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REPRESENTATION OF OPERATORS BY BILINEAR INTEGRALS

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Let us consider a locally compact Hausdorff topological space T and two complete locally convex Hausdorff spaces X and Z. Denote by $\mathscr{B}$ the family of all non empty bounded closed balanced and convex subsets of X, and by X_B the linear subspace of X generated by B $(\in\mathscr{B})$ equipped with the Minkowski functional q_B of B. The problem to be solved here is the following: Let $\mathscr{C}_B = \mathscr{C}_B(T, X_B)$ $(B \in \mathscr{B})$ be the space of all continuous functions tending to zero at infinity $f\colon T \to X_B$ endowed with the usual supremum norm

$$\|f\|_B = \sup \{q_B[f(t)]\colon t \in T\} , \tag{1}$$

$\mathscr{C} = \bigcup\{\mathscr{C}_B\colon B \in \mathscr{B}\}$ and $\mathscr{F}\colon \mathscr{C} \to Z$ a linear operator with continuous restrictions $\mathscr{F}_B = \mathscr{F} \mid \mathscr{C}_B$. The main object of this paper is to represent $\mathscr{F}$ by a bilinear integral. To this end we will consider the space Y of the linear mappings from X into Z with continuous restrictions to X_B, for all $B \in \mathscr{B}$, and the evaluation from $X \times Y$ into Z will be represented by xy $(x \in X,\ y \in Y)$.

If $\mathscr{R}$ is a generating family of seminorms on Z, for every $r \in \mathscr{R}$, $B \in \mathscr{B}$ and $y \in Y$, let us set

$$q_{B,r}(y) = \sup \{r(xy)\colon x \in B\} . \tag{2}$$

It is easily proved that $\{q_{B,r}\colon B \in \mathscr{B},\ r \in \mathscr{R}\}$ is a saturated family of seminorms defining on Y a topology (which henceforth will be the topology supposed to be defined on Y), making the evaluation mapping $X \times Y \to Z$ hypocontinuous.

Let Σ be the Borel σ-algebra of T and $\mu\colon \Sigma \to Y$ a countable additive measure. We define the semivariation $\|\mu\|_{B,r}$ and the variation $|\mu|_{B,r}$ $(B \in \mathscr{B},\ r \in \mathscr{R})$ in the usual way:

$$\|\mu\|_{B,r}(E) = \sup r\big(\sum_{i\in\mathscr{H}} x_i\, \mu(E_i)\big) \quad (E \in \Sigma) , \tag{3}$$

where the supremum is taken over all finite partitions $\{E_i\}_{i\in\mathscr{H}} \subset \Sigma$ of E and all finite families $\{x_i\}_{i\in\mathscr{H}} \subset B$, and

$$|\mu|_{B,r}(E) = \sup \sum_{C\in\pi} q_{B,r}[\mu(C)] \quad (E \in \Sigma) , \tag{4}$$

where the supremum is taken over all finite partitions $\pi \subset \Sigma$ of E.

A set $A \in \Sigma$ is said to be a *null set* if $\|\mu\|_{B,r}(A) = 0$ for all $B \in \mathscr{B}$ and $r \in \mathscr{R}$.

We will denote by $\mathscr{S} = \mathscr{S}(T, X)$ and $\mathscr{S}_B = \mathscr{S}_B(T, X_B)$ $(B \in \mathscr{B})$ the spaces of simple functions from T into X and X_B, respectively, and by $\mathscr{C} + \mathscr{S}$ or $\mathscr{C}_B + \mathscr{S}_B$ the algebraic sum of $\mathscr{C}$ and $\mathscr{S}$ or $\mathscr{C}_B$ and $\mathscr{S}_B$, respectively.

Definition 1. Let $\{\nu_{B,r}: B \in \mathscr{B},\ r \in \mathscr{R}\}$ be a family of positive and finite measures defined on Σ, and $\mu: \Sigma \to Y$ a countable additive measure. We say that μ is $(\nu_{B,r})$-*continuous* if

$$(5) \qquad \lim_{\nu_{B,r}(E) \to 0} \|\mu\|_{B,r}(E) = 0 .$$

In the case of μ of bounded variation, it is easily proved that μ is $\{|\mu|_{B,r}: B \in \mathscr{B}, r \in \mathscr{R}\}$-continuous.

Henceforth we will suppose to be given a fixed family $\{\nu_{B,r}: B \in \mathscr{B},\ r \in \mathscr{R}\}$ of positive and finite measures defined on Σ.

For the spaces X, Y and Z, and the evaluation mapping, the bilinear integral used here (with analogous properties as the bilinear integral given by Sivasankara in [14]) can be defined in the following way ([14]): Let $\mu: \Sigma \to Y$ be a $(\nu_{B,r})$-continuous measure.

A sequence of functions $f_n: T \to X$ is said to be *B-convergent* $(B \in \mathscr{B})$ to $f: T \to X$ if

$$\bigcup_{n=1}^{\infty} f_n(T) \cup f(T) \subset X_B$$

and $q_B(f_n - f) \to 0$ a.e..

A function $f: T \to X$ is said to be *B-measurable* $(B \in \mathscr{B})$ if $f(T) \subset X_B$ and there exists a sequence of simple functions (simple functions are defined as usual) which is B-convergent to f, and a function $g: T \to X$ is said to be measurable if it is B-measurable for some $B \in \mathscr{B}$.

We will say that a function $f: T \to X$ is *B-integrable* $(B \in \mathscr{B})$ if $f(T) \subset X_B$ and there exists a sequence (f_n) of simple functions which is B-convergent to f and for every $\varepsilon > 0$ and $r \in \mathscr{R}$ there exists $\delta = \delta(\varepsilon, r) > 0$ such that

$$r\left(\int_A f_n \,\mathrm{d}\mu\right) < \varepsilon$$

holds for all $n \in \mathbb{N}$ and every $A \in \Sigma$ with $\|\mu\|_{B,r}(A) < \delta$ (the integral of a simple function is defined as usual). A sequence (f_n) of the above type is called an approximating sequence of f.

A function $f: T \to X$ is said to be *integrable* if it is B-integrable for some $B \in \mathscr{B}$. It can be proved ([14]) that if $f: T \to X$ is integrable then the limit

$$\int_A f \,\mathrm{d}\mu = \lim_n \int_A f_n \,\mathrm{d}\mu$$

exists for every $A \in \Sigma$ and every approximating sequence (f_n) of f, and it is independent of the choice of the approximating sequence of f.

Definition 2. A linear operator $\mathscr{F}: \mathscr{C} \to Z$ is said to be $(\nu_{B,r})$-*continuous* if for every $B \in \mathscr{B}$, $r \in \mathscr{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $E \in \Sigma$ with $\nu_{B,r}(E) < \delta$, $r[\mathscr{F}(f)] < \varepsilon$ holds for all $f \in \mathscr{C}_B$ with $f(T) \subset B$ and $f \mid T - E \equiv 0$.

Analogous definitions can be given for linear Z-valued operators defined on $\mathcal{S}$ or $\mathcal{C} + \mathcal{S}$.

Proposition 3. *Let μ be a $(v_{B,r})$-continuous measure, then all functions belonging to $\mathcal{C}$ are μ-integrable and the linear functional $\mathcal{F}: \mathcal{C} \to Z$ defined by*

$$\mathcal{F}(f) = \int_T f \, d\mu$$

is (v_{Br})-continuous and its restrictions $\mathcal{F}_B$ are continuous.

Proof. Let $f \in \mathcal{C}$, then there exists $B \in \mathcal{B}$ with $f \in \mathcal{C}_B$. Let us prove that f is (μ, B)-integrable (and so, integrable).

$f: T \to X_B$ has a continuous extension $\bar{f}: \bar{T} \to X_B$ ($\bar{T}$ being Alexandroff's compactification of T) given by $\bar{f}(\infty) = 0$, and then $\bar{f}(T)$ is compact and therefore there exist $t_1, \ldots, t_n \in \bar{T}$ such that

$$f(T) \subset \bar{f}(T) \subset \bigcup_1^n B(\bar{f}(t_k), \varepsilon),$$

where $B(\bar{f}(t_k), \varepsilon)$ is the closed ball with center $\bar{f}(t_k)$ and radius ε.

Consider $A_1 = B(\bar{f}(t_1), \varepsilon)$, $A_2 = B(\bar{f}(t_2), \varepsilon) - A_1, \ldots, A_n = B(\bar{f}(t_n), \varepsilon) - \bigcup_{k=1}^{n-1} A_k$, $E_k = f^{-1}(A_k)$ and

$$g_\varepsilon = \sum_{k=1}^{n} x_k \chi_{E_k}$$

with $x_k \in A_k$. Obviously, $f(T) \subset \bigcup_{k=1}^{n} A_k$, and $T = \bigcup_{k=1}^{n} E_k$, so if $z \in T$ then there exists $k \in \{1, \ldots, n\}$ such that $t \in E_k$ and $f(t) \in A_k$. Then we have $q_B(x_k - f(t)) \leqq \varepsilon$ or $q_B(g_\varepsilon(t) - f(t)) \leqq \varepsilon$. By taking $\varepsilon = 1/n$, for $n \in \mathbb{N}$, we obtain that f is the uniform limit of simple functions, where from it is easily deduced that f is (μ, B)-integrable.

Moreover, for $B \in \mathcal{B}$, $r \in \mathcal{R}$ and $f \in \mathcal{C}_B$ we have

$$r(\mathcal{F}_B(f)) = r(\int_T f \, d\mu) \leqq \|f\|_B \, \|\mu\|_{B,r}(T),$$

and therefore, $\mathcal{F}_B$ is continuous.

Finally, $\mathcal{F}$ is $(v_{B,r})$-continuous because for every $B \in \mathcal{B}$, $r \in \mathcal{R}$, $\varepsilon > 0$, $E \in \Sigma$ and $f \in \mathcal{C}_B$ with $f(T) \subset B$ and $f \mid T - E \equiv 0$ we have

$$r(\int_T f \, d\mu) = r(\int_E f \, d\mu) \leqq \|f\|_B \, \|\mu\|_{B,r}(E) \leqq \|\mu\|_{B,r}(E),$$

and so, if $\delta > 0$ is such that $v_{Br}(E) < \delta$ implies $\|\mu\|_{B,r}(E) < \varepsilon$, then

$$r(\mathcal{F}(f)) \leqq \|\mu\|_{B,r}(E) < \varepsilon.$$

Proposition 4. *A linear operator $\mathcal{F}: \mathcal{C} \to Z$ with continuous restrictions $\mathcal{F}_B$ is $(v_{B,r})$-continuous if and only if for every $B \in \mathcal{B}$, $r \in \mathcal{R}$ and $\varepsilon > 0$ there exist $\delta > 0$, $1 \geqq \delta' > 0$ such that for all $E \in \Sigma$ with $v_{B,r}(E) < \delta$, $r(\mathcal{F}(f)) < \varepsilon$ holds for all $f \in \mathcal{C}_B$ with $q_B(f(t)) \leqq \delta'$ for all $t \in T - E$* [1]).

Proof. Let us suppose that $\mathcal{F}$ is $(v_{B,r})$-continuous, then for every $B \in \mathcal{B}$, $r \in \mathcal{R}$

[1]) The same result can be proved for Z-valued operators defined on $\mathcal{S}$.

and $\varepsilon > 0$ there exist $\delta > 0$ and $1 \geqq \delta' > 0$ such that for all $E \in \Sigma$ with $v_{B,r}(E) < \delta$, $r(\mathscr{F}(f)) < \varepsilon/2$ holds for all $f \in \mathscr{C}_B$ with $f \mid T - E \equiv 0$ or $\|f\|_B \leqq \delta'$. Now let $f \in \mathscr{C}_B$ with $q_B(f(t)) \leqq \delta'$ for all $t \in T - E$, then there exist $g, h \in \mathscr{C}_B$ such that $\|g\|_B \leqq \delta'$, $h \mid T - E \equiv 0$ and $f = g + h$ and therefore,

$$r(\mathscr{F}(f)) < \varepsilon .$$

Notice that if $U = \{t \in T: q_B(f(t)) \leqq \delta'\}$ then we may set

$$g(t) = \begin{cases} f(t) & \text{if} \quad t \in U \\ \dfrac{\delta' f(t)}{q_B(f(t))} & \text{if} \quad t \notin U \end{cases}$$

and $h = f - g$.

Proposition 5. *Let μ and $\mathscr{F}$ be as in Proposition 3, then there is an extension $\mathscr{F}^s: \mathscr{C} + \mathscr{S} \to Z$ of $\mathscr{F}$ such that $\mathscr{F}^s$ is $(v_{B,r})$-continuous and its restrictions $\mathscr{F}^s_B: \mathscr{C}_B + \mathscr{S}_B \to Z$ are continuous for all $B \in \mathscr{B}$ (the topologies of $\mathscr{S}$ and $\mathscr{S}_B$ are defined by the norm (1)).*

Proof. Let us define

$$\mathscr{F}^s(f) = \int_T f \, \mathrm{d}\mu$$

for $f \in \mathscr{C} + \mathscr{S}$, then $\mathscr{F}^s_B$ $(B \in \mathscr{B})$ is continuous because

$$r(\int_T f \, \mathrm{d}\mu) \leqq \|f\|_B \, \|\mu\|_{B,r} (T)$$

for $r \in \mathscr{R}$ and $f \in \mathscr{C}_B + \mathscr{S}_B$. To prove that $\mathscr{F}^s$ is $(v_{B,r})$-continuous it is enough to proceed as in Proposition 3.

Theorem 6. *Let $\mathscr{F}: \mathscr{C} \to Z$ be a linear operator with continuous restrictions $\mathscr{F}_B$ for all $B \in \mathscr{B}$. Then the following assertions are equivalent:*

6.1. *There exists a $(v_{B,r})$-continuous countable additive measure $\mu: \Sigma \to Y$ such that*

$$\mathscr{F}(f) = \int_T f \, \mathrm{d}\mu$$

for all $f \in \mathscr{C}$.

6.2. *There exists a $(v_{B,r})$-continuous operator $\mathscr{F}^s: \mathscr{C} + \mathscr{S} \to Z$ with continuous restrictions $\mathscr{F}^s_B$ $(B \in \mathscr{B})$, which extends fo $\mathscr{F}$.*

6.3. *There exists a linear $(v_{B,r})$-continuous operator $\mathscr{G}: \mathscr{S} \to Z$ with continuous restrictions $\mathscr{G}_B$ $(B \in \mathscr{B})$, such that for every B*

$$\lim_n \mathscr{G}_B(f_n) = \mathscr{F}_B(f) \tag{6}$$

holds for every sequence $(f_n)_n \subset \mathscr{S}_B$ which is uniformly convergent to $f \in \mathscr{C}_B$.

Proof. From Propositions 3 and 5 it is immediately deduced that 6.1 implies 6.2. Moreover, 6.2 clearly implies 6.3. Let us prove that 6.3 implies 6.1. This will be done in four steps:

i) *Construction of* μ. Let $E \in \Sigma$ and define

$$\mu(E)(x) = \mathscr{G}(x\chi_E)$$

for $x \in X$. Then $\mu(E)$ is a linear operator from X into Z such that if $B \in \mathscr{B}$, $x \in X_B$ and the sequence $(x_n)_{n\in\mathbb{N}} \subset X_B$ converges to x, then the sequence $(x_n\chi_E)$ is uniformly convergent to $x\chi_E$ and therefore

$$\mu(E)(x) = \mathscr{G}(x\chi_E) = \lim_n \mathscr{G}(x_n\chi_E) = \lim_n \mu(E)(x)$$

and $\mu(E) \in Y$.

ii) μ *si countably additive.* The finite additivity of μ results trivially from the linearity of $\mathscr{G}$. Let now $(E_n) \subset \Sigma$ be a disjoint sequence, then

$$\mu(\bigcup_{i=1}^{\infty} E_i) - \sum_{i=1}^{n} \mu(E_i) = \mu(\bigcup_{i>n} E_i)$$

holds for every $n \in \mathbb{N}$, and given $r \in \mathscr{R}$, $B \in \mathscr{B}$ and $\varepsilon > 0$ it follows from the $(v_{B,r})$-continuity of $\mathscr{G}$ that there exists $n_0 \in \mathbb{N}$ such that

$$r[\mathscr{G}(x\chi_{\bigcup_{i\geqq n_0} E_i})] < \varepsilon$$

for all $x \in B$, so

$$q_{B,r}[\mu(\bigcup_{i\geqq n_0} E_i)] < \varepsilon$$

and therefore

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

iii) μ *is* $(v_{B,r})$*-continuous.* Let $r \in \mathscr{R}$, $B \in \mathscr{B}$ and $\varepsilon > 0$. Since $\mathscr{G}$ is $(v_{B,r})$-continuous there exists $\delta > 0$ such that

$$r[\mathscr{G}_B(f)] < \varepsilon$$

for all $f \in \mathscr{S}_B$ with $f \mid T - E \equiv 0$ for some $E \in \Sigma$ of measure $v_{B,r}(E) < \delta$. Therefore, if $E \in \Sigma$ and $v_{B,r}(E) < \delta$ then for every finite family $\{x_1, \ldots, x_n\} \subset B$ and every finite partition $\{E_1, \ldots, E_n\} \subset \Sigma$ of E, we have

$$r[\sum_{i=1}^{n} x_i \, \mu(E_i)] = r[\mathscr{G}(\sum_{i=1}^{n} x_i\chi_{E_i})] \leqq \varepsilon$$

and consequently,

$$\|\mu\|_{B,r}(E) \leqq \varepsilon.$$

iv) μ *represents* $\mathscr{F}$. Let $B \in \mathscr{B}$ and $f \in \mathscr{C}_B$. As in Proposition 3 we can find a sequence $(f_n) \subset \mathscr{S}_B$ which is uniformly convergent to f, and therefore,

$$\mathscr{F}(f) = \lim_n \mathscr{G}_B(f_n) = \lim_n \int_T f_n \, d\mu = \int_T f \, d\mu.$$

Theorem 7. *If* $\{v_{B,r}: B \in \mathscr{B}, r \in \mathscr{R}\}$ *are Radon measures (i.e. regular Borel measures) and* $\mathscr{F}$ *is as in Theorem* 6 *verifying* 6.1, *then the measure* μ *of* 6.1 *is unique.*

Proof. Suppose that there exist two $(\nu_{B,r})$-continuous measures $\mu, \mu' : \Sigma \to Y$ such that

$$\mathscr{F}(f) = \int_T f \, d\mu = \int_T f \, d\mu'$$

holds for all $f \in \mathscr{C}$. Then Proposition 5 implies the existence of two $(\nu_{B,r})$-contiunous extensions $\mathscr{F}^s$ and $\mathscr{F}^{s'}$ of $\mathscr{F}$ to $\mathscr{C} + \mathscr{S}$. If $x \in X$ and $E \in \Sigma$, let us consider $B \in \mathscr{B}$ with $x \in X_B$ and $r \in \mathscr{R}$ arbitrary. Then two sequences (K_n) and (G_n) of compact and open subsets of T, respectively, can be found such that $K_n \subset E \subset G_n$ and

$$\nu_{B,r}(G_n - K_n) \leqq 1/n$$

for all $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, let $f_n: T \to [0, 1]$ be a continuous function with $f \mid K_n \equiv 1$ and $\operatorname{supp}(f_n) \subset G_n$, then

$$r[\mu(E)(x) - \mu'(E)(x)] = r[\mathscr{F}^s(x\chi_E) - \mathscr{F}^{s'}(x\chi_E)] \leqq$$
$$\leqq r[\mathscr{F}^s(x\chi_E - xf_n)] + r[\mathscr{F}^s(xf_n) - \mathscr{F}^{s'}(xf_n)] + r[\mathscr{F}^{s'}(xf_n - x\chi_E)] .$$

Hence it results that $r[\mu(E)(x) - \mu'(E)(x)] = 0$ (and therefore $\mu(E) = \mu'(E)$) because

$$r[\mathscr{F}^{s'}(xf_n) - \mathscr{F}^{s'}(xf_n)] = r[\mathscr{F}(xf_n) - \mathscr{F}(xf_n)] = 0$$

and

$$\lim_n r[\mathscr{F}^s(x\chi_E - xf_n)] = \lim_n r[\mathscr{F}^{s'}(xf_n - x\chi_E)] = 0$$

since $x\chi_E - xf_n$ takes non zero values in $G_n - K_n$,

$$\lim_n \nu_{B,r}(G_n - K_n) = 0 ,$$

and $\mathscr{F}^s$ and $\mathscr{F}^{s'}$ are $(\nu_{B,r})$-continuous.

Theorem 8. *Let us suppose that the family $\{\nu_{B,r}: B \in \mathscr{B}, r \in \mathscr{R}\}$ is uniformly tight (i.e., given $E \in \Sigma$ and $\varepsilon > 0$ there exists a compact $K \subset T$ such that $K \subset E$ and $\nu_{B,r}(E - K) < \varepsilon$ for all $B \in \mathscr{B}$ and $r \in \mathscr{R}$), and let $\mathscr{F}: \mathscr{C} \to Z$ be a linear operator with continuous restrictions $\mathscr{F}_B$ $(B \in \mathscr{B})$. Then there exists a $(\nu_{B,r})$-continuous measure $\mu: \Sigma \to Y$ such that*

$$\mathscr{F}(f) = \int_T f \, d\mu$$

for all $f \in \mathscr{C}$, if and only if the operator $\mathscr{F}$ is $(\nu_{B,r})$-continuous. In this case the measure μ is unique.

Proof. If such a measure exists, then the $(\nu_{B,r})$-continuity of $\mathscr{F}$ follows from Proposition 3, and the uniqueness of μ is deduced from Theorem 7.

Let us suppose that $\mathscr{F}$ is $(\nu_{B,r})$-continuous, then we will prove that 6.3 holds. If $E \in \Sigma$ we can find an increasing sequence of compact subsets $(K_n) \subset T$ and a decreasing sequence of open subsets $(G_n) \subset T$ such that $K_n \subset E \subset G_n$ and

$$\nu_{B,r}(G_n - K_n) \leqq 1/n$$

for all $r \in \mathscr{R}$ and $B \in \mathscr{B}$. Let $f_n: T \to [0, 1]$ be a continuous function such that $f_n \mid K_n \equiv 1$ and $f_n \mid T - G_n \equiv 0$, for all $n \in \mathbb{N}$.

Define

$$\mathscr{G}_0(x\chi_E) = \lim_n \mathscr{F}(xf_n) \tag{7}$$

for all $x \in X$. Let us prove that this limit exists and that it is independent of the sequence (f_n). If $n, m \in \mathbb{N}$ are such that $n \leqq m$, then for every $B \in \mathscr{B}$ and $x \in B$, the function $xf_m - xf_n$ belongs to $\mathscr{C}_B$ and vanishes outside of $G_m - K_n$. Moreover, for every $r \in \mathscr{R}$ we have

$$\lim_{\substack{m,n\to\infty \\ m\geqq n}} v_{B,r}(G_m - K_n) = 0\,,$$

and the $(v_{B,r})$-continuity of $\mathscr{F}$ yields

$$\lim_{\substack{m,n\to\infty \\ m\geqq n}} r[\mathscr{F}(xf_m - xf_n)] = 0\,.$$

So $\{\mathscr{F}(xf_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore the limit (7) exists.

Let us now consider other sequences (K_n'), (G_n') and (f_n') satisfying the above conditions. If $B \in \mathscr{B}$ and $x \in B$, then the function $xf_n - xf_n' \in \mathscr{C}_B$ vanishes outside $(G_n \cup G_n') - (K_n \cap K_n')$ and

$$\lim_n v_{B,r}[(G_n \cup G_n') - (K_n \cap K_n')] = 0$$

holds for all $r \in \mathscr{R}$, and therefore the $(v_{B,r})$-continuity of $\mathscr{F}$ implies

$$\lim_n \mathscr{F}(xf_n) = \lim_n \mathscr{F}(xf_n')\,.$$

For a simple function $f = \sum_{i=1}^{n} x_i\chi_{E_i}$ let us define an operator

$$\mathscr{G}(f) = \sum_{i=1}^{n} \mathscr{G}_0(x_i\chi_{E_i})\,,$$

which is clearly linear and with continuous restrictions $\mathscr{G}_B$ (since $\mathscr{F}$ has these properties), so the proof will be complete if we prove that 6.3 holds.

Since $\mathscr{F}$ is $(v_{B,r})$-continuous, then for every $B \in \mathscr{B}$, $r \in \mathscr{R}$ and $\varepsilon > 0$ there exist $\delta > 0$ and $1 \geqq \delta' > 0$ such that $r(\mathscr{F}(f)) < \varepsilon$ holds for all $f \in \mathscr{C}_B$ which verify $q_B \circ f \mid T - E \leqq \delta'$ for some $E \in \Sigma$ with $v_{B,r}(E) < \delta$. Then, if $g \in \mathscr{S}_B$ is such that $q_B \circ g \mid T - E \leqq \delta'$ for some $E \in \Sigma$ with $v_{B,r}(E) < \delta/2$, there exist $E' \in \Sigma$ and $f \in \mathscr{C}_B$ such that $v_{B,r}(E') < \delta/2$, $g \mid T - E' \equiv f \mid T - E'$ and

$$r(\mathscr{G}(g)) \leqq r(\mathscr{F}(f)) + \varepsilon\,.$$

Therefore, $q_B \circ f \mid T - (E \cup E') \leqq \delta'$, $v_{B,r}(E \cup E') < \delta$, $r(\mathscr{G}(g)) \leqq 2\varepsilon$ and $\mathscr{G}$ is $(v_{B,r})$-continuous.

Moreover, if $B \in \mathscr{B}$ and the sequence $(h_n) \subset \mathscr{S}_B$ is uniformly convergent to $f \in \mathscr{G}_B$, then for every $\varepsilon > 0$ and $r \in \mathscr{R}$ there exist $\delta > 0$ and $1 \geqq \delta' > 0$ such that $r(\mathscr{F}(g)) < \varepsilon$ for every $g \in \mathscr{C}_B$ which verifies $q_B \circ g \mid T - E \leqq \delta'$ for some $E \in \Sigma$ with $v_{B,r}(E) < \delta/2$. Moreover, we can find $n_0 \in \mathbb{N}$, $f_{n_0} \in \mathscr{C}_B$ and $E \in \Sigma$ with $v_{B,r}(E) < \delta$ such that

$q_B(h_{n_0} - f) < \delta'$, $f_{n_0} \mid T - E \equiv h_{n_0} \mid T - E$ and

$$r(\mathscr{G}(h_{n_0}) - \mathscr{F}(f_{n_0})) < \varepsilon .$$

Since $q_B(f_{n_0} - f) \mid T - E \leqq \delta'$ and $\mathscr{F}$ is $(v_{B,r})$-continuous, it results that $r(\mathscr{F}(f_{n_0} - f)) < \varepsilon$. Therefore, $r(\mathscr{G}(h_{n_0}) - \mathscr{F}(f)) < 2\varepsilon$ holds and 6.3 is verified.

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