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THE HILBERT PROJECTIVE METRIC AND AN EQUATION IN A C^*-ALGEBRA

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In [1], P. J. Bushell has proved the following theorem.

Theorem 0. *Let T be a real non-singular $n \times n$ matrix and $k \geqq 1$ a fixed integer. Then there exists a unique real positive definite matrix A such that $T^*A^{2^k}T = A$.*

In this paper we extend this theorem in two directions. Firstly, we consider a more general equation than $T^*A^{2^k}T = A$ and, secondly, we work in a C^*-algebra. To realize this program we derive some fundamental properties of the Hilbert projective pseudometric defined on the cone of invertible positive elements of a C^*-algebra. This is done in the first eleven lemmas which are also of independent interest.

Let A be a (nonzero) unital C^*-algebra (with unit e), $\mathrm{Inv}(A)$ the (multiplicative) group of all invertible elements of A, A_h the (real) linear subspace of A consisting of all Hermitian elements of A, $A_+ = \{a \in A_h: \sigma(a) \geqq 0\}$ (the set of all positive elemenst of A; here $\sigma(a)$ is the spectrum of a and $\sigma(a) \geqq 0$ means $t \geqq 0$ for each $t \in \sigma(a)$), A_+^0 and A_+^b the interior and the boundary of A_+ in A_h, respectively. For $a \in A$ with $\sigma(a) \subset R$, set $m(a) = \min \sigma(a)$ and $M(a) = \max \sigma(a)$; in this case, $m(a) \leqq M(a)$ and the spectral radius $r(a) = \max \{M(a), -m(a)\}$. For $a, b \in A$ write $a \leqq b$ iff $b - a \in A_+$.

It is well known that:

(i) A_+ is a (real) closed convex cone in A_h with $A_h = A_+ - A_+$ and $A_+ \cap \cap (-A_+) = 0$; hence A_h is a partially ordered real linear space (but not a vector lattice if A is not commutative);

(ii) if $a \in A_h$, then
$m(a) = \max \{t \in R: te \leqq a\}$,
$M(a) = \min \{t \in R: a \leqq te\}$;

(iii) if $a \in A_h$, then $\|a\| = r(a)$ $(= \max \{M(a), -m(a)\})$;

(iv) if $a, b, c \in A$ and $a \leqq b$, then $c^*ac \leqq c^*bc$; if, in addition, $a \in A_+$, then $\|a\| \leqq \|b\|$ (that is, the norm is monotone);

(v) if $a \in A_+$, then $a \in \mathrm{Inv}(A)$ iff $m(a) > 0$;

(vi) if $0 \leqq a \leqq b$ and $a \in \mathrm{Inv}(A)$, then $b \in \mathrm{Inv}(A)$ and $0 \leqq b^{-1} \leqq a^{-1}$;

(vii) if $0 \leqq a \leqq b$ and $0 < p \leqq 1$, then $0 \leqq a^p \leqq b^p$.

Note also that:

(viii) if $a, b \in A_h$ and $a \leqq b$, then $m(a) \leqq m(b)$ and $M(a) \leqq M(b)$ (this follows from $m(a)\, e \leqq a \leqq b \leqq M(b)\, e$ and (ii));

(ix) if $a \in A_h$ and $t \in R$, then $m(ta) = \min\{t\, m(a), t\, M(a)\}$ and $M(ta) = \max\{t\, m(a), t\, M(a)\}$;

(x) if $a, b \in A_h$, then $|m(a) - m(b)| \leqq \|a - b\|$ and $|M(a) - M(b)| \leqq \|a - b\|$ (this follows from (viii) and $-\|a - b\|\, e + b \leqq a \leqq \|a - b\|\, e + b$);

(xi) if $a \in \operatorname{Inv}(A)$ and $b \in A$, then $\sigma(ab) = \sigma(ba)$ (this follows from $ab - \lambda e = a(ba - \lambda e)\, a^{-1}$, $\lambda \in C$);

(xii) if $a \in \operatorname{Inv}(A) \cap (A_+ \cup (-A_+))$, then $m(a^{-1}) = M(a)^{-1}$ and $M(a^{-1}) = m(a)^{-1}$.

Most of the above assertions may be found in [2].

For $a \in A_h$ and $r \geqq 0$, let $B_h(a, r)$ be the closed r-ball in A_h centered at a.

Lemma 1. (1) *Let* $a \in A_+$. *Then* $B_h(a, m(a)) \subset A_+$ *and* $\operatorname{dist}(a, A_+^b) = m(a)$, *where* dist *is the distance function.*

(2) $A_+^0 = \{a \in A_+ : m(a) > 0\} = A_+ \cap \operatorname{Inv}(A)$ *and* $A_+^b = \{a \in A_+ : m(a) = 0\} = A_+ \setminus \operatorname{Inv}(A)$.

Proof. By (v), we have $\{a \in A_+ : m(a) > 0\} = A_+ \cap \operatorname{Inv}(A)$ and $\{a \in A_+ : m(a) = 0\} = A_+ \setminus \operatorname{Inv}(A)$.

If $a, b \in A_h$, then $a + b \geqq (m(a) - \|b\|)\, e$. This shows that $B_h(a, m(a)) \subset A_+$ and $\operatorname{dist}(a, A_+^b) \geqq m(a)$ for each $a \in A_+$, and $\{a \in A_+ : m(a) > 0\} \subset A_+^0$. Since $a - (m(a) + r)\, e \notin A_+$ for each $a \in A_+$ and $r > 0$ (because $m(a - (m(a) + r)\, e) = -r < 0$), we also have $\operatorname{dist}(a, A_+^b) \leqq m(a)$. This completes the proof of (1).

Now let $a \in A_+^0$. Then $B_h(a, r) \subset A_+^0$ for some $r > 0$; since $a - re \in B_h(a, r)$, we have $m(a) = m(a - re + re) = m(a - re) + r \geqq r > 0$ which completes the proof of the first equality in (2). The second equality in (2) follows from the first one and the equality $A_+^b = A_+ \setminus A_+^0$.

Let $a, b \in A_+^0$. Then $m(a)\, M(b)^{-1}\, b \leqq m(a)\, e \leqq a \leqq M(a)\, e \leqq M(a)\, m(b)^{-1}\, b$. This and $A_+ \cap (-A_+) = 0$ make it possible to define

$$m(a/b) = \sup\{t \in (0, \infty) : tb \leqq a\} \in (0, \infty),$$
$$M(a/b) = \inf\{t \in (0, \infty) : a \leqq tb\} \in (0, \infty),$$

and

$$d(a, b) = \log\left(M(a/b)\, m(a/b)^{-1}\right).$$

Lemma 2. *Let* $a, b, c \in A_+^0$. *Then*

(1) $m(a/b) = \max\{t \in (0, \infty) : tb \leqq a\}$, $m(a/e) = m(a)$,
$M(a/b) = \min\{t \in (0, \infty) : a \leqq tb\}$, $M(a/e) = M(a)$;

(2) $M(a/b)\, m(b/a) = 1$, $d(a, b) = d(b, a)$;

(3) $M(a/a) = m(a/a) = 1$, $d(a, a) = 0$;

(4) if $t, s \in (0, \infty)$, then
$m(ta/sb) = ts^{-1} m(a/b)$,

$M(ta/sb) = ts^{-1}M(a/b)$,
$d(ta, sb) = d(a, b)$;

(5) $m(a/b) \leqq M(a/b)$, $d(a, b) \geqq 0$;

(6) $m(a/c) \geqq m(a/b)\, m(b/c)$,
$M(a/c) \leqq M(a/b)\, M(b/c)$,
$d(a, c) \leqq d(a, b) + d(b, c)$;

(7) $m(a)\, M(b)^{-1} \leqq m(a/b) \leqq \min\{m(a)\, m(b)^{-1},\ M(a)\, M(b)^{-1}\}$,
$M(a)\, m(b)^{-1} \geqq M(a/b) \geqq \max\{m(a)\, m(b)^{-1},\ M(a)\, M(b)^{-1}\}$;

(8) $d(a, b) = 0$ iff $a = tb$ for some $t \in (0, \infty)$;

(9) if $p \in (0, 1]$, then $a^p \in A^0_+$ and
$m(a^p/b^p) \geqq m(a/b)^p$,
$M(a^p/b^p) \leqq M(a/b)^p$,
$d(a^p, b^p) \leqq p d(a, b)$.

(10) d is a pseudometric on A^0_+ (called *the Hilbert projective pseudometric* on A^0_+).

Proof. (1) follows from the closedness of A_+. (2)–(6) and the first inequality in each row of (7) are trivial. The remaining inequalities in (7) follow by applying the property (viii) to $m(a/b)\, b \leqq a \leqq M(a/b)\, b$. Similarly, (vii) implies (9). (8) is a consequence of (3), (4) and $A_+ \cap (-A_+) = 0$. Finally, (10) follows from (2), (3), (5) and (6).

Lemma 3. (1) *If* $a, b \in A^0_+$, *then*

$$1 - \|a - b\|\, m(b)^{-1} \leqq 1 + \min\{m(a - b)\, m(b)^{-1}, m(a - b)\, M(b)^{-1}\} \leqq$$
$$\leqq m(a/b) \leqq M(a/b) \leqq$$
$$\leqq 1 + \max\{M(a - b)\, m(b)^{-1},\ M(a - b)\, M(b)^{-1}\} \leqq 1 + \|a - b\|\, m(b)^{-1}.$$

(2) *If* $a, b \in A^0_+$ *and* $\|a - b\| < m(b)$, *then*

$$d(a, b) \leqq \log\left((m(b) + \|a - b\|)(m(b) - \|a - b\|)^{-1}\right) \leqq$$
$$\leqq 2\|a - b\|\, (m(b) - \|a - b\|)^{-1}.$$

(3) *The identity mapping* $\mathrm{id}: (A^0_+, \|\cdot\|) \to (A^0_+, d)$ *is Lipschitz continuous on each ball* $B_h(c, r)$ *with* $c \in A^0_+$ *and* $r < m(c)$.

Proof. (1) The fourth inequality follows from

$$a = a - b + b \leqq M(a - b)\, e + b \leqq$$
$$\leqq (1 + \max\{M(a - b)\, m(b)^{-1},\ M(a - b)\, M(b)^{-1}\})\, b.$$

The second inequality follows similarly and the other ones are trivial.

(2) is a direct consequence of (1) and the inequality $\log(1 + t) \leqq t$, $t \in [0, \infty)$.

(3) Let $c \in A^0_+$, $r \in (0, m(c))$ and $s \in (0, m(c) - r)$. Take any $a, b \in B_h(c, r)$ with $\|a - b\| \leqq s$; it is clear that $a, b \in A^0_+$. By (x), $m(c) - r \leqq m(b)$, $\|a - b\| \leqq s < m(c) - r \leqq m(b)$ and we may apply (2) to obtain

$$d(a, b) \leqq 2\|a - b\|\, (m(b) - \|a - b\|)^{-1} \leqq 2(m(c) - r - s)^{-1}\, \|a - b\|.$$

We have shown that $a, b \in B_h(c, r)$ and $\|a - b\| \leqq s$ imply $d(a, b) \leqq 2(m(c) - - r - s)^{-1} \|a - b\|$.

Now take $a, b \in B_h(c, r)$ arbitrarily. Let n be any integer such that $\|a - b\| \leqq ns$, and define $a_i = a + in^{-1}(b - a)$ $(i = 0, 1, \ldots, n)$. Then $\|a_i - a_{i+1}\| \leqq s$ and hence

$$d(a_i, a_{i+1}) \leqq 2(m(c) - r - s)^{-1} \|a_i - a_{i+1}\| = \\ = 2n^{-1}(m(c) - r - s)^{-1} \|a - b\| \quad \text{for all} \quad i = 0, 1, \ldots, n - 1,$$

and consequently,

$$d(a, b) \leqq \sum_{n=0}^{n-1} d(a_i, a_{i+1}) \leqq 2(m(c) - r - s)^{-1} \|a - b\| .$$

As $s \in (0, m(c) - r)$ was arbitrary, we have

$$d(a, b) \leqq 2(m(c) - r)^{-1} \|a - b\| \quad \text{for all} \quad a, b \in B_h(c, r) .$$

Lemma 4. *Let* $a, b \in A^0_+$. *Then*:

(1) $$M(a - b) \leqq \min \{\max \{(M(a/b) - 1) M(b), (M(a/b) - 1) m(b)\}, \\ \max \{(1 - m(b/a)) m(a), (1 - m(b/a)) M(a)\}\} = \\ = \begin{cases} (1 - m(b/a)) M(a) & \text{if} \quad M(a/b) \geqq 1 , \\ (M(a/b) - 1) m(b) & \text{if} \quad M(a/b) \leqq 1 ; \end{cases}$$

(2) $$m(a - b) \geqq \max \{\min \{(m(a/b) - 1) m(b), (m(a/b) - 1) M(b)\}, \\ \min \{(1 - M(b/a)) M(a), (1 - M(b/a)) m(a)\}\} = \\ = \begin{cases} (1 - M(b/a)) m(a) & \text{if} \quad m(a/b) \geqq 1 , \\ (m(a/b) - 1) M(b) & \text{if} \quad m(a/b) \leqq 1 ; \end{cases}$$

(3) $$\|a - b\| \leqq \max \{(1 - m(b/a)) M(a), (1 - m(a/b)) M(b)\} \leqq \\ \leqq |M(a) - M(b)| + \\ + \max \{M(b) - M(a) m(b/a), M(a) - M(b) m(a/b)\} \leqq \\ \leqq |M(a) - M(b)| + \\ + \max \{M(a) (M(b/a) - m(b/a)), M(b) (M(a/b) - m(a/b))\} \leqq \\ \leqq |M(a) - M(b)| + \max \{M(a), M(b)\} (\exp (d(a, b)) - 1) .$$

Proof. From $a - b \leqq (M(a/b) - 1) b$ and $a - b \leqq (1 - m(b/a)) a$ one has

$$M(a - b) \leqq \max \{(M(a/b) - 1) M(b), (M(a/b) - 1) m(b)\}$$

and

$$M(a - b) \leqq \max \{(1 - m(b/a)) m(a), (1 - m(b/a)) M(a)\}$$

which gives the inequality in (1). If $M(a/b) \geqq 1$ (or $M(a/b) \leqq 1$), then $m(b/a) = = M(a/b)^{-1} \leqq 1$ ($m(b/a) \geqq 1$, respectively) and, by Lemma 2, (7), the right hand side of the inequality in (1) equals

$$\min \{(M(a/b) - 1) M(b), (1 - m(b/a)) M(a)\} = \\ = (1 - m(b/a)) . \min \{M(b) m(b/a)^{-1}, M(a)\} = (1 - m(b/a)) M(a)$$

(or, respectively $\min\{(M(a/b)-1)\,m(b),\ (1-m(b/a))\,m(a)\} =$
$= (M(a/b)-1)\,.\,\max\{m(b),\ m(a)\,M(a/b)^{-1}\} = (M(a/b)-1)\,m(b))$.

(2) Follows similarly (or also from (1) by using $m(a-b) = -M(b-a)$).

Since $\|a-b\| = \max\{M(a-b),\ -m(a-b)\}$, we have by (1) and (2),

a) $\|a-b\| \leqq \max\{(1-m(b/a))\,M(a),\ (M(b/a)-1)\,m(a)\} =$
$= (1-m(b/a))\,M(a)$ if $m(a/b) \geqq 1$;

b) $\|a-b\| \leqq \max\{(1-m(b/a))\,M(a),\ (1-m(a/b))\,M(b)\}$
if $m(a/b) \leqq 1 \leqq M(a/b)$;

c) $\|a-b\| \leqq \max\{(M(a/b)-1)\,m(b),\ (1-m(a/b))\,M(b)\} =$
$= (1-m(a/b))\,M(b)$ if $M(a/b) \leqq 1$.

This proves the first inequality in (3). Since

$$\begin{aligned}
&\max\{(1-m(b/a))\,M(a),\ (1-m(a/b))\,M(b)\} = \\
&= \max\{M(a)-M(b)+M(b)-M(a)\,m(b/a), \\
&\quad M(b)-M(a)+M(a)-M(b)\,m(a/b)\} \leqq \\
&\leqq \max\{M(a)-M(b),\ M(b)-M(a)\} + \\
&+ \max\{M(b)-M(a)\,m(b/a),\ M(a)-M(b)\,m(a/b)\}\,,
\end{aligned}$$

we have the second inequality in (3). As, by Lemma 2, (7), $M(b) \leqq M(b/a)\,M(a)$ and $M(a) \leqq M(a/b)\,M(b)$, we have the third inequality in (3). Again by Lemma 2, (7),

$$\begin{aligned}
M(a)\,(M(b/a)-m(b/a)) &= M(a)\,m(b/a)\,(\exp(d(a,b))-1) \leqq \\
&\leqq M(b)\,(\exp(d(a,b))-1)\,,
\end{aligned}$$

and similarly

$$M(b)\,(M(a/b)-m(a/b)) \leqq M(a)\,(\exp(d(a,b))-1)\,,$$

which gives the fourth inequality in (3).

Lemma 5. *Let $c \in A$. Then the following are equivalent*:

(1) $c^*ac \in A_+^0$ *for all* $a \in A_+^0$;

(2) $m(c^*c) > 0$;

(3) $c^*c \in A_+^0$;

(4) $c = uh$, *where $h \in A_+^0$ and $u \in A$ is an isometry* (i.e. $u^*u = e$).

Proof. (1) $\Rightarrow$ (3). As $e \in A_+^0$, we have $c^*c = c^*ec \in A_+^0$.

(2) and (3) are equivalent by Lemma 1.

(2) $\Rightarrow$ (1). Let $a \in A_+^0$. Then $a \geqq m(a)\,e$, where by Lemma 1 $m(a) > 0$. By (iv), $c^*ac \geqq m(a)\,c^*c$, and hence, by (viii) and (ix), $m(c^*ac) \geqq m(a)\,m(c^*c) > 0$. By Lemma 1 we conclude that $c^*ac \in A_+^0$.

(4) $\Rightarrow$ (2). We have $c^*c = hu^*uh = h^2$ and hence $m(c^*c) = m(h)^2 > 0$.

(2) $\Rightarrow$ (4). Set $h = (c^*c)^{1/2}$. Then $m(h) = m(c^*c)^{1/2} > 0$ and hence $h \in A_+^0$. Set $u = ch^{-1}$. Then $u^*u = h^{-1}c^*ch^{-1} = h^{-1}h^2h^{-1} = e$.

Lemma 6. *Let* $a, b \in A_+^0$ *and* $c \in A$ *with* $c^*c \in A_+^0$ *(i.e.* $m(c^*c) > 0$*). Then*
$M(c^*ac/c^*bc) \leqq M(a/b)$,
$m(c^*ac/c^*bc) \geqq m(a/b)$,
$d(c^*ac, c^*bc) \leqq d(a, b)$.

Proof. Since $m(a/b)\, b \leqq a \leqq M(a/b)\, b$, we have by (iv) $m(a/b)\, c^*bc \leqq c^*ac \leqq$ $\leqq M(a/b)\, c^*bc$, which gives the first two inequalities in the lemma. The third one follows from the preceding inequalities.

Lemma 7. *Let* $c \in A$. *Then the following are equivalent:*
(1) $c^*ac \in A_+^0$ *and* $M(c^*ac/c^*bc) = M(a/b)$ *for all* $a, b \in A_+^0$;
(2) $c^*ac \in A_+^0$ *and* $m(c^*ac/c^*bc) = m(a/b)$ *for all* $a, b \in A_+^0$;
(3) $c^*ac \in A_+^0$ *and* $d(c^*ac, c^*bc) = d(a, b)$ *for all* $a, b \in A_+^0$;
(4) $c \in \mathrm{Inv}(A)$.

Proof. (1) and (2) are equivalent by Lemma 2, (2). Hence each of (1) and (2) implies (3). From the definition of the Hilbert projective pseudometric and Lemma 6 it is also easy to see that (3) implies both (1) and (2).

(4) $\Rightarrow$ (1). Let $c \in \mathrm{Inv}(A)$ and $a, b \in A_+^0$. By Lemma 1, c^*c, $(c^{-1})^*\, c^{-1} \in A_+ \cap$ $\cap \mathrm{Inv}(A) = A_+^0$. Now by Lemma 6 we have

$$M(c^*ac/c^*bc) \leqq M(a/b) =$$
$$= M((c^{-1})^*\,(c^*ac)\, c^{-1}, (c^{-1})^*\,(c^*bc)\, c^{-1}) \leqq M(c^*ac/c^*bc)$$

and hence $M(c^*ac/c^*bc) = M(a/b)$.

(2) $\Rightarrow$ (4). We have $c^*c = c^*ec \in A_+^0$ so that $m(c^*c) > 0$. Further, by Lemma 2, (7),

$$m(cc^*) = m(cc^*/e) = m(c^*cc^*c/c^*ec) \geqq m(c^*c)^2/M(c^*c) > 0\,.$$

We have proved that both c^*c and cc^* are invertible and hence $c \in \mathrm{Inv}(A)$.

Lemma 8. *Let* $a \in A_+^0$ *and* $b \in A_h$. *Then* $\sigma(ab) = \sigma(ba) \subset R$, $M(ab) = M(ba) \leqq$ $\leqq \max\{m(a)\, M(b), M(a)\, M(b)\}$ *and* $m(ab) = m(ba) \geqq \min\{m(a)\, m(b), M(a)\, m(b)\}$

Proof. Set $c = a^{1/2}$. As $c \in A_+^0$ and $cbc \in A_h$, we have by (xi), $\sigma(ab) = \sigma(ba) =$ $= \sigma(cbc) \subset R$,
(a) $m(ab) = m(ba) = m(cbc)$,
and
(b) $M(ab) = M(ba) = M(cbc)$.

By (iv), $m(b)\, e \leqq b \leqq M(b)\, e$ implies

$$m(b)\, a = m(b)\, c^*c \leqq c^*bc \leqq M(b)\, c^*c = M(b)\, a\,.$$

By (ix) and (viii) we have

$$\min\{m(b)m(a), m(b)M(a)\} = m(m(b)a) \leqq m(c^*bc) = m(cbc)$$

and similarly

$$\max\{M(b)\, m(a), M(b)\, M(a)\} \geqq M(cbc)\,.$$

This together with (a) and (b) gives the inequalities in the lemma.

Corollary 1. *Let* $a, b \in A_+^0$. *Then* $\sigma(b^{-1}a) = \sigma(ab^{-1}) \subset (0, \infty)$, $M(a/b) = M(b^{-1}a) = M(ab^{-1})$, $m(a/b) = m(b^{-1}a) = m(ab^{-1})$.

Proof. By Lemma 8 (or by (xi)) we have $\sigma(b^{-1}a) = \sigma(ab^{-1}) = \sigma(b^{-1/2}ab^{-1/2}) \subset (0, \infty)$. By Lemma 7, $M(a/b) = M(b^{-1/2}ab^{-1/2}/b^{-1/2}bb^{-1/2}) = M(b^{-1/2}ab^{-1/2}) = M(b^{-1}a) = M(ab^{-1})$. The assertion concerning $m(a/b)$ follows similarly (or from that for $M(b/a)$).

Set $E = \{a \in A_+^0 : \|a\| = 1\}$ $(= \{a \in A_+^0 : M(a) = 1\})$. By Lemma 2, (E, d) is a metric space.

Lemma 9. (1) *The identity mapping* id: $(E, \|\cdot\|) \to (E, d)$ *is Lipschitz continuous on each ball* $E \cap B_h(c, r)$ *with* $c \in E$ *and* $r < m(c)$;

(2) *the identity mapping* id: $(E, d) \to (E, \|\cdot\|)$ *is Lipschitz continuous*, $\|a - b\| \leqq d(a, b)$ *for* $a, b \in E$;

(3) *if* $A \neq C$, *then the identity mapping* id: $(E, \|\cdot\|) \to (E, d)$ *is not uniformly continuous and the (best) Lipschitz constant of its inverse equals one.*

Proof. (1) is a special case of Lemma 3, (3).

(2) Let $a, b \in E$. By Lemma 2, (7), $m(a/b) \leqq 1 \leqq M(a/b)$ and $m(b/a)$ $1 \leqq M(b/a)$. Note also that $1 - t \leqq \log(t^{-1})$ for $t \in (0, 1]$. Thus, by Lemma 4, (3), we have

$$\begin{aligned}\|a - b\| &\leqq \max\{1 - m(b/a),\ 1 - m(a/b)\} \leqq \\ &\leqq \max\{\log(m(b/a)^{-1}),\ \log(m(a/b)^{-1})\} \leqq \\ &\leqq \max\{\log(M(b/a)\, m(b/a)^{-1}),\ \log(M(a/b)\, m(a/b)^{-1})\} = d(a, b)\,.\end{aligned}$$

(3) Assume that $A \neq C$. Since $A = A_+ - A_+ + \mathrm{i}(A_+ - A_+)$, we have $A_+ \neq [0, \infty)\, e$, $A_+^0 \neq (0, \infty)\, e$ and hence $E \neq \{e\}$. Take any $a \in E$ with $a \neq e$. Then $m(a) < 1$. For simplicity, we shall write here m for $m(a)$. For $t \in (-\infty, m)$ set $b_t = a - te$. Since $m(b_t) = m - t > 0$, we have $b_t \in A_+^0$ and $a_t = \|b_t\|^{-1}\, b_t \in E$ for all $t \in (-\infty, m)$. One easily computes (for example, by the spectral mapping theorem and Corollary 1) that

$$\|a_t - a_s\| = (1 - m)(1 - s)^{-1}(1 - t)^{-1}(t - s)$$

and

$$d(a_t, a_s) = \log(1 + (1 - m)(1 - s)^{-1}(m - t)^{-1}(t - s))$$

for $s \leqq t < m$.

Let $\varepsilon > 0$ be given. Set

$$r = \min\{(1 - m)^{-1}, ((3 - m)(\exp(\varepsilon) - 1))^{-1}\}\,.$$

Take any $h \in (0, (1 - m)\, r)$ and set $t = m - h^2$ and $s = m - h - h^2$. Then

$$\|a_t - a_s\| \leqq (1 - m)^{-1}(t - s) = (1 - m)^{-1}\, h\,, \quad h \leqq 1\,,$$

but

$$\begin{aligned}d(a_t, a_s) &= \log(1 + (1 - m)(1 - m + h + h^2)^{-1}\, h^{-1}) \geqq \\ &\geqq \log(1 + (1 - m)(3 - m)^{-1}\, h^{-1}) \geqq \log(1 + (3 - m)^{-1}\, r^{-1}) \geqq \varepsilon\,.\end{aligned}$$

This shows that the mapping id: $(E, \|\cdot\|) \to (E, d)$ is not uniformly continuous.

Since $M(a^t) = 1$ and $m(a^t) = m^t$, we have $a^t \in E$ and $d(a^t, e) = -t \,.\, \log m$ for all $t \in (0, \infty)$. As $\|a^t - e\| = 1 - m^t$, we have

$$\lim_{t \to 0_+} \|a^t - e\| \, d(a^t, e)^{-1} = 1$$

and hence, by (2), the (best) Lipschitz constant of the mapping id: $(E, d) \to (E, \|\cdot\|)$ equals one.

Remark. Lemma 9 implies that the topologies of $(E, \|\cdot\|)$ and (E, d) coincide, but the corresponding uniformities do not (if $A \neq C$). Therefore we may speak about the topological space E.

One may easily see that the topological space E is arcwise connected. Indeed, for given $a, b \in E$ and $t \in [0, 1]$, set $f(t) = \|(1 - t)\, a + tb\|^{-1} ((1 - t)\, a + tb)$. Then f is a continuous arc in E joining a to b.

Let us also show that (E, d) is metrically convex. Since $d(a, b) = d(b^{-1/2} a b^{-1/2}, e)$ for each $a, b \in A_+^0$, it is sufficient to show that for each $a \in E$ and $t \in [0, 1]$ there exists $c \in E$ with $d(a, c) = (1 - t)\, d(a, e)$ and $d(c, e) = td(a, e)$. But this is easy, set $c = a^t$. In general, for given a and t, this point c is not determined uniquely provided $t \in (0, 1)$. (For example, if $\sigma(a)$ contains at least three points, then there exist infinitely many such points c even in $E \cap C^*(a)$, where $C^*(a)$ is the unital C^*-subalgebra of A generated by a.)

On the other hand, the metric space $(E, \|\cdot\|)$ is not generally metrically convex. (For example, consider $A = C(X)$, where X is a compact space with at least two isolated points.) Nevertheless, for each $a \in E$, the point $(a + e)/2$ is a midpoint in $(E, \|\cdot\|)$ between a and e.

Lemma 10. *The metric space (E, d) is complete.*

Proof. Let $\{a_k\}$ be a Cauchy sequence in (E, d). By Lemma 9, (2), the sequence $\{a_k\}$ is also Cauchy in $(E, \|\cdot\|)$ and hence it converges in the norm to some $a \in A_+$ with $\|a\| = 1$. By Lemma 2, (7), we have

$$m(a_k) \geqq m(a_k/a_n)\, m(a_n) =$$
$$= m(a_n)\, M(a_k/a_n) \,.\, \exp\left(-d(a_k, a_n)\right) \geqq m(a_n) \,.\, \exp\left(-d(a_k, a_n)\right)$$

for all k, n. This and the Cauchy property of the sequence $\{a_k\}$ in (E, d) implies that the sequence $\{m(a_k)\}$ is bounded away from zero. By (x), $m(a) = \lim m(a_k)$ and hence $m(a) > 0$. Thus, the element a lies in E. By Lemma 9, (1), we conclude that $a_k \to a$ in (E, d).

Let $G\colon A_+^0 \to A$ and $p\colon A_+^0 \to (0, \infty)$. We say that G is 1) increasing if $a, b \in A_+^0$, $a \leqq b$ implies $Ga \leqq Gb$; 2) p-homogeneous if $G(ta) = t^{p(a)} Ga$ for all $a \in A_+^0$ and $t \in (0, \infty)$.

Lemma 11. *Let $p, f\colon A_+^0 \to (0, \infty)$ and $G\colon A_+^0 \to A_+^0$ be given. Define $T\colon A_+^0 \to A_+^0$ and $S\colon E \to E$ by*

$Ta = f(a)\, Ga, \ a \in A^0_+,$
$Sa = \|Ta\|^{-1}\, Ta \ (= \|Ga\|^{-1}\, Ga), \ a \in E.$
Further, for $a \in A^0_+$ *define* $f_a: (0, \infty) \to (0, \infty)$ *by* $f_a(t) = f(ta) \,.\, t^{p(a)-1}$, $t \in (0, \infty)$.
Finally, assume that G *is increasing and p-homogeneous. Then*:
(1) $p(ta) = p(a)$ *for* $a \in A^0_+$, $t \in (0, \infty)$;
(2) $M(Ga/Gb) = f(a)^{-1} f(b)\, M(Ta/Tb) \leqq \min \{M(a/b)^{p(a)}, M(a/b)^{p(b)}\}$,
$m(Ga/Gb) = f(a)^{-1} f(b)\, m(Ta/Tb) \geqq \max \{m(a/b)^{p(a)}, m(a/b)^{p(b)}\}$,
$d(Ga, Gb) = d(Ta, Tb) \leqq \min \{p(a), p(b)\}\, d(a, b)$ *for all* $a, b \in A^0_+$;
(3) $M(Sa/Sb) = \|Ga\|^{-1} \|Gb\|\, M(Ga/Gb) \leqq$
$\leqq \|Ga\|^{-1} \|Gb\| \min \{M(a/b)^{p(a)}, M(a/b)^{p(b)}\}$,
$m(Sa/Sb) = \|Ga\|^{-1} \|Gb\|\, m(Ga/Gb) \geqq$
$\geqq \|Ga\|^{-1} \|Gb\| \max \{m(a/b)^{p(a)}, m(a/b)^{p(b)}\}$,
$d(Sa, Sb) \leqq \min \{p(a), p(b)\}\, d(a, b)$ *for all* $a, b \in E$;
(4) $\|Ga - Gb\| \leqq \|Gb\| \max \{(1 + \|a - b\|\, m(b)^{-1})^{p(b)} - 1,$
$1 - (1 - \|a - b\|\, m(b)^{-1})^{p(b)}\}$
for $a, b \in A^0_+$ *with* $\|a - b\| < m(b)$;
(5) $G: A^0_+ \to A^0_+$ *and* $p: A^0_+ \to (0, \infty)$ *are continuous*;
(6) *if* u *is a fixed point of* S *and* $t \in (0, \infty)$, *then* $x = tu$ *is a fixed point of* T *iff* $f_u(t) = \|Gu\|^{-1}$;
(7) *if* S *has a fixed point and* $f_a(0, \infty) = (0, \infty)$ *for each* $a \in A^0_+$, *then* T *has a fixed point*;
(8) *if* x *is a fixed point of* T, *then* $u = \|x\|^{-1}\, x$ *is a fixed point of* S;
(9) *if* S *has at most one fixed point and* $f_a(t) = f(a)$ *for each* $a \in A^0_+$ *and* $t \in (0, \infty)$ *with* $t \neq 1$, *then* T *has at most one fixed point*;
(10) *if* S *has a unique fixed point*, $f_a(0, \infty) = (0, \infty)$ *and* $f_a(t) \neq f(a)$ *for each* $a \in A^0_+$ *and* $t \in (0, \infty)$ *with* $t \neq 1$, *then* T *has a unique fixed point.*

Proof. (1) Let $a \in A^0_+$ and $t \in (0, \infty)$. Then $G(ta) = t^{p(a)} Ga = t^{p(a)} G(t^{-1} ta) = t^{p(a) - p(ta)} G(ta)$ and thus $p(a) = p(ta)$.

(2) Let $a, b \in A^0_+$. By the assumption on G, the inequalities $m(a/b)\, b \leqq a \leqq M(a/b)\, b$ imply $m(a/b)^{p(b)} Gb = G(m(a/b)\, b) \leqq Ga \leqq G(M(a/b)\, b) = M(a/b)^{p(b)} Gb$. Similarly, the inequalities $m(b/a)\, a \leqq b \leqq M(b/a)\, a$ imply $m(b/a)^{p(a)} Ga \leqq Gb \leqq M(b/a)^{p(a)} Ga$. These inequalities and Lemma 2 give the result of (2).

(3) is a consequence of (2).

(4) Let $a, b \in A^0_+$ and $\|a - b\| < m(b)$. Using Lemma 3, (1) and the properties of G we obtain

$$(1 - \|a - b\|\, m(b)^{-1})^{p(b)}\, Gb \leqq Ga \leqq (1 - \|a - b\|\, m(b)^{-1})^{p(b)}\, Gb\,,$$

that is

$$-(1 - (1 - \|a - b\|\, m(b)^{-1})^{p(b)}\, Gb \leqq Ga - Gb \leqq$$
$$\leqq ((1 + \|a - b\|\, m(b)^{-1})^{p(b)} - 1)\, Gb\,.$$

Using (iii), (viii) and (ix) we obtain the result.

(5) The continuity of G is a direct consequence of (4). Let $a, b \in A_+^0$ and $t \in (0, \infty)$, $t \neq 1$. Then $\|G(ta)\| = t^{p(a)}\|Ga\|$, $\|G(tb)\| = t^{p(b)}\|Gb\|$ and hence

$$p(a) - p(b) = (\log t)^{-1} \log (\|G(ta)\| \, \|Gb\| \, \|G(tb)\|^{-1} \, \|Ga\|^{-1}) .$$

This and the continuity of G imply that p is also continuous.

(6) Let u be a fixed point of S, $t \in (0, \infty)$ and $x = tu$. Then $Tx = f(x)\,Fx = f(tu) \,.\, t^{p(u)}Gu = f(tu) \,.\, t^{p(u)}\|Gu\| \, Su = f_u(t) \,.\, t\|Gu\| \, Su = f_u(t) \,.\, t\|Gu\| \, u = f_u(t) \, \|Gu\| \, x$. Hence x is a fixed point of T iff $f_u(t) \, \|Gu\| = 1$.

(7) is an immediate consequence of (6).

(8) Let x be a fixed point of T and set $u = \|x\|^{-1} x$. Then $u \in E$ and $Su = \|Tu\|^{-1} \, Tu = \|Tx\|^{-1} \, Tx = \|x\|^{-1} \, x = u$.

(9) Let $x, y \in A_+^0$ be fixed points of T. Then both $\|x\|^{-1} x$ and $\|y\|^{-1} y$ are fixed points of S and hence $\|x\|^{-1} x = \|y\|^{-1} y$. Set $t = \|y\| \, \|x\|^{-1}$. Then

$$tx = y = Ty = f(x)^{-1} f(tx) \,.\, t^{p(x)}Tx = t\, f(x)^{-1} f_x(t)\, x$$

and hence $f_x(t) = f(x)$. By assumption, $t = 1$ and $x = y$. Therefore T has at most one fixed point.

(10) is a consequence of (7) and (9).

Now we are prepared to prove the following abstract fixed point theorem.

Theorem 1. *Let $p, f\colon A_+^0 \to (0, \infty)$ and $G\colon A_+^0 \to A_+^0$ be given. Define $T\colon A_+^0 \to A_+^0$ by $Ta = f(a)\, Ga$, $a \in A_+^0$, and, for each $a \in A_+^0$, define $f_a\colon (0, \infty) \to (0, \infty)$ by $f_a(t) = f(ta) \,.\, t^{p(a)-1}$, $t \in (0, \infty)$. Assume that G is increasing and p-homogeneous with $\sup p(A_+^0) < 1$. Suppose that $f_a(0, \infty) = (0, \infty)$ for each $a \in A_+^0$. Then T has a fixed point. If, in addition, $f_a(t) \neq f(a)$ for each $a \in A_+^0$ and $t \in (0, \infty)$ with $t \neq 1$, then T has a unique fixed point.*

Proof. Define $S\colon E \to E$ as in Lemma 11. By Lemma 11, (3), the mapping $S\colon (E, d) \to (E, d)$ is an L-contraction, where $L = \sup p(A_+^0) < 1$. Since (E, d) is complete by Lemma 10, we may apply the Banach contraction principle to obtain a unique fixed point of S. Now the theorem follows from Lemma 11.

A direct consequence of Theorem 1 is the main existence result of this paper.

Theorem 2. *Let $p_0, p_1, \ldots, p_n \in (0, 1]$ with $p = p_0 p_1 \ldots p_n < 1$, $c_i \in A_+^0$ with $c_i^* c_i \in A_+^0$ $(i = 1, \ldots, n)$ be given. Let $f\colon A_+^0 \to (0, \infty)$ be such that for each $a \in A_+^0$ and $s \in (0, \infty)$ there exists $t \in (0, \infty)$ satisfying $f(ta)\, t^{p-1} = s$. Then there exists $x \in A_+^0$ such that*

$$f(x) \,.\, (c_n^*(c_*^{n-1} \ldots (c_1^* x^{p_0} c_1)^{p_1} \ldots c_{n-1})^{p_{n-1}} c_n)^{p_n} = x .$$

If, in addition, $f(ta)\, t^{p-1} \neq f(a)$ for all $a \in A_+^0$ and $t \in (0, \infty)$ with $t \neq 1$, then such x is unique.

Proof. By Lemmas 2 and 6 we may define a mapping $G\colon A_+^0 \to A_+^0$ by

$$Ga = (c_n^*(c_{n-1}^* \ldots (c_1^* a^{p_0} c_1)^{p_1} \ldots c_{n-1})^{p_{n-1}} \, c_n)^{p_n} , \quad a \in A_+^0 .$$

The mapping G is clearly p-homogeneous and, by (iv) and (vii), also increasing. Now it remains to apply Theorem 1.

Corollary 2. *Let H be a complex or real Hilbert space, $L(H)$ the algebra of all bounded linear operators on H, $C_i \in L(H)$ with $C_i^* C_i \in L(H)_+^0$ $(i = 1, \ldots, n)$, $p_0, p_1, \ldots, p_n \in (0, 1]$ with $p = p_0 p_1 \ldots p_n < 1$, and let f be a positive function on $L(H)_+^0$ such that for each $T \in L(H)_+^0$ and $s \in (0, \infty)$ there exists $t \in (0, \infty)$ satisfying $f(tT) \, . \, t^{p-1} = s$. Then there exists an operator $X \in L(H)_+^0$ such that*

$$f(X) \, . \, (C_n^*(C_{n-1}^* \ldots (C_1^* X^{p_0} C_1)^{p_1} \ldots C_{n-1})^{p_{n-1}} C_n)^{p_n} = X \, .$$

If, in addition, $f(tT) \, . \, t^{p-1} \neq f(T)$ for each $T \in L(H)_+^0$ and $t \in (0, \infty)$ with $t \neq 1$, then such operator X is unique.

Proof. If H is a complex Hilbert space, then the corollary is a special case of Theorem 2.

Assume that H is a real Hilbert space and let H^c be its complexification. For $T \in L(H)$, let T^c $(= T + iT)$ be the complexification of T. Set $A = L(H^c)$ and $D = \{T^c : T \in L(H)_+^0, \|T\| = 1\}$. One easily sees that $(L(H)_+)^c$ is a closed subset of A_+, $(L(H)_+^0)^c \subset A_+^0$ and $D \subset E$. Since $M(T^c/S^c) = M(T/S)$ and $m(T^c/S^c) = m(T/S)$, we have $d(T^c, S^c) = d(T, S)$ for all $T, S \in L(H)_+^0$. By the "real" variant of Lemma 10, the metric space (D, d) is complete and hence closed in (E, d). Now it is sufficient to note that the complexification of the mapping, defined by the left hand side of the equation in the statement of the corollary, maps $(L(H)_+^0)^c$ into itself, and that any fixed point of this mapping is of the form X^c for some $X \in L(H)_+^0$.

It is clear that Bushell's theorem (see Theorem 0) is a special case of this corollary because the equation $T^* A^{2^k} T = A$ may be transformed into the equation $B = (T^{-1})^* B^{2^{-k}} (T^{-1})$.

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