ (h^*)^* \circ \chi_M)\,(z) = \eta_{R^n,A}(\chi_M(z) \circ h^* =$$
$$= (\chi_M(z\ (x^1 \circ h\,, \ldots, \chi_M(z\ (x^n \circ h)) = (\chi_M(z)\,(h^1), \ldots, \chi_M(z)\,(h^n)) =$$
$$= (F(h^1)\,(z), \ldots, F(h^n)\,(z)) = F(h)\,(z)\,.$$

This is smooth in $z \in F(M)$. Since $\eta_{M,A}$ is a diffeomorphism and $\mathrm{Fu}_A(h)$ is a closed embedding, χ_M is smooth. qed.

3.5. Let F be a multiplicative functor. We will construct another multiplicative functor as follows: for any manifold M choose a universal cover $q_M: \tilde{M} \to M$ (over each connected component of M separately) and let $\pi_1(M)$ denote the group of deck transformations of $\tilde{M} \to M$, which is isomorphic to the product of all fundamental groups of the connected components of M. It is easy to see that $\pi_1(M)$ acts strictly discontinuously on $\mathrm{Fu}(\mathrm{Al}(F))(\tilde{M})$ and by 3.4 also on $F(\tilde{M})$. So $\tilde{F}(M) := F(\tilde{M})/\pi_1(M)$ is a smooth manifold. For $f: M_1 \to M_2$ choose any lift $\tilde{f}: \tilde{M}_1 \to \tilde{M}_2$. Then $F(\tilde{f})$ factors as follows:

$$\begin{tikzcd} F(\tilde{M}_1) \arrow[r, "F(\tilde{f})"] \arrow[d] & F(\tilde{M}_2) \arrow[d] \\ \tilde{F}(M_1) \arrow[r, dashed, "\tilde{F}(f)"] & \tilde{F}(M_2) \end{tikzcd}$$

The resulting smooth mapping $\tilde{F}(f)$ does not depend on the the choice of the lift $\tilde{f}$. So we get a functor $\tilde{F}: \mathcal{Mf} \to \mathcal{Mf}$ and a natural transformation $q: \tilde{F} \to F$, induced from $F(q_M): F(\tilde{M}) \to F(M)$. The functor $\tilde{F}$ is again multiplicative, because we may choose $(M_1 \times M_2)^\sim = \tilde{M}_1 \times \tilde{M}_2$ and $\pi_1(M_1 \times M_2) = \pi_1(M_1) \times \pi_1(M_2)$, thus $\tilde{F}(M_1 \times M_2) = F((M_1 \times M_2)^\sim)/\pi_1(M_1 \times M_2) = F(\tilde{M}_1)/\pi_1(M_1) \times F(\tilde{M}_2)/\pi_1(M_2) = \tilde{F}(M_1) \times \tilde{F}(M_2)$.

Theorem. 1. *If M is connected then there is a unique smooth mapping ψ_M: $(\mathrm{Fu}(\mathrm{Al}(F)))^\sim(M) \to F(M)$ which is natural in F and M and satisfies $\chi_{F,M} \circ \psi_{F,M} = q_{\mathrm{Fu}(\mathrm{Al}(F)),M}$:*

$$\begin{tikzcd} & F(M) \arrow[d, "\chi"] \\ (\mathrm{Fu}(\mathrm{Al}(F)))^\sim(M) \arrow[ur, "\psi"] \arrow[r, "q"'] & \mathrm{Fu}(\mathrm{Al}(F))(M). \end{tikzcd}$$

2. *If M is connected and F is local then $\chi_{F,M}$ and $\psi_{F,M}$ are covering mappings.*

Proof. Put $A := \mathrm{Al}(F)$ for short.

Sublemma. *If M is connected then there $\chi_{F,M}$ is surjective and near each φ in $\mathrm{Fu}_A(M)$ there is a smooth local section of $\chi_{F,M}$.*

Let $\varphi \in \mathrm{Hom}(C^\infty(M), A)$, $\varphi = \varphi_1 + \ldots + \varphi_k$, $\varphi_i \in \mathrm{Hom}(C^\infty(M), A_i)$, where $A = \bigoplus^k A_i$ is the decomposition into irreducible components. For each i there is exactly one point x_i in M such that $\varphi_i(f)$ depends only on a finite jet of f at x_i by 1.1.

Since M is connected there is a smoothly contractible open set U in M containing all x_i. Let $g\colon R^m \to M$ be a diffeomorphism onto U. Then $g^{**}\colon \mathrm{Hom}\,(C^\infty(R^m), A) \to \to \mathrm{Hom}\,(C^\infty(M), A)$ is an embedding of an open neighbourhood of φ, so there is $\bar\varphi$ in $\mathrm{Hom}\,(C^\infty(R^m), A)$ depending smoothly on φ such that $g^{**}(\bar\varphi) = \varphi$. Now consider

$$\mathrm{Hom}\,(C^\infty(R^m), A) \xrightarrow{\eta_{R^m}} \mathrm{Fu}_A(R^m) = F(R^m) \xrightarrow{F(g)} F(M) \xrightarrow{\chi_M} \mathrm{Hom}\,(C^\infty(M), A)\,.$$

We have $(\chi_M \circ F(g) \circ \eta_R m)\,\bar\varphi = (g^{**} \circ \chi_R m \circ \eta_R m)\,\bar\varphi = g^{**}(\bar\varphi) = \varphi$, since it follows from 3.4 that $\chi \circ \eta = \mathrm{Id}$ for all R^m's. Note that $F(g) \circ \eta_R m \circ (g^{**})^{-1} =: s_U$ is a local smooth section of χ_M near φ. We may also write $s_U = F(i_U) \circ \chi_{F,U}^{-1}\colon \mathrm{Fu}_A(U) \subset \subset \mathrm{Fu}_A(M) \to F(M)$, where $i_U\colon U \to M$ is the embedding, since for contractible U the mapping $\chi_{F,U}$ is clearly a diffeomorphism.

Now we start with the construction of $\psi_{F,M}$. Note first that it suffices to construct $\psi_{F,M}$ for simply connected M, because the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{Fu})_A^\sim(\tilde M) = \mathrm{Fu}_A(\tilde M) \arrow[r, "\psi_{F,\tilde M}"] \arrow[d] & F(\tilde M) \arrow[d] \\
(\mathrm{Fu})_A^\sim(M) \arrow[r, dashed, "\psi_{F,M}"] & F(M)\,.
\end{tikzcd}$$

Furthermore it suffices to construct $\psi_{F,M}$ for high-dimensional M, since we have

$$\begin{tikzcd}
(\mathrm{Fu})_A^\sim(M \times R) \arrow[r, "\psi_{F,M\times R}"] \arrow[d] & F(M \times R) \arrow[d] \\
(\mathrm{Fu})_A^\sim(M) \times F(R) \arrow[r, "\psi_{F,M} \times \mathrm{Id}_{F(R)}"] & F(M) \times F(R)
\end{tikzcd}$$

So we may assume that M is connected, simply connected and of high dimension. For any contractible subset U of M we consider the local section s_U of $\chi_{F,M}$ constructed in the sublemma above and we just put $\psi_{F,M}(\varphi) = s_U(\varphi)$ for φ in $\mathrm{Fu}_A(U) \subset \mathrm{Fu}_A(M)$. We have to show that $\psi_{F,M}$ is well defined. So consider contractible U and U' in M with φ in $\mathrm{Fu}_A(U) \cap \mathrm{Fu}_A(U')$. If $p(\varphi) = (x_1, \dots, x_k)$ in M^k as in the sublemma, this means that $x_1, \dots, x_k$ are in $U \cap U'$. We claim that there are contractible open subsets V, V', W of M such that $x_1, \dots, x_k$ are in $V \cap V' \cap W$ and $V \subset U \cap W$, $V' \subset U' \cap W$. Then by the naturality of χ we have $s_U(\varphi) = s_V(\varphi) = s_W(\varphi) = s_{V'}(\varphi) = s_{U'}(\varphi)$ as required. For the existence of these sets choose an embedding $H\colon R^2 \to M$ such that $c(t) := H(t, \sin t)$ is in U, $c'(t) := H(t, -\sin t)$ is in U' and $H(2\pi j, 0) = x_j$ for $j = 1, \dots, k$. This exists by the following argument: connect the points x_j by a smooth curve in U and one in U', choose a homotopy between these curves fixing the x_j's, and approximate the homotopy by an embedding (again fixing the x_j's) using transversality. For this one needs that $\dim M \geqq 5$. Then V, V', W are just small tubular neighbourhoods of c, c', H. If F is a local functor in the sense of 3.3, then $F(M)$ is the union of all $F(U)$, where U runs through all contractible subsets of M. Since for contractible U the map $\chi_{F,U}$ is a diffeomorphism, this implies that $\psi_{F,M}$ is surjective.

Since $\chi \circ \psi = q$, q is a covering map, and ψ is surjective, it follows, that both χ and ψ are covering maps. qed.

3.6. Now we will determine all local multiplicative functors F on connected manifolds with $\mathrm{Al}(F)$ equal to some given Weil algebra A with k irreducible components. For a connected manifold M define $C(M)$ by the following transversal pullback:

$$\begin{tikzcd}
C(M) \arrow[r] \arrow[d] & F(M) \arrow[d, "\chi"] \\
M^k \arrow[r, "0"] & \mathrm{Fu}_A(M)
\end{tikzcd}$$

where 0 is the natural transformation induced by the inclusion of the subalgebra generated by all idempotents into A.

Now consider the following diagram. In it every square is a pullback, and each vertical mapping is a covering mapping, if F is local, by 3.5.

$$\begin{tikzcd}
\tilde{M}^k \arrow[r, "0"] \arrow[d] & \mathrm{Fu}_A(\tilde{M}) \arrow[d] \\
\tilde{M}^k/\pi_1(M) \arrow[r] \arrow[d] & (\mathrm{Fu})^{\sim}_A(M) \arrow[d, "\psi"] \\
C(M) \arrow[r] \arrow[d] & F(M) \arrow[d, "\chi"] \\
M^k \arrow[r] & \mathrm{Fu}_A(M)
\end{tikzcd}$$

Thus $F(M)$ equals $\mathrm{Fu}_A(\tilde{M})/G$, where G is the group of deck transformations of the covering $\tilde{M}^k \to C(M)$, a subgroup of $\pi_1(M)^k$ containing $\pi_1(M)$ (with it's diagonal action on $\tilde{M}^k$). Here $g = (g_1, \ldots, g_k)$ in $\pi_1(M)^k$ acts on $\mathrm{Fu}_A(\tilde{M}) = \mathrm{Fu}_{A_1}(\tilde{M}) \times \times \mathrm{Fu}_{A_2}(\tilde{M}) \times \ldots \times \mathrm{Fu}_{A_k}(\tilde{M})$ via $\mathrm{Fu}_{A_1}(g_1) \times \ldots \times \mathrm{Fu}_{A_k}(g_k)$. Thus we have:

Theorem. *A local multiplicative functor F on connected manifolds is uniquely determined by specifying its Weil algebra A and a multiplicative functor G: $Mf \to \to$ groups satisfying $\pi_1 \subseteq G \subseteq \pi_1^k$, where π_1 is now the "fundamental group" – functor, sitting as diagonal in π_1^k, and k is the number of irreducible components of* $\mathrm{Al}(F)$.

The statement of this theorem is not completely rigorous, since π_1 depends on the choice of a base point of the manifold in question. So we have to identify isomorphic groups in the category *groups*, i.e. we have to consider the skeleton of this category.

Corollary. *On the category of simply connected manifolds a local functor is completely determined by its Weil algebra.*

3.7. Proposition. *Let F_1, F_2 be local multiplicative functors: $Mf \to Mf$. Then $\mathrm{Al}(F_2 \circ F_1) = \mathrm{Al}(F_1) \otimes \mathrm{Al}(F_2)$, naturally in F_1 and F_2.*

Proof. Let B be a real basis for $\mathrm{Al}(F_1)$, then

$$\mathrm{Al}(F_2 \circ F_1) = F_2(F_1(\boldsymbol{R})) = F_2(\prod_{b \in B} \boldsymbol{R} \,.\, b) \cong \prod_{b \in B} F_2(\boldsymbol{R}) \,.\, b \,,$$

so the formula holds for the underlying vector spaces. Now express the multiplication $F_1(m)$: $\mathrm{Al}(F_1) \times \mathrm{Al}(F_1) \to \mathrm{Al}(F_1)$ in terms of the basis B: $b_i b_j = \sum_k c_{ij}^k b_k$, and use

$$F_2(F_1(m)) = (F_1(m)^*)^*\colon \mathrm{Hom}\,(C^\infty(\mathrm{Al}(F_1) \times \mathrm{Al}(F_1))\,,\, \mathrm{Al}(F_2)) \to$$
$$\to \mathrm{Hom}\,(C^\infty(\mathrm{Al}\,(F_1)), \mathrm{Al}(F_2))$$

to see that the formula holds also for the multiplication.

3.8. Remark. We choose the order $\mathrm{Al}(F_1) \otimes \mathrm{Al}(F_2)$, so that the elements of $\mathrm{Al}(F_2)$ stand on the right hand side. This coincides with the usual convention of writing an atlas for the second tangent bundle. See also 4.3.

3.9. Remark. Any natural transformation $\varphi\colon F_1 \to F_2$ between local multiplicative functors factors uniquely as $\varphi = \bar{\bar{\varphi}} \circ \bar{\varphi}\colon F_1 \to \mathrm{Im}\,\varphi \to F_2$, where $\mathrm{Im}\,\varphi$ is a local multiplicative functor, $\bar{\varphi}_M\colon F_1(M) \to (\mathrm{Im}\,\varphi)(M)$ is a fibre bundle projection for each M, and $\bar{\bar{\varphi}}_M\colon (\mathrm{Im}\,\varphi)(M) \to F_2(M)$ is a closed embedding.

3.10. We sketch here the results of David Eck [15] concerning the behaviour of local multiplicative functors on not connected manifolds. First one contracts each connected component to a point, then F acts on this discrete manifold. Out of this combinatorial data one may conclude how each irreducible part of F acts on each component and how these pieces are to be multiplied together to get the action of F on the manifold.

4. APPLICATIONS IN DIFFERENTIAL GEOMETRY

4.1. In the following we explain the best known natural transformations in differential geometry and show that these are indeed the most important ones. One may say that natural transformations owe their existence to three properties of differentiation on $\boldsymbol{R}^n$: (i) each k-th derivative is k-linear, (ii) each k-th derivative is symmetric, (iii) differentiation is linear.

In the following we consider only irreducible Weil algebras (but see 2.4.3) and their induced functors.

4.2. The tangent functor T: $T\boldsymbol{R} = \boldsymbol{R}^2$, $T(+)\,(a, a')\,(b, b') = (a + b,\ a' + b')$, $T(m)\,(a, a')\,(b, b') = (ab, ab' + a'b)$, $T(m_\lambda)\,(a, a') = (\lambda a, \lambda a')$. So $D = \mathrm{Al}(T) = T\boldsymbol{R}$ is the algebra generated by 1, δ with $\delta^2 = 0$.

Lemma. *Let F be a local multiplicative functor which is a natural vector bundle* *over* Id_{Mf}, *then* $F(M) = V \otimes TM$ *for a vector space* V (fibre wise tensor product). *Furthermore* $\mathrm{Nat}\,(V \otimes T,\ W \otimes T) \cong L(V, W)$.

Proof. By 2.3.1 $\mathrm{Al}(F) = \boldsymbol{R}\,.\,1 \oplus V$, where the multiplication in V is 0, but $F = \mathrm{Fu}(\boldsymbol{R}\,.\,1 \oplus V) = V \otimes T(\cdot)$. Clearly $\mathrm{Nat}\,(V \otimes T,\ W \otimes T) = \mathrm{Hom}\,(\boldsymbol{R}\,.\,1 \oplus V, \boldsymbol{R}\,.\,1 \oplus W) \cong L(V, W)$. qed.

Remark. This expresses the linearity of the first derivative. So $\mathrm{Nat}\,(T, T)$ consists of all fibre scalar multiplication m_λ, $\lambda \in \boldsymbol{R}$; $m_\lambda(1) = 1$, $m_\lambda(\delta) = \lambda\,.\,\delta$. $\mathrm{Nat}\,(T \times_{\mathrm{id}} T, T)$ contains the fibre addition $+: x1 + y\delta + z\bar{\delta} \to x1 + (y + z)\,\delta$.

$\mathrm{Nat}\,(T, \mathrm{Id})$ consists of $\pi\colon T \to \mathrm{Id}$ alone.

4.3. Let F, F_1, F_2 be functors on Mf (local multiplicative with irreducible Weil algebras $\mathrm{Al}(F) = \boldsymbol{R}\,.\,1 \oplus \mathfrak{N}(F)$, $\mathfrak{N}(F)$ the nilradical). By 3.6 we have $\mathrm{Al}(F_2 \circ F_1) = \mathrm{Al}(F_1) \otimes \mathrm{Al}(F_2)$. Using this and 2.3.6 we define the following natural transformations:

(1) $\pi_1\colon F_1 \to \mathrm{Id}$, $\pi_2\colon F_2 \to \mathrm{Id}$, in general $\pi_F\colon F \to \mathrm{Id}\,(\pi_F = \mathrm{Fu}(p)$ as in 2.3.1.) Thus $F_2\pi_1\colon F_2 \circ F_1 \to F_2$; $\pi_2 F_1\colon F_2 \circ F_1 \to F_1$.

(2) $O_1\colon \mathrm{Id} \to F_1$, $O_2\colon \mathrm{Id} \to F_2$ (from $\boldsymbol{R} \to \mathrm{Al}(F_i)$, $\lambda \to \lambda.1$). Thus $F_2 O_1\colon F_2 \to F_2 \circ F_1$, $O_2 F_1\colon F_1 \to F_2 \circ F_1$.

(3) $\mathrm{Al}(F_1) \otimes \mathrm{Al}(F_2) \cong \mathrm{Al}(F_2) \otimes \mathrm{Al}(F_1)$ induces the "canonical flip" $\varkappa_{F_2,F_1} = \varkappa$, $F_2 \circ F_1 \to F_1 \circ F_2$; clearly $\varkappa_{F_1,F_2} = \varkappa^{-1}_{F_2,F_1}$.

(4) The multiplication m in $\mathrm{Al}(F)$ is a homomorphism $m\colon \mathrm{Al}(F) \otimes \mathrm{Al}(F) \to \mathrm{Al}(F)$ and we get $\mu_F = \mu := \mathrm{Fu}(m)\colon \mathrm{F} \circ \mathrm{F} \to F$.

(5) $F_1 \times_{\mathrm{Id}} F_2 = \mathrm{Fu}(\boldsymbol{R}\,.\,1 \oplus \mathfrak{N}(F_1) \oplus \mathfrak{N}(F_2))$, as is easily seen. Consider the natural transformations

$$(\pi_2 F_1, F_2\pi_1)\,, \quad O_{F_1 \times_{\mathrm{Id}} F_2} \circ \pi_{F_2 \circ F_1}\colon F_2 \circ F_1 \to (F_1 \times_{\mathrm{Id}} F_2)\,.$$

The "equalizer" of these two transformations is denoted by $V\colon F_2 * F_1 \to F_2 \circ F_1$ and called *vertical lift*.

Clearly $F_2 * F_1 = \mathrm{Fu}(\boldsymbol{R}\,.\,1 \oplus (\mathfrak{N}(F_1) \otimes \mathfrak{N}(F_2)))\,.$

(6) $\varkappa$ and μ induce mappings

$$\varkappa_{F_2 * F_1}\colon F_2 * F_1 \to F_1 * F_2 \quad \text{with} \quad V \circ \varkappa_{F_2 * F_1} = \varkappa_{F_2,F_1} \circ V \quad \text{and} \quad \mu \circ V\colon F * F \to F\,.$$

It is clear that $\varkappa$ expresses the symmetry of higher derivatives. We will see that V expresses linearity of differentiation. We advise the reader to work out the Weil algebra side of all these transformations.

4.4. Now we put $F_1 = F_2 = T$ in 4.2 and consider the second tangent functor $T^2 = T \circ T$. Then $D_2 := \mathrm{Al}(T^2) = D \otimes D = \boldsymbol{R}^4$ with generators $1, \delta_1, \delta_2$ and relations $\delta_1^2 = \delta_2^2 = 0$, and $(1, \delta_1; \delta_2, \delta_1\delta_2)$ is the standard basis of $\boldsymbol{R}^4 = T^2\boldsymbol{R}$ in the usual description.

Making use of 4.2 we get $\pi T\colon (\delta_1, \delta_2) \to (\delta, 0)$, $T\pi\colon (\delta_1, \delta_2) \to (0, \delta)$, $\mu = + \circ (\pi T, T\pi)\colon (\delta_1, \delta_2) \to (\delta, \delta)$ since $\mathfrak{N}(T)^2 = 0$.

$T * T = T$ since $\mathfrak{N}(T) \otimes \mathfrak{N}(T) = \mathfrak{N}(T)$. Furthermore we have

$\varkappa\colon T^2 \to T^2$, $\varkappa(a1 + x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_2) = a1 + x_2\delta_1 + x_1\delta_2 + x_3\delta_1\delta_2$,
$V\colon T \to T^2$, $V(a1 + x\delta) = a1 + x\delta_1\delta_2$,
$m_\lambda T\colon T^2 \to T^2$, $m_\lambda T(a1 + x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_2) = a1 + x_1\delta_1 + \lambda x_2\delta_2 + \lambda x_3\delta_1\delta_2$,
$Tm_\lambda(a1 + x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_3) = a1 + \lambda x_1\delta_1 + x_2\delta_2 + \lambda x_3\delta_1\delta_2 +$
$+ T((a1 + x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_2), (a1 + x_1\delta_1 + y_2\delta_2 + y_3\delta_1\delta_2)) =$
$= a1 + x_1\delta_1 + (x_2 + y_2)\,\delta_2 + (x_3 + y_3)\,\delta_1\delta_2$,
$T + ((a1 + x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_2), (a1 + y_1\delta_1 + x_2\delta_2 + y_3\delta_1\delta_2)) =$
$= a1 + (x_1 + y_2)\,\delta_2 + x_2\delta_3 + (x_3 + y_3)\,\delta_1\delta_2$.

Now $\mathrm{Nat}\,(T, T^2) \cong \mathrm{Hom}\,(D, D_2) \cong \boldsymbol{R}^2 \cup_{\boldsymbol{R}} \boldsymbol{R}^2$ consists of all

$\delta \to x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_2$ with $x_1x_2 = 0$ (since $\delta^2 = 0$),
$\delta \to x\delta_1 + y\delta_1\delta_2$ corresponds to $+T \circ (V \circ m_y, OT \circ m_x)$,
$\delta \to x\delta_2 + y\delta_1\delta_2$ corresponds to $T+ \circ (V \circ m_y, TO \circ m_x)$.

So any element in $\mathrm{Nat}\,(T, T^2)$ can be expressed by

$$\{OT, TO, T+, +T, T\pi, \pi T, V, m_\lambda, (\lambda \in \boldsymbol{R})\}$$

Similarly $\mathrm{Nat}\,(T^2, T^2) \cong \mathrm{Hom}\,(D_2, D_2) \cong (\boldsymbol{R}^2 \cup_{\boldsymbol{R}} \boldsymbol{R}^2) \times (\boldsymbol{R}^2 \cup_{\boldsymbol{R}} \boldsymbol{R}^2)$ consists of all

$$\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} \to \begin{pmatrix}x_1\delta_1 + x_2\delta_2 + x_3\delta_1\delta_2\\ y_1\delta_1 + y_2\delta_2 + y_3\delta_1\delta_2\end{pmatrix}$$

with $x_1x_2 = y_1y_2 = 0$.

If e.g. $x_2 = y_1 = 0$, then the corresponding transformation is

$$+T \circ (m_{y_2}T \circ Tm_{x_1},\ T+ \circ (V \circ + \circ (m_{x_3} \circ \pi T, m_{y_3} \circ T\pi), OT \circ m_{x_1} \circ \pi T)).$$

Again any element of $\mathrm{Nat}\,(T^2, T^2)$ can be expressed by

$$\{OT, TO, T+, +T, T\pi, \pi T, V, m_\lambda T, Tm_\lambda(\lambda \in \boldsymbol{R}), \varkappa\}.$$

Note also the relations $T\pi \circ \varkappa = \pi T$, $\varkappa \circ T+ = +T \circ (\varkappa \times \varkappa)$, $\varkappa \circ V = V$, $\varkappa \circ Tm_\lambda = m_\lambda T \circ \varkappa$, that T^2 has two natural vector bundle structures over T, namely $(+T, m_\lambda T, \pi T)$, $(T+, Tm_\lambda, T\pi)$, and $V\colon T \to T^2$ is linear for both of them.

4.5. In the situation of 4.3 we let F_1 be general and $F_2 = T$. So we consider $T \circ F$ which is isomorphic to $F \circ T$ via $\varkappa_{T,F}$. In general we have $(F_1 \times_{\mathrm{Id}} F_2) * F = F_1 * F \times_{\mathrm{Id}} F_2 * F$, so $+\colon T \times_{\mathrm{Id}} T \to T$ induces a fibre addition.

$+ * F\colon T * F \times_{\mathrm{Id}} T * F \to T * F$, and $m_\lambda * F\colon T * F \to T * F$ is a fibre scalar multiplication. So $T * F$ is a natural vector bundle over Id.

Lemma. *In the notation of lemma* 4.2 *we have* $T * F \cong \overline{\mathfrak{N}}(F) \otimes T$, *where* $\overline{\mathfrak{N}}(F)$ *is the underlying vector space of the nilradical* $\mathfrak{N}(F)$.

Proof. $\mathrm{Al}(T * F) = \boldsymbol{R}\,.\,1 \oplus \mathfrak{N}(F) \otimes \mathfrak{N}(T)$ by 4.3.6, $\mathfrak{N}(F) \otimes \mathfrak{N}(T) = \mathfrak{N}(F) \otimes \boldsymbol{R}\,.\,\delta = \overline{\mathfrak{N}}(F)$ as vector space, and the multiplication on $\mathfrak{N}(F) \otimes \mathfrak{N}(T)$ is zero.

qed.

4.6. For a local multiplicative functor F with $\mathrm{Al}(F)$ irreducible let $\mathfrak{X}_F(M)$ be the space of all smooth sections of $\mathrm{Fu}(p_{\mathrm{Al}(F)})\colon F(M) \to M$. Note that it is infinite dimensional.

Theorem. 1. *$\mathfrak{X}_F(M)$ is a group with multiplication $X \square Y = \mu_F \circ F(Y) \circ X$ and identity O_F.*

2. *$\mathfrak{X}_{T*F}(M)$ is a Lie algebra with bracket induced from the usual Lie bracket on T and the multiplication $m\colon \mathfrak{N}(F) \times \mathfrak{N}(F) \to \mathfrak{N}(F)$: $[a \otimes X, \ b \otimes Y]_{T*F} = a \,.\, b \otimes [X, Y]$.*

3. *There is a bijective mapping* exp: $\mathfrak{X}_{T*F}(M) \to \mathfrak{X}_F(M)$ *expressing the multiplication $\square$ by the Campell-Baker-Hausdorff-formula.*

4. *The multiplication $\square$, the Lie bracket $[,]$ and* exp *are natural in F and M.*

Remark. If $F = T$, then $\mathfrak{X}_T(M)$ is the space of all smooth vector fields on M, $X \square Y = X + Y$, $[X, Y]_{T*T} = 0$, and exp is the identity. So the multiplication in 1, which is commutative only if F is a natural vector bundle, generalizes the linear structure on $\mathfrak{X}_T(M)$.

4.7. For the proof of 4.6 we need some preparation. Put $A := \mathrm{Al}(F) = \boldsymbol{R} \,.\, 1 \oplus N$, where $N = \mathfrak{N}(F)$ is the nilradical.

Recall from 2.6.3 that $\mathfrak{X}_F(M)$ is isomorphic to the space of all $\boldsymbol{R}$-linear mappings $\xi\colon C^\infty(M) \to C^\infty(M, N) = N \otimes C^\infty(M)$ such that

(1) $$\xi(f \,.\, g) = g \,.\, \xi(f) + f \,.\, \xi(g) + \xi(f) \,.\, \xi(g) \quad \text{holds}\,.$$

$\boldsymbol{R}$-linear mappings satisfying (1) will be called *expansions* (if N is generated by δ with $\delta^{k+1} = 0$, then these are parametrized Taylor expansions). The relation between $X \in \mathfrak{X}_F(M)$ and

$$D_X f = D_X^F f = \xi\colon C^\infty(M) \to N \otimes C^\infty(M)$$

is given by

(2) $$F(f) \circ X = f \,.\, 1 + D_X(f) = (f \,.\, 1, D_X f) \quad \text{or}$$
$$f(x) \,.\, 1 + D_X(f)(x) = F(f)(X(x)) = \eta_{M,A}(X(x))(f)$$

(see 2.4).

If $a \in N$ and X is a smooth vector field on M, then

$$a \otimes X \in \mathfrak{X}_{T*F}(M)\,, \quad (T * F)(f) \circ (a \otimes X) =$$
$$= (\mathrm{Id}_N \otimes Tf)(a \otimes X) = f \,.\, 1 + a \,.\, df(X) = f \,.\, 1 + a \,.\, X(f)$$

on the one hand, and equals $f \,.\, 1 + D_{a \otimes X}(f)$ by (2), so

(3) $$D_{a\otimes X}^{T*F}(f) = D_{a \otimes X}(f) = a \,.\, X(f) = a \,.\, \mathrm{d}f(X)\,.$$

Clearly $\mathfrak{X}_{T*F}(M)$ is isomorphic to the space of all $\boldsymbol{R}$-linear $\xi\colon C^\infty(M) \to N \otimes C^\infty(M)$ with

(4) $$\xi(f \,.\, g) = \xi(f) \,.\, g + f \,.\, \xi(g)\,, \quad \text{called } \textit{derivations}, \text{ by } 4.5\,.$$

Now denote $\mathfrak{L} := \mathrm{Lin}_{\boldsymbol{R}}(C^\infty(M), N \otimes C^\infty(M))$ for short, and for $\xi, \eta \in \mathfrak{L}$ define $\xi \,.\, \eta \in \mathfrak{L}$ by

(5) $\xi . \eta = (m \otimes \mathrm{Id}_{C^\infty(M)}) \circ (\mathrm{Id}_N \otimes \xi) \circ \eta \colon C^\infty(M) \to N \otimes C^\infty(M) \to N \otimes N \otimes$ $\otimes C^\infty(M) \to N \otimes C^\infty(M)$, where $m \colon N \otimes N \to N$ is the (nilpotent) multiplication on N.

4.8. Lemma. 1. *$\mathfrak{L}$ is a real nilpotent algebra with multiplication* 4.7.5, *not commutative if $m \neq 0$, and without unit.*

2. *For $X, Y \in \mathfrak{X}_F(M)$ we have*

$$D_{X \square Y} = D_X . D_Y + D_X + D_Y .$$

3. *For $X, Y \in \mathfrak{X}_{T*F}(M)$ we have*

$$D_{[X,Y]_{T*F}} = D_X . D_Y - D_Y . D_X .$$

4. *For $\xi \in \mathfrak{L}$ define $\overline{\exp}(\xi) = \sum_{i=1}^{\infty} (1/i!)\, \xi^i$ and $\overline{\log}(\xi) = \sum_{i=1}^{\infty} ((-1)^{i-1})/i)\, \xi^i$. Then $\overline{\exp}, \overline{\log} \colon \mathfrak{L} \to \mathfrak{L}$ are bijective and inverse to each other and $\overline{\exp}(\xi)$ satisfies* 4.7.1 *if and only if ξ satisfies* 4.7.4. *Note that $i = 0$ lacks in the formulas, since we have no unit.*

Proof. 1.

$$\begin{aligned}
\xi . (\eta . \zeta) &= (m \otimes \mathrm{Id}_{C^\infty(M)}) \circ (\mathrm{Id}_N \otimes \xi) \circ (\eta . \zeta) \\
&= (m \otimes \mathrm{Id}) \circ (\mathrm{Id}_N \otimes \xi) \circ (m \otimes \mathrm{Id}) \circ (\mathrm{Id}_N \otimes \eta) \circ \zeta \\
&= (m \otimes \mathrm{Id}) \circ (m \otimes \mathrm{Id}_N \otimes \mathrm{Id}) \circ (\mathrm{Id}_{N \otimes N} \otimes \xi) \circ (\mathrm{Id}_N \otimes \eta) \circ \zeta \\
&= (m \otimes \mathrm{Id}) \circ (\mathrm{Id}_N \otimes m \otimes \mathrm{Id}) \circ (\mathrm{Id}_N \otimes \mathrm{Id}_N \otimes \xi) \circ (\mathrm{Id}_N \otimes \eta) \circ \zeta \\
&\qquad\qquad (m \text{ is associative}) \\
&= (m \otimes \mathrm{Id}) \circ (\mathrm{Id}_N \otimes [(m \otimes \mathrm{Id}) \circ (\mathrm{Id}_N \circ \xi) \circ \eta]) \circ \zeta = (\xi . \eta) . \zeta .
\end{aligned}$$

So the multiplication is associative. Clearly it is $\boldsymbol{R}$-bilinear. The order of nilpotence equals that of N.

2. Recall from 3.7 (and 4.3) that

$$\begin{aligned}
F^2(\boldsymbol{R}) = A \otimes A &= (\boldsymbol{R} . 1 \otimes \boldsymbol{R} . 1 \oplus \boldsymbol{R} . 1 \otimes N) \oplus (N \otimes \boldsymbol{R} . 1 \oplus N \otimes N) \cong \\
&\cong A \oplus F(N) \cong F(\boldsymbol{R} . 1) \times F(N) = F(\boldsymbol{R} . 1 \times N) = F(A) .
\end{aligned}$$

We will use this decomposition in exactly this order in the following computation

$$\begin{aligned}
f . 1 + D_{X \square Y}(f) &= F(f) \circ (X \square Y) = \\
&= F(f) \circ \mu_{F,M} \circ F(Y) \circ X && \text{by 4.6.1} \\
&= \mu_{F,\boldsymbol{R}} \circ F^2(f) \circ F(Y) \circ X \text{ since } \mu \colon F^2 \to F \text{ is natural} \\
&= m \circ F(F(f) \circ Y) \circ X \\
&= m \circ F((f . 1, D_Y f)) \circ X && \text{by 4.7.2} \\
&= m \circ (1 \otimes F(f) \circ X \oplus F(D_Y(f)) \circ X) \\
&= m \circ (1 \otimes (f . 1 + D_X(f)) + (D_Y(f) \otimes 1 + (\mathrm{Id}_N \otimes D_X)(D_Y(f))) \\
&= f . 1 + D_X(f) + D_Y(f) + (D_X . D_Y)(f) .
\end{aligned}$$

3. For vector fields X, Y on M and $a, b \in N$ we have

$$\begin{aligned} D_{[a\otimes X, b\otimes Y]_{T*F}}(f) &= D_{a.b\otimes[X,Y]}(f) && \text{by 4.6.2} \\ &= a\,.\,b\,.\,[X, Y]\,(f) && \text{by 4.7.3} \\ &= a\,.\,b\,.\,(X(Y(f)) - Y(X(f))) \\ &= (m \otimes \mathrm{Id}_{C^\infty(M)})\,(\mathrm{Id}_N \otimes D_{a\otimes X})\, D_{b\otimes Y}(f) = \dots \\ &= (D_{a\otimes X}\,.\,D_{b\otimes Y} - D_{b\otimes Y}\,.\,D_{a\otimes X})\,(f)\,. \end{aligned}$$

4. After adjoining a unit to $\mathfrak{L}$ we see that $\overline{\exp}\,(\xi) = e^\xi - 1$, $\overline{\log}\,(\xi) = \log(1 + \xi)$. So $\overline{\exp}$, $\overline{\log}$ are inverse to each other in the ring of formal power series of 1 variable. 1 and ξ generate a quotient of the power series ring in $\boldsymbol{R}1 \oplus \mathfrak{L}$ and the formal expressions $\overline{\exp}$ and $\overline{\log}$ commutate with taking quotients. So $\overline{\exp} = \overline{\log}^{-1}$.

The second assertion follows by a direct (formal) computation, or from the observation 4.9 below. qed.

4.9. Consider the $\boldsymbol{R}$-linear mapping C of $\mathfrak{L}$ into the ring of all $\boldsymbol{R}$-linear endomorphisms of the algebra $A \otimes C^\infty(M)$,

$$C_\xi = m \circ (\mathrm{Id}_A \otimes \xi)\colon A \otimes C^\infty(M) \to$$
$$\to A \otimes N \otimes C^\infty(M) \subset A \otimes A \otimes C^\infty(M) \to A \otimes C^\infty(M)\,,$$

where $m\colon A \otimes A \to A$ is the multiplication, so $C_\xi(a \otimes f) = a\,.\,\xi(f)$.

Lemma. 1. *$C_{\xi.\eta} = C_\xi \circ C_\eta$, so C is an algebra homomorphism.*

2. *$\xi \in \mathfrak{L}$ is an expansion if and only if* $\mathrm{Id} + C_\xi$ *is an automorphism of $A \otimes \otimes C^\infty(M)$.*

3. *$\xi \in \mathfrak{L}$ is a derivation if and only if C_ξ is a derivation of the algebra $A \otimes \otimes C^\infty(M)$.*

The proof is obvious.

Now let $\eta = \overline{\exp}\,(\xi) = \sum_{i=1}^{\infty} (1/i!)\, \xi^i$ in $\mathfrak{L}$.

Then η is an expansion (so $\eta = D_Y$ for $Y \in \mathfrak{X}_F(M)$)

$\Leftrightarrow 1 + C_\eta = \sum_{i=0}^{\infty} (1/i!)\,(C_\xi)^i$ is an automorphism (C_ξ is nilpotent)

$\Leftrightarrow C_\xi$ is a derivation

$\Leftrightarrow \xi$ is a derivation in $\mathfrak{L}$, so $\xi = D_X$ for $X \in \mathfrak{X}_{T*F}(M)$.

4.10. Proof of theorem 4.6. 1. Note first that $\mathfrak{L}$ is group with multiplication $\xi \mathbin{\square} \eta = \xi\,.\,\eta + \xi + \eta$, identity 0, and inverse $\eta = \sum_{i=1}^{\infty} (-\xi)^i$. $D\colon \mathfrak{X}_F(M) \to \mathfrak{L}$ is an isomorphism onto the subgroup of expansions, because $\mathrm{Id} + C \circ D\colon \mathfrak{X}_F(M) \to \mathfrak{L} \to \to \mathrm{End}\,(A \otimes C^\infty(M))$ is an isomorphism onto the subgroup of automorphisms.

2. Is obvious. $C \circ D\colon \mathfrak{X}_{T*F}(M) \to \mathrm{End}\,(A \otimes C^\infty(M))$ is a Lie algebra isomorphism onto the sub Lie algebra of derivations of $A \otimes C^\infty(M)$.

3. Define $\exp(X)$ by $D_{\exp(X)} = \overline{\exp}\, D_X$, $X \in \mathfrak{X}_{T*F}(M)$. The Campell-Baker-Hausdorff formula holds in $\mathfrak{L}$. qed.

4.11. Now we come again back to the functor T and its iterates. The material in 4.5.—4.10. gives only back the addition of vector fields, so for $X, Y \in \mathfrak{X}_T(M)$ we have $X \square Y = X + Y$, $[X, Y]_{T*F} = 0$, since $T * T = T$.

We will develop more structure now.

$\mathrm{Al}(T) = \boldsymbol{R} . 1 + \boldsymbol{R} . \delta$, $\delta^2 = 0$. So $N \cong \boldsymbol{R}$ now, with the nilpotent multiplication zero, but we have still the usual multiplication, now called m, on $\boldsymbol{R}$.

For $X, Y \in \mathfrak{X}_T(M)$ we have (4.7.2 with slightly different notation) $D_X \in \mathfrak{L} = \mathrm{Lin}_{\boldsymbol{R}}(C^\infty(M), C^\infty(M))$, defined by $f . 1 + D_X(f) . \delta = Tf \circ X$. Thus $D_X(f) = X(f) = df(X)$ in the usual sense. $\mathfrak{L}$ has one more structure now, *composition* (given by specifying a generator δ of $\mathrm{Al}(T)$). The usual Lie bracket $[X, Y]$ of vector fields now is given by $D_{[X,Y]} = D_X \circ D_Y - D_Y \circ D_X$.

Lemma. $TY \circ X \ -T \ \varkappa_T \circ TX \circ Y = V \circ [X, Y] \ T+ \ OT \circ Y.$

Remark. One may also express $[X, Y]$ by using the *vertical projection* ϱ: Ver $\to T$, where Al(Ver) is the subalgebra of $\mathrm{Al}(T^2)$ generated by δ_1 and $\delta_1\delta_2$, and $\varrho(\delta_1) = 0$, $\varrho(\delta_1\delta_2) = \delta$. So $\mathrm{Ver}(M)$ is the vertical subbundle of T^2M. Then the formula above is equivalent to (with a little abuse of notation):

$$[X, Y] = \varrho \circ (TY \circ X \ -T \ \varkappa \circ TX \circ Y)$$

Proof. For $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}_T(M)$ we have:

$$\begin{aligned}
T^2f \circ TY \circ X &= T(Tf \circ Y) \circ X = T(f . 1 \oplus D_Y f . \delta_1) \circ X \\
&= Tf \circ X . 1 \oplus T(D_Y(f)) \circ X . \delta_1 \\
&= f . 1 + D_X f . \delta_2 + (D_Y(f) . 1 + D_X D_Y f . \delta_2) . \delta_1 \\
&= f . 1 + D_Y f . \delta_1 + D_X f . \delta_2 + D_X D_Y f . \delta_1\delta_2 .
\end{aligned}$$

$$\begin{aligned}
&T^2 f \circ (-T) \circ (TY \circ X, \varkappa_T \circ TX \circ Y) = \\
&= (-T) \circ (T^2 f \circ TY \circ X, \ \varkappa_T \circ T^2 f \circ TX \circ Y) \\
&= (-T) \circ (f . 1 + D_Y f . \delta_1 + D_X f . \delta_2 + \\
&\quad + D_X D_Y f . \delta_1\delta_2, \varkappa_T(f . 1 + D_X f . \delta_1 + D_Y f . \delta_2 + D_Y D_X f . \delta_1\delta_2) \\
&= (-T) \circ (f . 1 + D_Y f . \delta_1 + D_X f . \delta_2 + D_X D_Y f . \delta_1\delta_2, f . 1 + \\
&\quad + D_Y f . \delta_1 + D_X f . \delta_2 + D_Y D_X f . \delta_1\delta_2) \\
&= f . 1 + D_Y f . \delta_1 + (D_X D_Y f - D_Y D_X f) . \delta_1\delta_2 \\
&= (T+) \circ (OT \circ (f . 1 + D_Y f . \delta), \ V \circ (f . 1 + D_{[X,Y]} f . \delta)) \\
&= (T+) \circ (OT \circ Tf \circ Y, \ V \circ Tf \circ [X, Y]) \\
&= (T+) \circ (T^2 f \circ OT \circ Y, \ T^2 f \circ V \circ [X, Y]) \\
&= T^2 f \circ (T+) \circ (OT \circ Y, \ V \circ [X, Y]) .
\end{aligned}$$

qed.

4.12. A *connector* K on a manifold M is a mapping $K\colon T^2M \to TM$ which is linear for both vector bundle structures on T^2M and is a left inverse to the natural vertical lift $V_M: TM \to T^2M: K \circ V = \mathrm{Id}$. K defines a *covariant derivative* $\nabla: \mathfrak{X}_T(M) \times \times \mathfrak{X}_T(M) \to \mathfrak{X}_T(M)$ by $\nabla_X Y := K \circ TY \circ X$.

The projection $(\pi T, T\pi): T^2M \to TM \times_M TM$, restricted to the kernel of K, becomes bijective, and its inverse gives the *connection* $C: TM \times_M TM \to T^2M$ of K, where $C(\cdot, v)$ is fibre linear for the $T+$-structure, and $C(v, \cdot)$ is fibre linear for the $+T$-structure on T^2M.

Corollary. 1. *The Riemannian curvature of* ∇ *is then given by*

$$R(X, Y)Z = (K \circ TK \circ \varkappa T - K \circ TK) \circ T^2Z \circ TX \circ Y.$$

2. *The torsion is given by* $\mathrm{Tor}(X, Y) = (K \circ \varkappa - K) \circ TX \circ Y$.

Proof. 1. Note first that $\mathrm{d}/\mathrm{d}\lambda|_{\lambda=0}\, m_\lambda = V: T \to T^2$, where $m_\lambda: T \to T$ is the fibre scalar multiplication by λ.

$$\text{So } V \circ K = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} m_\lambda \circ K =$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} K \circ m_\lambda T, \text{ since } K \text{ is linear}$$
$$= TK \circ \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} m_\lambda T$$
$$= TK \circ VT.$$

$$\begin{aligned} R(X, Y)Z &= \nabla_X\nabla_Y Z = \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \\ &= K \circ T(K \circ TZ \circ Y) \circ X - K \circ T(K \circ TZ \circ X) \circ Y - K \circ TZ \circ [X, Y]. \end{aligned}$$

$$\begin{aligned} K \circ TZ \circ [X, Y] &= K \circ V \circ K \circ TZ \circ [X, Y] = \\ &= K \circ TK \circ VT \circ TZ \circ [X, Y] \\ &= K \circ TK \circ T^2Z \circ V \circ [X, Y] \\ &= K \circ TK \circ T^2Z \circ ((TY \circ X(-T)\varkappa \circ TX \circ Y)(T-)\, OT \circ Y) \qquad \text{by 4.11} \\ &= K \circ TK \circ T^2Z \circ TY \circ X - K \circ TK \circ T^2Z \circ \varkappa \circ TX \circ Y - 0 \end{aligned}$$

Summing up the formula follows.

$$\begin{aligned} 2.\ \mathrm{Tor}(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= K \circ TY \circ X - K \circ TX \circ Y - K \circ V \circ [X, Y] \end{aligned}$$

$$\begin{aligned} K \circ V \circ [X, Y] &= K \circ ((TY \circ X(-T)\varkappa \circ TX \circ Y)'(T-)\, OT \circ Y) \\ &= K \circ TY \circ X - K \circ \varkappa \circ TX \circ Y - 0 \qquad \text{qed.} \end{aligned}$$

So the formula follows.

4.13. Now we apply the general theory of 4.6.–4.10. for the functor $F = T^2$. So in the following we put:

$\mathfrak{X}, \mathfrak{D}, \mathfrak{B} \in \mathfrak{X}_{T^2}(M)$, sections of $T^2M \to M$, X, Y, Z vector fields on M.

$\square$ ... Multiplication in $\mathfrak{X}_{T^2}(M)$.

$A = \mathrm{Al}(T^2)$, generators δ_1, δ_2, with $\delta_1^2 = \delta_2^2 = 0$.
$N = \mathfrak{N}(T^2)$. $\overline{N}$, the underlying subspace of N, with zero multiplication, generators v_1, v_2, v_3.
$\exp = \exp^{T^2}: \mathfrak{X}_{T*T^2}(M) = \mathfrak{X}_{N\otimes T}(M) \to \mathfrak{X}_{T^2}(M)$.
$[\,,\,]_{T*T^2}$ ... the Lie bracket in $\mathfrak{X}_{T*T^2}(M)$.
$[\,,\,]$ the usual Lie bracket in $\mathfrak{X}_T(M)$.
$\mathfrak{L} = \mathrm{Lin}_{\boldsymbol{R}}(C^\infty(M), N \otimes C^\infty(M))$.

1. Clearly $D_{\mathfrak{X}}^{T^2}(f) = D_{\mathfrak{X}}(f) = D_{\mathfrak{X}}^1 f \,.\, \delta_1 + D_{\mathfrak{X}}^1 f \,.\, \delta_2 + D_{\mathfrak{X}}^2 f \,.\, \delta_1\delta_2$, since $\delta_1, \delta_2, \delta_1\delta_2$ are an $\boldsymbol{R}$-linear basis of N.

$\pi T \circ (f\,.\,1 + D_{\mathfrak{X}}^{T^2}(f)) = \pi T \circ T^2 f \circ \mathfrak{X} = Tf \circ \pi T \circ \mathfrak{X} = f\,.\,1 + D_{\pi T\circ\mathfrak{X}}^T(f)$, so $D_{\mathfrak{X}}^1 = D_{\pi T\circ\mathfrak{X}}^T$, and similarly $D_{\mathfrak{X}}^2 = D_{T\pi\circ\mathfrak{X}}^T$.

More general, for $\xi \in \mathfrak{L}$ we have

$$\xi(f) = \xi^1(f)\,.\,\delta_1 + \xi^2(f)\,.\,\delta_2 + \xi^3(f)\,.\,\delta_1\delta_2\,.$$

Then $(\xi\,.\,\eta)(f) = (\xi^1\eta^2(f) + \xi^2\eta^1(f))\,.\,\delta_1\delta_2$.

So

$$\begin{aligned} D_{\mathfrak{X}\square\mathfrak{D}}^{T^2}(f) &= (D_{\mathfrak{X}}\,.\,D_{\mathfrak{D}} + D_{\mathfrak{X}} + D_{\mathfrak{D}})(f) = \\ &= ((D_{\pi T\circ\mathfrak{X}}^T \circ D_{T\pi\circ\mathfrak{D}}^T + D_{T\pi\circ\mathfrak{X}}^T \circ D_{\pi T\circ\mathfrak{D}}^T)(f) + D_{\mathfrak{X}}^3(f) + D_{\mathfrak{D}}^3(f))\,\delta_1\delta_2 \\ &\quad + (D_{\pi T\circ\mathfrak{X}}^T + D_{\pi T\circ\mathfrak{D}}^T)(f)\,.\,\delta_1 + (D_{T\pi\circ\mathfrak{X}}^T + D_{T\pi\circ\mathfrak{D}}^T)(f)\,.\,\delta_2\,. \end{aligned}$$

Finally

$$D_{\mathfrak{X}}^3(f\,.\,g) = f\,.\,D_{\mathfrak{X}}^3(g) + g\,.\,D_{\mathfrak{X}}^3(f) + D_{\pi T\circ\mathfrak{X}}^T(f)\,.\,D_{T\pi\circ\mathfrak{X}}^T(g) + D_{T\pi\circ\mathfrak{X}}^T(f)\,.\,D_{\pi T\circ\mathfrak{X}}^T(g)$$

from the expansion property of $D_{\mathfrak{X}}^{T^2}$.

2. Let $\overline{X} = X_1v_1 + X_2v_2 + X_3v_3 \in \mathfrak{X}_{N\otimes T}(M)$. Then

$$D_{\overline{X}}^{T*T^2}(f) = D_{X_1}^T(f)\,.\,\delta_1 + D_{X_2}^T(f)\,.\,\delta_2 + D_{X_3}^T(f)\,.\,\delta_1\delta_2\,,$$

so

$$\begin{aligned} D_{[\overline{X},\overline{Y}]_{T*T^2}}(f) &= (D_{\overline{X}}\,.\,D_{\overline{Y}} - D_{\overline{Y}}\,.\,D_{\overline{X}})(f) \\ &= (D_{X_1}^T D_{Y_2}^T + D_{X_2}^T D_{Y_1}^T - D_{Y_1}^T D_{X_2}^T - D_{Y_2}^T D_{X_1}^T)(f)\,\delta_1\delta_2 \\ &= D_{[X_1,Y_2]+[X_2,Y_1]}^T(f)\,.\,\delta_1\delta_2\,. \end{aligned}$$

Thus we have finally

$$[X_1v_1 + X_2v_2 + X_3v_3,\, Y_1v_1 + Y_2v_2 + Y_3v_3]_{T*T^2} = ([X_1, Y_2] + [X_2, Y_1])\,v_3\,.$$

3.

$$\begin{aligned} D_{\exp^{T^2}\overline{X}}^{T^2} &= D_{\overline{X}} + (1/2!)\,D_{\overline{X}}\,.\,D_{\overline{X}} + (1/3!)\,D_{\overline{X}}\,.\,D_{\overline{X}}\,.\,D_{\overline{X}} + \dots \\ &= D_{X_1}^T \otimes \delta_1 + D_{X_2}^T \otimes \delta_2 + (D_{X_3}^T + \tfrac{1}{2}(D_{X_1}^T D_{X_2}^T + D_{X_2}^T D_{X_1}^T)) \otimes \delta_1\delta_2\,. \end{aligned}$$

4.

$$\begin{aligned} f\,.\,1 + D_{TX\circ Y}^{T^2}(f) &= T^2 f \circ TX \circ Y = T(Tf\,.\,X)\,.\,Y \\ &= T(f\,.\,1 + D_X^T(f)\,.\,\delta_1) \circ Y \\ &= f\,.\,1 + D_X^T(f)\,.\,\delta_1 + D_Y^T(f\,.\,1 + D_X^T(f)\,.\,\delta_1)\,.\,\delta_2 \\ &= f\,.\,1 + D_X^T(f)\,.\,\delta_1 + D_Y^T(f)\,.\,\delta_2 + D_Y^T D_X^T(f)\,\delta_1\delta_2\,. \end{aligned}$$

So $D^{T^2}_{TX\cdot Y} = D^T_X \otimes \delta_1 + D^T_Y \otimes \delta_2 + D^T_Y D^T_X \otimes \delta_1\delta_2$.

Now we use 3 to compute

$$D^{T^2}_{\exp^{T^*}(X v_1 + Y v_2 - \frac{1}{2}[X,Y] v_3)} =$$

$$= D^T_X \otimes \delta_1 + D^T_Y \otimes \delta_2 + \left(-\tfrac{1}{2}D^T_{[X,Y]} + \tfrac{1}{2}(D^T_X D^T_Y + D^T_Y D^T_X)\right) \otimes \delta_1\delta_2 = D^{T^2}_{TX\cdot Y}$$

and get finally the formula

$$TX \circ Y = \exp^{T^2}(X.\, v_1 + Y.\, v_2 - \tfrac{1}{2}[X, Y]\, v_3)\,.$$

5. We compute $(TX \circ Y) \square (TY \circ X)$ with the help of the Campell-Baker-Hausdorff formula now:

$$\begin{aligned}(TX \circ Y) \square (TY \circ X) &= \exp\left(X\,.\, v_1 + Y.\, v_2 - \tfrac{1}{2}[X, Y]\,.\, v_3 + Y.\, v_1 + X\,.\, v_2 - \right.\\ &\quad \left. - \tfrac{1}{2}[Y, X]\, v_3 + ([X, X] + [Y, Y])\, v_3 + 0\right)\\ &= \exp\left((X + Y)\, v_1 + (X + Y)\, v_2\right)\end{aligned}$$

$$\begin{aligned}D^{T^2}_{(TX\circ Y)\square(TY\circ X)} &= D^{T^2}_{\exp((X+Y)v_1+(X+Y)v_2)} =\\ &= D^T_{X+Y} \otimes \delta_1 + D^T_{X+Y} \otimes \delta_2 + D^T_{X+Y} D^T_{X+Y} \otimes \delta_1\delta_2\,.\end{aligned}$$

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